A TEXTBOOK

ON

METALLURGY

OF

GOLD, SILVER, COPPER, LEAD, AND ZINC

INTERNATIONAL CORRESPONDENCE SCHOOLS
SCRANTON, PA.

ARITHMETIC
MENSURATION
ELEMENTARY ALGEBRA AND TRIGONOMETRIC FUNCTIONS
MECHANICS
HYDRAULICS AND HYDRAULIC MACHINERY

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PREFACE

There has long been a demand for some work, written from the practical rather than from the theoretical point of view, on the metallurgy of gold, silver, copper, lead, and zinc. Our long experience in connection with our Metal Mining and Metal Prospector's Courses has made us familiar with the conditions that prevail in the regions where mining and metallurgical operations are carried on, and has enabled us to prepare a thoroughly practical Course on the metallurgy of the above-named metals. In the preparation of this Course, as in our other Courses, the various papers have been written by specialists, each metal being handled by some writer who was thoroughly familiar with its reduction, and many of the papers being then submitted to other experts for criticism and suggestions.

The preliminary studies include the necessary mathematics, papers on Steam Boilers, Steam Engines, and Hydraulics and Hydraulic Machinery. The last-named subject also treats of the development and measurement of water-power, and will be of great value to any one operating in regions where the mills are driven by water-power. This section is followed by papers on Elementary Chemistry, Chemistry and Chemical Operations, Blowpiping, Mineralogy, and Assaying. Mineralogy is introduced to familiarize the metallurgist with the ores from which the metals are obtained. The paper on Surface Arrangements at Reduction Works treats of the general arrangement of reduction works. The paper on Ore Sampling treats of both mechanical and hand sampling. The paper on Ore Dressing and

Milling treats of the reduction and concentration of ores, and also of the amalgamation of gold and silver ores. Following this there are special papers on the following subjects: The Cyanide Process, treating of the recovery of gold and silver; Hyposulphite Lixiviation, treating of the recovery of silver; The Chlorination Process, treating of the recovery of gold; Copper Smelting and Refining, including the treatment of by-products; Lead Smelting and Refining, including the treatment of by-products; and Zinc Smelting and Refining. These are followed by three papers entitled Electrometallurgy, treating of the separation and recovery of different metals by electrolysis.

In preparing this Course, it has been our aim to produce something that would be of use to the practical millman and metallurgist, and also to the theoretical chemist. The practical working of the different processes is described in such a manner that those who are now engaged at metallurgical plants, or who wish to become familiar with the processes in use, can obtain a thorough knowledge of them without consulting other authorities. The whole work is the most thoroughly up-to-date and practical textbook that has yet appeared on the metallurgy of the metals specified.

The following are some of the writers not of our own staff who have assisted us in this work: Prof. L. J. Stabler, University of Southern California; W. T. Weightman, Electric Reduction Co., Buffalo, New York; W. P. Cleveland, Joplin, Missouri. In addition to the writers named, we have been assisted by several practical metallurgists and college professors. W. H. Graves and H. M. Lane, among our own force, have contributed to the work or assisted in the revision, preparation, etc., the whole being edited by E. B. Wilson, Principal of our School of Metal Mining.

We wish to extend our thanks to the many metallurgical works and metallurgists throughout the country who have cheerfully answered questions, admitted the different members of our staff to their works, or shown us other courtesies.

The method of numbering the pages, cuts, articles, etc. is such that each paper and part is complete in itself; hence,

in order to make the indexes intelligible, it was necessary to give each paper and part a number. This number is placed at the top of each page, on the headline, opposite the page number; to distinguish it from the page number, it is preceded by the printer's section mark §. Consequently, a reference such as § 33, page 28, is readily found as follows: Look along the inside edges of the headlines until § 33 is found, and then through § 33 until page 28 is found.

The Examination Questions and their answers are grouped together at the ends of the volumes containing the papers to which they refer, and are given the same section numbers.

INTERNATIONAL CORRESPONDENCE SCHOOLS.

CONTENTS

ARITHMETIC.

(PART 1.)

DEFINITIONS.

1. Arithmetic is the art of reckoning, or the study of numbers.

2. A **unit** is *one*, or a single thing, as *one*, *one* car, *one* prop, *one* drill.

3. A **number** is a unit, or a collection of units, as *one*, *three* rails, *five* wheels.

4. The **unit of a number** is one of the collection of units which constitutes the number. Thus, the unit of *twelve* is *one*, of *twenty* picks is *one* pick.

5. A **concrete number** is a number applied to some particular kind of object or quantity, as three *mules*, five *shovels*, ten *tons*.

6. An **abstract number** is a number that is not applied to any object or quantity, as *three*, *five*, *ten*.

7. Like numbers are numbers which express units of the *same kind*, as six *drills* and ten *drills*, two *feet* and five *feet*.

8. Unlike numbers are numbers which express units of *different kinds*, as ten *stamps* and eight *molds*, seven *slag pots* and five *tappets*.

NOTATION AND NUMERATION.

9. Numbers are expressed in three ways: (1) By words; (2) by figures; (3) by letters.

10. Notation is the art of expressing numbers by figures or letters.

11. Numeration is the art of reading the numbers which have been expressed by figures or letters.

12. The **Arabic notation** is the method of expressing numbers by figures. This method employs ten different **figures** to represent numbers, viz.:

Figures	0	1	2	3	4	5	6	7	8	9
Names	*naught, cipher, or zero.*	*one*	*two*	*three*	*four*	*five*	*six*	*seven*	*eight*	*nine*

The first character (0) is called **naught, cipher,** or **zero,** and, when standing alone, has no value.

The other nine figures are called **digits,** and each one has a value of its own.

Any whole number is called an **integer.**

13. As there are only ten *figures* used in expressing numbers, each *figure* must express a different *value* at different times.

14. The value of a figure depends upon its *position* in relation to others.

15. Figures have **simple** values and **local** or **relative** values.

16. The **simple** value of a figure is the value it expresses when standing alone.

17. The **local** or **relative** value is the *increased* value it expresses by having other figures placed on its right.

For instance, if we see the figure 6 standing alone, thus. 6
we consider it as *six units*, or simply **six.**

Place another 6 to the *left* of it; thus. 66

The original figure is still *six units*, but the second one is *ten times* 6, or 6 **tens.**

If a third 6 be now placed still one place further to the *left*, it is increased in value *ten times* more, thus making it 6 **hundreds** 666

A fourth 6 would be 6 **thousands** 6666

A fifth 6 would be 6 **tens of thousands,** or **sixty thousand** 66666

A sixth 6 would be 6 **hundreds of thousands** . 666666

A seventh 6 would be 6 **millions** 6666666

The entire line of seven figures is read *six millions, six hundred sixty-six thousands, six hundred sixty-six.*

18. The **increased value** of each of these figures is its *local* or *relative* value. Each figure is *ten times* greater in value than the one immediately on its *right*.

19. The **cipher** (0) has no value itself, but it is useful in determining the place of other figures. To represent the number *four hundred five*, two digits only are necessary, one to represent *four hundred*, and the other to represent *five units;* but if these two digits are placed together, as 45, the 4 (being in the second place) will mean 4 *tens*. To mean 4 *hundreds*, the 4 should have two figures on its right, and a *cipher* is therefore inserted in the place usually given to *tens*, to show that the number is composed of *hundreds* and *units* only, and that there are no *tens*. *Four hundred five* is therefore expressed as 405. If the number were *four thousand and five*, two ciphers would be inserted; thus, 4005. If it were *four hundred fifty*, it would have the *cipher* at the right-hand side to show that there were no *units*, and only *hundreds and tens;* thus, 450. *Four thousand and fifty* would be expressed 4050, the first cipher indicating that there are no hundreds and the second that there are no units.

NOTE.—When speaking of the figures of a number by referring to them as first figure, second figure, etc., always begin to count at the *left*. Thus, in the number 41,625, 4 is the first figure, 6 the third figure, 5 the fifth or last figure, etc.

20. In *reading* figures, it is usual to point off the number into groups of three figures each, beginning with the right-hand or **units** column, a comma (,) being used to point off these groups.

Billions.			*Millions.*			*Thousands.*			*Units.*		
Hundreds of Billions.	Tens of Billions.	Billions.	Hundreds of Millions.	Tens of Millions.	Millions.	Hundreds of Thousands.	Tens of Thousands.	Thousands.	Hundreds of Units.	Tens of Units.	Units.
4	3	2,	1	9	8,	7	6	5,	4	3	2

In *pointing off* these figures, begin at the right-hand figure and count—*units, tens, hundreds;* the next group of three figures is *thousands*, therefore, we insert a comma (,) before beginning with them. Beginning at the figure 5, we say *thousands, tens of thousands, hundreds of thousands*, and insert another comma; we next read *millions, tens of millions, hundreds of millions*, and insert another comma; we then read *billions, tens of billions, hundreds of billions.*

The entire line of figures would be read: *Four hundred thirty-two billions, one hundred ninety-eight millions, seven hundred sixty-five thousands, four hundred thirty-two.* When we thus *read* a line of figures it is called **numeration,** and if the **numeration** be changed back to *figures*, it is called **notation.**

For instance, the writing of the figures,

72,584,623,

would be the **notation,** and the **numeration** would be *seventy-two millions, five hundred eighty-four thousands, six hundred twenty-three.*

21. Note.—It is customary to leave the *s* off the words millions, thousands, etc., in cases like the above, both in speaking and writing; hence, the above would usually be expressed, seventy-two million, five hundred eighty-four thousand, six hundred twenty-three.

22. The four fundamental processes of Arithmetic are **addition, subtraction, multiplication,** and **division.** They are called fundamental processes, because all operations in Arithmetic are based upon them.

ADDITION.

23. Addition is the *process* of *finding* the *sum* of *two* or *more* numbers. The sign of addition is $+$. It is read *plus*, and means *more*. Thus, $5 + 6$ is read 5 *plus* 6, and means that 5 and 6 are to be added.

24. The sign of equality is $=$. It is read *equals* or *is equal to*. Thus, $5 + 6 = 11$ may be read 5 plus 6 *equals* 11.

25. *Like numbers* can be added, but *unlike numbers* cannot. Thus, 6 dollars *can* be added to 7 dollars, and the *sum* will be 13 dollars, but 6 dollars *cannot* be added to 7 *feet.*

26. The following table gives the sum of any two numbers from 1 to 12:

TABLE 1.

1 and 1 are 2	2 and 1 are 3	3 and 1 are 4	4 and 1 are 5
1 and 2 are 3	2 and 2 are 4	3 and 2 are 5	4 and 2 are 6
1 and 3 are 4	2 and 3 are 5	3 and 3 are 6	4 and 3 are 7
1 and 4 are 5	2 and 4 are 6	3 and 4 are 7	4 and 4 are 8
1 and 5 are 6	2 and 5 are 7	3 and 5 are 8	4 and 5 are 9
1 and 6 are 7	2 and 6 are 8	3 and 6 are 9	4 and 6 are 10
1 and 7 are 8	2 and 7 are 9	3 and 7 are 10	4 and 7 are 11
1 and 8 are 9	2 and 8 are 10	3 and 8 are 11	4 and 8 are 12
1 and 9 are 10	2 and 9 are 11	3 and 9 are 12	4 and 9 are 13
1 and 10 are 11	2 and 10 are 12	3 and 10 are 13	4 and 10 are 14
1 and 11 are 12	2 and 11 are 13	3 and 11 are 14	4 and 11 are 15
1 and 12 are 13	2 and 12 are 14	3 and 12 are 15	4 and 12 are 16
5 and 1 are 6	6 and 1 are 7	7 and 1 are 8	8 and 1 are 9
5 and 2 are 7	6 and 2 are 8	7 and 2 are 9	8 and 2 are 10
5 and 3 are 8	6 and 3 are 9	7 and 3 are 10	8 and 3 are 11
5 and 4 are 9	6 and 4 are 10	7 and 4 are 11	8 and 4 are 12
5 and 5 are 10	6 and 5 are 11	7 and 5 are 12	8 and 5 are 13
5 and 6 are 11	6 and 6 are 12	7 and 6 are 13	8 and 6 are 14
5 and 7 are 12	6 and 7 are 13	7 and 7 are 14	8 and 7 are 15
5 and 8 are 13	6 and 8 are 14	7 and 8 are 15	8 and 8 are 16
5 and 9 are 14	6 and 9 are 15	7 and 9 are 16	8 and 9 are 17
5 and 10 are 15	6 and 10 are 16	7 and 10 are 17	8 and 10 are 18
5 and 11 are 16	6 and 11 are 17	7 and 11 are 18	8 and 11 are 19
5 and 12 are 17	6 and 12 are 18	7 and 12 are 19	8 and 12 are 20
9 and 1 are 10	10 and 1 are 11	11 and 1 are 12	12 and 1 are 13
9 and 2 are 11	10 and 2 are 12	11 and 2 are 13	12 and 2 are 14
9 and 3 are 12	10 and 3 are 13	11 and 3 are 14	12 and 3 are 15
9 and 4 are 13	10 and 4 are 14	11 and 4 are 15	12 and 4 are 16
9 and 5 are 14	10 and 5 are 15	11 and 5 are 16	12 and 5 are 17
9 and 6 are 15	10 and 6 are 16	11 and 6 are 17	12 and 6 are 18
9 and 7 are 16	10 and 7 are 17	11 and 7 are 18	12 and 7 are 19
9 and 8 are 17	10 and 8 are 18	11 and 8 are 19	12 and 8 are 20
9 and 9 are 18	10 and 9 are 19	11 and 9 are 20	12 and 9 are 21
9 and 10 are 19	10 and 10 are 20	11 and 10 are 21	12 and 10 are 22
9 and 11 are 20	10 and 11 are 21	11 and 11 are 22	12 and 11 are 23
9 and 12 are 21	10 and 12 are 22	11 and 12 are 23	12 and 12 are 24

This table should be carefully committed to memory. Since 0 has no value, the sum of any number and 0 is the number itself; thus, 17 and 0 are 17.

27. For *addition*, place the numbers to be added directly under each other, taking care to place *units* under *units*, *tens* under *tens*, *hundreds* under *hundreds*, and so on.

When the numbers are thus written, the *right-hand figure* of *one number* is placed *directly under the right-hand figure*

of the *number above it*, thus bringing the unit figures of all the numbers to be added in the same vertical line. Proceed as in the following examples:

28. EXAMPLE.—What is the sum of 131, 222, 21, 2, and 413?

SOLUTION.—

```
      131
      222
       21
        2
      413
      ---
sum   789  Ans.
```

EXPLANATION.—After placing the numbers in proper order, begin at the bottom of the right-hand or *units* column, and add, mentally repeating the different sums. Thus, three and two are five and one are six and two are eight and one are nine, the sum of the numbers in *units* column. Place the 9 directly beneath as the first or *units* figure in the sum.

The sum of the numbers in the next or *tens* column equals 8 *tens*, which is the second or *tens* figure in the sum.

The sum of the numbers in the next or *hundreds* column equals 7 *hundreds*, which is the third or *hundreds* figure in the sum.

The sum or answer is 789.

29. EXAMPLE.—What is the sum of 425, 36, 9,215, 4, and 907?

SOLUTION.—

```
       425
        36
      9215
         4
       907
     -----
        27
        60
      1500
      9000
     -----
sum  10587  Ans.
```

EXPLANATION.—The sum of the numbers in the first or units column is seven and four are eleven and five are sixteen and six are twenty-two and five are twenty-seven, or 27 units; i. e., two tens and seven units. Write 27 as shown.

The sum of the numbers in the second or tens column is six tens, or 60. Write 60 underneath 27 as shown. The sum of the numbers in the third or hundreds column is 15 hundreds, or 1,500. Write 1,500 under the two preceding results as shown. There is only one number in the fourth or thousands column, 9, which represents 9,000. Write 9,000 under the three preceding results. Adding these four results, the sum is 10,587, which is the sum of 425, 36, 9,215, 4, and 907.

NOTE.—It frequently happens, when adding a long column of figures, that the sum of two numbers, one of which does not occur in the addition table, is required. Thus, in the first column above, the sum of 16 and 6 was required. We know from the table that $6 + 6 = 12$; hence, the first figure of the sum is 2. Now, the sum of any number less than 20 and of any number less than 10 must be less than 30, since $20 + 10 = 30$; therefore, the sum is 22. Consequently, in cases of this kind, add the first figure of the larger number to the smaller number, and, if the result is greater than 9, increase the second figure of the larger number by 1. Thus, $44 + 7 = ?$ $4 + 7 = 11$; hence, $44 + 7 = 51$.

30. The addition may also be performed as follows:

$$\begin{array}{rr} & 425 \\ & 36 \\ & 9215 \\ & 4 \\ & 907 \\ \hline \textit{sum} & 10587 \text{ Ans.} \end{array}$$

EXPLANATION.—The sum of the numbers in units column is 27 units, or 2 tens and 7 units. Write the 7 units as the first or right-hand figure in the sum. Reserve the two tens and add them to the figures in tens column. The sum of the figures in the tens column plus the 2 tens reserved and carried from the units column is 8, which is written down as the second figure in the sum. There is nothing to carry to the next column, because 8 is less than 10. The sum of the numbers in the next column is 15 hundreds, or 1 thousand and 5 hundreds. Write down the 5 as the third or hundreds figure in the sum and carry the 1 to the next column. $1 + 9 = 10$, which is written down at the left of the other figures.

The second method saves space and figures, but the first is to be preferred when adding a long column.

31. EXAMPLE.—Add the numbers in the column below:

SOLUTION.—

```
      890
       82
       90
      393
      281
       80
      770
       83
      492
       80
      383
       84
      191
     ----
sum  3899 Ans.
```

EXPLANATION.—The sum of the digits in the first column equals 19 *units*, or 1 *ten* and 9 *units*. Write down the 9 and carry 1 to the next column. The sum of the digits in the second column + 1 = 109 *tens*, or 10 *hundreds* and 9 *tens*. Write down the 9 and carry the 10 to the next column. The sum of the digits in this column plus the 10 reserved = 38. The entire sum is 3,899.

32. Rule.—I. *Begin at the right, add each column separately, and write the sum, if it be only one figure, under the column added.*

II. *If the sum of any column consists of two or more figures, put the right-hand figure of the sum under that column, and add the remaining figure or figures to the next column.*

33. Proof.—*To prove addition, add each column from top to bottom. If you obtain the same result as by adding from bottom to top, the work is probably correct.*

EXAMPLES FOR PRACTICE.

34. Find the sum of

(*a*) 104 + 203 + 613 + 214.	Ans.	(*a*) 1,134.
(*b*) 1,875 + 3,143 + 5,826 + 10,832.		(*b*) 21,676.
(*c*) 4,865 + 2,145 + 8,173 + 40,084.		(*c*) 55,267.
(*d*) 14,204 + 8,173 + 1,065 + 10,042.		(*d*) 33,484.

(*e*) 10,832 + 4,145 + 3,133 + 5,872.
(*f*) 214 + 1,231 + 141 + 5,000.
(*g*) 123 + 104 + 425 + 126 + 327.
(*h*) 6,354 + 2,145 + 2,042 + 1,111 + 3,333.

Ans. (*e*) 23,982. (*f*) 6,586. (*g*) 1,105. (*h*) 14,985.

1. A smelting company received 701 tons of ore in the first week in January, 723 tons in the second week, 634 tons in the third week, and 254 tons in the fourth week; how many tons were received in the entire month? Ans. 2,312 tons.

2. A smelting company received during one month 384 tons of limestone, 785 tons of coke, and 1,056 tons of ore. What number of tons were received during the month? Ans. 2,225 tons.

3. A week's record of coal burned in an engine room is as follows: Monday, 1,800 pounds; Tuesday, 1,655 pounds; Wednesday, 1,725 pounds; Thursday, 1,690 pounds; Friday, 1,648 pounds; Saturday, 1,020 pounds. How much coal was burned during the week?
Ans. 9,538 pounds.

SUBTRACTION.

35. In Arithmetic, **subtraction** is the process of finding how much greater one number is than another.

The greater of the two numbers is called the **minuend.**

The smaller of the two numbers is called the **subtrahend.**

The number left after subtracting the subtrahend from the minuend is called the **difference** or **remainder.**

36. The sign of subtraction is —. It is read **minus,** and means *less*. Thus, 12 — 7 is read *12 minus 7*, and means that 7 is to be taken from 12.

37. Example.—From 7,568 take 3,425.

Solution.—

minuend	7568
subtrahend	3425
remainder	4143 Ans.

Explanation.—Begin at the right-hand or *units* column and subtract in succession each figure in the subtrahend from the one directly above it in the minuend, and write the remainders below the line. The result is the entire remainder.

38. When there are more figures in the *minuend* than in the *subtrahend*, and when some figures in the minuend are *less* than the figures directly under them in the subtrahend, proceed as in the following example:

EXAMPLE.—From 8,453 take 844.

SOLUTION.—

minuend	8453
subtrahend	844
remainder	7609 Ans.

EXPLANATION.—Begin to subtract at the right-hand or *units* column. We can not take 4 from 3, and must, therefore, borrow 1 from 5 in *tens* column and annex it to the 3 in *units* column. The 1 *ten* = 10 *units*, which added to the 3 in *units* column = 13 *units*. 4 from 13 = 9, the first or *units* figure in the remainder.

Since we borrowed 1 from the 5, only 4 remains; 4 from 4 = 0, the second or *tens* figure. We can not take 8 from 4, and must, therefore, borrow 1 from 8 in *thousands* column. Since 1 *thousand* = 10 *hundreds*, 10 *hundreds* + 4 *hundreds* = 14 *hundreds*, and 8 from 14 = 6, the third or *hundreds* figure in the remainder.

Since we borrowed 1 from 8, only 7 remains, from which there is nothing to subtract; therefore, 7 is the next figure in the remainder or answer.

The operation of borrowing is performed by mentally placing 1 before the figure following the one from which it is borrowed. In the above example the 1 borrowed from 5 is placed before 3, making it 13, from which we subtract 4. The 1 borrowed from 8 is placed before 4, making 14, from which 8 is taken.

39. EXAMPLE.—Find the difference between 10,000 and 8,763.

SOLUTION.—

minuend	10000
subtrahend	8763
remainder	1237 Ans.

EXPLANATION.—In the above example we borrow 1 from the second column and place it before 0, making 10; 3 from

10 = 7. In the same way we borrow 1 and place it before the next cipher, making 10; but as we have borrowed 1 from this column and have taken it to the units column, only 9 remains, from which to subtract 6; 6 from 9 = 3. For the same reason we subtract 7 from 9 and 8 from 9 for the next two figures, and obtain a total remainder of 1,237.

40. Rule.—*Place the subtrahend (or smaller number) under the minuend or larger number, in the same manner as for addition, and proceed as in Arts.* ***37, 38,*** *and* ***39.***

41. Proof.—*To prove an example in subtraction, add the remainder to the subtrahend. The sum should equal the minuend. If it does not, a mistake has been made, and the work should be done over.*

Proof of the above example:

$$\begin{array}{rr} \textit{subtrahend} & 8768 \\ \textit{remainder} & 1237 \\ \hline \textit{minuend} & 10000 \end{array} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

42. From

(*a*) 94,278 take 62,574.	Ans.	(*a*) 31,704.
(*b*) 53,714 take 25,824.		(*b*) 27,890.
(*c*) 71,832 take 58,109.		(*c*) 13,723.
(*d*) 20,804 take 10,408.		(*d*) 10,396.
(*e*) 310,465 take 102,141.		(*e*) 208,324.
(*f*) (81,043 + 1,041) take 14,831.		(*f*) 67,253.
(*g*) (20,482 + 18,216) take 21,214.		(*g*) 17,484.
(*h*) (2,040 + 1,213 + 542) take 3,791.		(*h*) 4.

1. The total weight of a charging barrow loaded with coke is 1,300 pounds; the empty barrow weighs 700 pounds; what is the weight of the coke? Ans. 600 lb.

2. One shift of bottom fillers shoveled 3,875 tons of coke, flux, and ore in a month; another shift shoveled 2,325 tons of coke, flux, and ore; how many more tons did the first shift handle than the second? Ans. 1,550 tons.

3. Two furnaces smelted 127,750 tons of ore in one year; one smelted 54,750 tons; how many tons did the other smelt?

Ans. 73,000 tons.

4. A battery of four furnaces produced 202,500 pounds of copper matte daily. Numbers 1 and 2 furnaces produced 120,000 pounds; number 3 furnace, 45,000 pounds. What quantity of matte did number 4 furnace turn out? Ans. 37,000 lb.

5. The total cost of erecting a steam plant was $2,675. The engine cost $900; the boiler, $775; the fittings and connections, $225. The remainder was expended in erecting the engine house. What was the cost of the engine house? Ans. $775.

MULTIPLICATION.

43. To **multiply** a number is to *add* it to itself a certain number of times.

44. **Multiplication** is the process of multiplying one number by another.

The number thus added to itself, or the number to be multiplied, is called the **multiplicand.**

The number which shows how many times the *multiplicand* is to be taken, or the number by which we *multiply*, is called the **multiplier.**

The result obtained by multiplying is called the **product.**

45. The sign of multiplication is ×. It is read *times* or *multiplied by*. Thus, 9 × 6 is read *9 times 6*, or *9 multiplied by 6*.

46. It matters not in what order the numbers to be multiplied together are placed. Thus, 6 × 9 is the same as 9 × 6.

47. In the following table, the product of any two numbers (neither of which exceeds 12) may be found:

TABLE 2.

1 times 1 is 1	2 times 1 are 2	3 times 1 are 3
1 times 2 are 2	2 times 2 are 4	3 times 2 are 6
1 times 3 are 3	2 times 3 are 6	3 times 3 are 9
1 times 4 are 4	2 times 4 are 8	3 times 4 are 12
1 times 5 are 5	2 times 5 are 10	3 times 5 are 15
1 times 6 are 6	2 times 6 are 12	3 times 6 are 18
1 times 7 are 7	2 times 7 are 14	3 times 7 are 21
1 times 8 are 8	2 times 8 are 16	3 times 8 are 24
1 times 9 are 9	2 times 9 are 18	3 times 9 are 27
1 times 10 are 10	2 times 10 are 20	3 times 10 are 30
1 times 11 are 11	2 times 11 are 22	3 times 11 are 33
1 times 12 are 12	2 times 12 are 24	3 times 12 are 36
4 times 1 are 4	5 times 1 are 5	6 times 1 are 6
4 times 2 are 8	5 times 2 are 10	6 times 2 are 12
4 times 3 are 12	5 times 3 are 15	6 times 3 are 18
4 times 4 are 16	5 times 4 are 20	6 times 4 are 24
4 times 5 are 20	5 times 5 are 25	6 times 5 are 30
4 times 6 are 24	5 times 6 are 30	6 times 6 are 36
4 times 7 are 28	5 times 7 are 35	6 times 7 are 42
4 times 8 are 32	5 times 8 are 40	6 times 8 are 48
4 times 9 are 36	5 times 9 are 45	6 times 9 are 54
4 times 10 are 40	5 times 10 are 50	6 times 10 are 60
4 times 11 are 44	5 times 11 are 55	6 times 11 are 66
4 times 12 are 48	5 times 12 are 60	6 times 12 are 72
7 times 1 are 7	8 times 1 are 8	9 times 1 are 9
7 times 2 are 14	8 times 2 are 16	9 times 2 are 18
7 times 3 are 21	8 times 3 are 24	9 times 3 are 27
7 times 4 are 28	8 times 4 are 32	9 times 4 are 36
7 times 5 are 35	8 times 5 are 40	9 times 5 are 45
7 times 6 are 42	8 times 6 are 48	9 times 6 are 54
7 times 7 are 49	8 times 7 are 56	9 times 7 are 63
7 times 8 are 56	8 times 8 are 64	9 times 8 are 72
7 times 9 are 63	8 times 9 are 72	9 times 9 are 81
7 times 10 are 70	8 times 10 are 80	9 times 10 are 90
7 times 11 are 77	8 times 11 are 88	9 times 11 are 99
7 times 12 are 84	8 times 12 are 96	9 times 12 are 108
10 times 1 are 10	11 times 1 are 11	12 times 1 are 12
10 times 2 are 20	11 times 2 are 22	12 times 2 are 24
10 times 3 are 30	11 times 3 are 33	12 times 3 are 36
10 times 4 are 40	11 times 4 are 44	12 times 4 are 48
10 times 5 are 50	11 times 5 are 55	12 times 5 are 60
10 times 6 are 60	11 times 6 are 66	12 times 6 are 72
10 times 7 are 70	11 times 7 are 77	12 times 7 are 84
10 times 8 are 80	11 times 8 are 88	12 times 8 are 96
10 times 9 are 90	11 times 9 are 99	12 times 9 are 108
10 times 10 are 100	11 times 10 are 110	12 times 10 are 120
10 times 11 are 110	11 times 11 are 121	12 times 11 are 132
10 times 12 are 120	11 times 12 are 132	12 times 12 are 144

This table should be carefully committed to memory.

Since 0 has no value, the product of 0 and any number is 0.

48. To multiply a number by one figure only:

EXAMPLE.—Multiply 425 by 5.

SOLUTION.—	*multiplicand*	425	
	multiplier	5	
	product	2125	Ans.

EXPLANATION.—For convenience, the *multiplier* is generally written *under* the *right-hand figure* of the *multiplicand.* On looking in the multiplication table, we see that 5 × 5 are 25. Multiplying the *first figure* at the *right* of the *multiplicand,* or 5, by the *multiplier* 5, it is seen that 5 times 5 units are 25 units, or 2 tens and 5 units. Write the *5 units* in *units* place in the *product,* and *reserve* the 2 tens to add to the product of tens. Looking in the multiplication table again, we see that 5 × 2 are 10. Multiplying the *second figure* of the multiplicand by the multiplier 5, we see that 5 times 2 tens are 10 tens, and 10 tens plus the 2 tens reserved, are 12 tens, or 1 hundred plus 2 tens. Write the *2 tens* in *tens* place, and reserve the 100 to add to the product of hundreds. Again, we see by the multiplication table that 5 × 4 are 20. Multiplying the *third* or *last figure* of the multiplicand by the multiplier 5, we see that 5 times 4 hundreds are 20 hundreds, and 20 hundreds plus the 1 hundred reserved, are 21 hundreds, or 2 thousands plus 1 hundred, which we write in *thousands* and *hundreds* places, respectively.

Hence, the product is 2,125.

This result is the same as adding 425 five times. Thus,

	425	
	425	
	425	
	425	
	425	
sum	2125	Ans.

EXAMPLES FOR PRACTICE.

49. Find the product of

(*a*) 61,483 × 6.	Ans.	(*a*) 368,898.
(*b*) 12,375 × 5.		(*b*) 61,875.
(*c*) 10,426 × 7.		(*c*) 72,982.
(*d*) 10,835 × 3.		(*d*) 32,505.

(*e*) 98,376 × 4.		(*e*) 393,504.
(*f*) 10,873 × 8.	Ans.	(*f*) 86,984.
(*g*) 71,543 × 9.		(*g*) 643,887.
(*h*) 218,734 × 2.		(*h*) 437,468.

1. If two men can cob 5 tons of ore in one day, how many tons can they cob in 389 days? Ans. 1,945 tons.

2. If a yard locomotive can haul 3 cars of slag a trip, how many cars can it haul in 169 trips? Ans. 507 cars.

3. A stationary engine makes 5,520 revolutions per hour. Running 9 hours a day, 5 days in the week, and 5 hours on Saturday, how many revolutions would it make in 4 weeks? Ans. 1,104,000 revolutions.

50. To multiply a number by two or more figures:

EXAMPLE.—Multiply 475 by 234.

SOLUTION.— *multiplicand* 475
multiplier 234

```
    475
    234
   ----
   1900
  1425
  950
 ------
 111150  Ans.
```

product 111150 Ans.

EXPLANATION.—For convenience, the multiplier is generally written *under* the multiplicand, placing units under units, tens under tens, etc.

We *can not* multiply by 234 at one operation; we must, therefore, multiply by the *parts* and then *add* the **partial products.**

The parts by which we are to multiply are 4 units, 3 tens, and 2 hundreds. 4 times 475 = 1,900, the first partial product; 3 times 475 = 1,425, the second partial product, the *right-hand figure of which is written directly under the figure multiplied by*, or 3; 2 times 475 = 950, the third partial product, the right-hand figure of which is written directly under the figure multiplied by, or 2.

The sum of these three partial products is 111,150, which is the *entire product.*

51. Rule.—I. *Write the multiplier under the multiplicand, so that units are under units, tens under tens, etc.*

II. *Begin at the right and multiply each figure of the multiplicand by each successive figure of the multiplier, placing the right-hand figure of each partial product directly under the figure used as a multiplier.*

III. *The sum of the partial products will equal the required product.*

52. Proof.—*Review the work carefully, or multiply the multiplier by the multiplicand; if the results agree, the work is correct.*

53. When there is a *cipher* in the *multiplier*, multiply the entire multiplicand by it; since the result will be zero, place a cipher under the cipher in the multiplier. Thus,

(*a*)	(*b*)	(*c*)	(*d*)
0	2	15	708
× 0	× 0	× 0	× 0
0 Ans.	0 Ans.	0 Ans.	0 Ans.

(*e*)	(*f*)	(*g*)
3114	4008	31264
203	305	1002
9342	20040	62528
62280	120240	3126400
632142 Ans.	1222440 Ans.	31326528 Ans.

In examples (*e*), (*f*), and (*g*), we multiply by 0 as directed above; then multiply by the next figure of the multiplier and place the first figure of the product alongside the 0, as shown.

EXAMPLES FOR PRACTICE.

54. Find the product of

		Ans.	
(*a*)	3,842 × 26.	(*a*)	99,892.
(*b*)	3,716 × 45.	(*b*)	167,220.
(*c*)	1,817 × 124.	(*c*)	225,308.
(*d*)	675 × 38.	(*d*)	25,650.
(*e*)	1,875 × 33.	(*e*)	61,875.
(*f*)	4,836 × 47.	(*f*)	227,292.

(*g*) 5,682 × 543.		(*g*) 3,085,326.
(*h*) 3,257 × 246.		(*h*) 801,222.
(*i*) 2,875 × 302.		(*i*) 868,250.
(*j*) 17,819 × 1,004.		(*j*) 17,890,276.
(*k*) 38,674 × 205.	Ans.	(*k*) 7,928,170.
(*l*) 18,304 × 100.		(*l*) 1,830,400.
(*m*) 7,834 × 10.		(*m*) 78,340.
(*n*) 87,543 × 1,000.		(*n*) 87,543,000.
(*o*) 48,763 × 100.		(*o*) 4,876,300.

1. If two furnaces can roast 12 tons of ore in one day, how many tons can they roast in 18 days? Ans. 216 tons.

2. If it requires 8 men to unload one R. R. car in one hour, how many men will be required to fill 37 R. R. cars in one hour? Ans. 296 men.

3. The output of a mine is 123 tons per day; what is its output for a month of 31 days? Ans. 3,813 tons.

4. One bottom filler at a furnace loads and wheels 56 barrows of ore daily. If each barrow contains 420 pounds of ore, how many pounds does he load? Ans. 23,520 lb.

DIVISION.

55. **Division** is the process of finding how many times one number is contained in another of the same kind.

The number to be *divided* is called the **dividend.**

The number by which we *divide* is called the **divisor.**

The number which shows how many times the *divisor is contained in the dividend* is called the **quotient.**

56. The sign of division is ÷. It is read *divided by.* 54 ÷ 9 is read *54 divided by 9.* Another way to write *54 divided by 9* is $\frac{54}{9}$. Thus, $54 \div 9 = 6$, or $\frac{54}{9} = 6$.

In both of these cases 54 is the dividend and 9 is the divisor.

Division is the *reverse* of **multiplication.**

57. **To divide when the divisor consists of but one figure,** proceed as in the following example:

EXAMPLE.—What is the quotient of 875 ÷ 7?

SOLUTION.—

```
divisor dividend quotient
        7)875(125  Ans.
          7
          --
          17
          14
          --
           35
           35
           --
remainder   0
```

EXPLANATION.—7 is contained in 8 *hundreds* 1 *hundred* times. Place the one as the first or *left-hand figure* of the quotient. Multiply the divisor 7 by the 1 *hundred* of the quotient, and place the product 7 *hundreds* under the 8 *hundreds* in the dividend, and subtract. Beside the remainder 1, bring down the next or *tens* figure of the quotient, in this case 7, making 17 *tens;* 7 is contained in 17, 2 times. Write the 2 as the *second figure* of the quotient. Multiply the divisor 7 by the 2 in the quotient, and subtract the product from 17. Beside the remainder 3, bring down the next or *units* figure of the dividend, in this case 5, making 35 *units.* 7 is contained in 35, 5 times, which is placed in the quotient. Multiplying the divisor by the last figure of the quotient, 5 times $7 = 35$, which subtracted from 35, under which it is placed, leaves 0. Therefore, the quotient is 125. This method is called **long division.**

58. In **short division,** only the divisor, dividend, and quotient are written, the operations being performed mentally.

```
                  dividend
        divisor 7 ) 8 ¹7 ³5
                  ---------
        quotient    1  2  5   Ans.
```

The mental operation is as follows: 7 is contained in 8, once and one remainder; imagine 1 to be placed before 7 making 17; 7 is contained in 17, 2 times and 3 over; imagine 3 to be placed before 5 making 35; 7 is contained in 35, 5 times. These partial quotients, placed in order as they are found, make the entire quotient 125.

The small figures are placed in the example given to better illustrate the explanation; they are never written when actually performing division in this way.

59. If the divisor consists of *two or more* figures, proceed as in the following example:

EXAMPLE.—Divide 2,702,826 by 63.

```
            divisor   dividend     quotient
SOLUTION.—      63)2702826(42902  Ans.
                   252
                   ---
                    182
                    126
                    ---
                     568
                     567
                     ---
                       126
                       126
                       ---
                         0
```

EXPLANATION.—As 63 is not contained in the first two figures, 27, we must use the first three figures, 270. Now, by trial, we must find how many times 63 is contained in 270; 6 is contained in the first two figures of 270, 4 times. Place the 4 as the first or left-hand figure in the quotient. Multiply the divisor 63 by 4, and subtract the product 252 from 270. The remainder is 18, beside which we write the next figure of the dividend, 2, making 182. Now, 6 is contained in the first two figures of 182, 3 times, but on multiplying 63 by 3, we see that the product 189 is too great, so we try 2 as the second figure of the quotient. Multiplying the divisor 63 by 2, and subtracting the product 126 from 182, the remainder is 56, beside which we bring down the next figure of the dividend, making 568; 6 is contained in 56 about 9 times. Multiply the divisor 63 by 9 and subtract the product 567 from 568. The remainder is 1, and bringing down the next figure of the dividend, 2, gives 12. As 12 is smaller than 63, we write 0 in the quotient and bring down the next figure, 6, making 126. 63 is contained in 126, 2

times without a remainder. Therefore, 42,902 is the quotient.

60. Rule.—I. *Write the divisor at the left of the dividend, with a line between them.*

II. *Find how many times the divisor is contained in the lowest number of the left-hand figures of the dividend that will contain it, and write the result at the right of the dividend, with a line between, for the first figure of the quotient.*

III. *Multiply the divisor by this quotient; write the product under the partial dividend used, and subtract, annexing to the remainder the next figure of the dividend. Divide as before, and thus continue until all the figures of the dividend have been used.*

IV. *If any partial dividend will not contain the divisor, write a cipher in the quotient, annex the next figure of the dividend, and proceed as before.*

V. *If there be a remainder at last, write it after the quotient, with the divisor underneath.*

61. Proof.—*Multiply the quotient by the divisor, and add the remainder, if there be any, to the product. The result will be the dividend.*

		divisor dividend quotient	
Thus,		63) 4235 ($67\frac{14}{63}$	Ans.
		378	
		455	
		441	
	remainder	14	
Proof,	*quotient*	67	
	divisor	63	
		201	
		402	
		4221	
	remainder	14	
	dividend	4235	

EXAMPLES FOR PRACTICE.

62. Divide the following:

(*a*) 126,498 by 58.		(*a*) 2,181.
(*b*) 3,207,594 by 767.		(*b*) 4,182.
(*c*) 11,408,202 by 234.		(*c*) 48,753.
(*d*) 2,100,315 by 581.		(*d*) 3,615.
(*e*) 969,936 by 4,008.	Ans.	(*e*) 242.
(*f*) 7,481,888 by 1,021.		(*f*) 7,328.
(*g*) 1,525,915 by 5,003.		(*g*) 305.
(*h*) 1,646,301 by 381		(*h*) 4,321.
(*i*) 1,486,968 by 371.		(*i*) 4,008.

1. At a certain furnace there were 674 tons of ore bedded in one day. How many tram cars must have been dumped if each held 2 tons? Ans. 337 cars.

2. If one miner can break 6 tons of ore in one day, how many miners will be required to break 1,326 tons in one day? Ans. 221 miners.

3. There were 344 tons of ore delivered at a stamp mill in one day. If 172 wagon loads were required to deliver this ore, how many tons were put in each wagon load? Ans. 2 tons.

4. How many R. R. cars will a siding 1,792 feet long hold, the cars each being 32 feet long? Ans. 56 R. R. cars.

5. A wire-rope tramway delivers 28,800 pounds of ore in 8 hours to a chlorination mill. How many pounds can it deliver at this rate in one hour? Ans. 3,600 pounds.

CANCELATION.

63. **Cancelation** is the process of shortening operations in division by casting out equal factors from both dividend and divisor.

64. The **factors** of a number are those numbers which, when *multiplied* together, *equal the given number*. Thus, 5 and 3 are factors of 15, since $5 \times 3 = 15$. Likewise, 8 and 7 are the factors of 56, since $8 \times 7 = 56$.

65. A **prime number** is one which cannot be divided by any number except itself and 1. Thus, 2, 3, 11, 29, etc. are prime numbers.

66. A **prime factor** is any factor that is a prime number.

Any number that is not a prime is called a **composite** number, and may be produced by multiplying together its prime factors. Thus, 60 is a composite number, and is equal to the product of its prime factors, $2 \times 2 \times 3 \times 5$.

Numbers are said to be **prime to each other** when no two of them can be divided by any number except 1; the numbers themselves *may* be either prime or composite. Thus, the numbers 3, 5, and 11 are prime to each other, so also are 22, 25, and 21—all composite numbers.

67. Canceling *equal factors* from *both dividend and divisor* does *not* change the *quotient*.

The *canceling* of a *factor* in *both dividend and divisor* is the *same* as *dividing them both* by the *same number*, which, by the principle of division, does not *change the quotient*.

Write the *numbers* which make the *dividend* above the *line*, and those which make the *divisor* below it.

68. EXAMPLE.—Divide $4 \times 45 \times 60$ by 9×24.

SOLUTION.—Placing the dividend over the divisor, and canceling

$$\frac{\cancel{4} \times \overset{5}{\cancel{45}} \times \overset{10}{\cancel{60}}}{\cancel{9} \times \underset{\underset{1}{\cancel{6}}}{\cancel{24}}} = \frac{50}{1} = 50. \quad \text{Ans.}$$

EXPLANATION.—The 4 in the dividend and 24 in the divisor are both divisible by 4, since 4 divided by 4 equals 1, and 24 divided by 4 equals 6. Cross off the four and write the 1 over it· also, cross off the 24 and write the 6 under it. Thus,

$$\frac{\overset{1}{\cancel{4}} \times 45 \times 60}{9 \times \underset{6}{\cancel{24}}}.$$

60 in the dividend and 6 in the divisor are divisible by 6, since 60 divided by 6 equals 10, and 6 divided by 6 equals 1. Cross off the 60 and write 10 over it; also, cross off the 6 and write 1 under it. Thus,

$$\frac{\overset{1}{\cancel{4}} \times 45 \times \overset{10}{\cancel{60}}}{9 \times \underset{\underset{1}{\cancel{6}}}{\cancel{24}}}.$$

Again, 45 in the dividend and 9 in the divisor are divisible by 9, since 45 divided by 9 equals 5, and 9 divided by 9 equals 1. Cross off the 45 and write the 5 over it; also, cross off the 9 and write the 1 under it. Thus,

$$\frac{\overset{1}{\cancel{4}} \times \overset{5}{\cancel{45}} \times \overset{10}{\cancel{60}}}{\underset{1}{\cancel{9}} \times \underset{\underset{1}{\cancel{6}}}{\cancel{24}}}.$$

Since there are no two remaining numbers (one in the dividend and one in the divisor) divisible by any number except 1, without a remainder, it is impossible to cancel further.

Multiply all the uncanceled numbers in the dividend together, and divide their product by the product of all the uncanceled numbers in the divisor. The result will be the quotient. The product of all the uncanceled numbers in the dividend equals $5 \times 1 \times 10 = 50$; the product of all the uncanceled numbers in the divisor equals $1 \times 1 = 1$.

$$\text{Hence, } \frac{\overset{1}{\cancel{4}} \times \overset{5}{\cancel{45}} \times \overset{10}{\cancel{60}}}{\underset{1}{\cancel{9}} \times \underset{\underset{1}{\cancel{6}}}{\cancel{24}}} = \frac{1 \times 5 \times 10}{1 \times 1} = 50. \quad \text{Ans.}$$

It is usual to omit the 1's when canceling them, instead of writing them as above.

69. Rule.—I. *Cancel the common factors from both the dividend and divisor.*

II. *Then divide the product of the remaining factors of the dividend by the product of the remaining factors of the divisor, and the result will be the quotient.*

EXAMPLES FOR PRACTICE.

70. Divide

	Ans.
(*a*) $14 \times 18 \times 16 \times 40$ by $7 \times 8 \times 6 \times 5 \times 3$.	(*a*) 32.
(*b*) $3 \times 65 \times 50 \times 100 \times 60$ by $30 \times 60 \times 13 \times 10$.	(*b*) 250.
(*c*) $8 \times 4 \times 3 \times 9 \times 11$ by $11 \times 9 \times 4 \times 3 \times 8$.	(*c*) 1.
(*d*) $164 \times 321 \times 6 \times 7 \times 4$ by $82 \times 321 \times 7$.	(*d*) 48.
(*e*) $50 \times 100 \times 200 \times 72$ by $1,000 \times 144 \times 100$.	(*e*) 5.
(*f*) $48 \times 63 \times 55 \times 49$ by $7 \times 21 \times 11 \times 48$.	(*f*) 105
(*g*) $110 \times 150 \times 84 \times 32$ by $11 \times 15 \times 100 \times 64$.	(*g*) 42.
(*h*) $115 \times 120 \times 400 \times 1,000$ by $23 \times 1,000 \times 60 \times 800$.	(*h*) 5.

ARITHMETIC.

(PART 2.)

FRACTIONS.

71. A **fraction** is one or more equal parts of a unit: *One-half*, *one-third*, *two-fifths* are fractions.

72. *Two* numbers are required to express a fraction; one is called the *numerator*, and the other the *denominator*.

73. The numerator is placed above the denominator, with a line between them; as, $\frac{2}{3}$. 3 is the **denominator**, and shows into how many *equal parts* the *unit* or *one* is divided. The **numerator** 2 shows how many of these equal parts are taken or considered. The denominator also indicates the names of the parts.

$\frac{1}{2}$ is read one-half; $\frac{3}{4}$ is read three-fourths; $\frac{3}{8}$ is read three-eighths; $\frac{5}{16}$ is read five-sixteenths; $\frac{29}{47}$ is read twenty-nine-forty-sevenths.

74. In the expression "$\frac{3}{4}$ of an apple," the denominator 4 shows that the apple is to be (or has been) cut into 4 *equal parts*, and the numerator 3 shows that *three of these parts*, or *fourths*, are taken or considered.

If each of the parts, or fourths, of the apple were cut in *two equal pieces*, there would then be twice as many pieces as before, or $4 \times 2 = 8$ pieces in all; one of these pieces would be called one-eighth, and would be expressed in figures as $\frac{1}{8}$. Three of these pieces would be called three-eighths, and written $\frac{3}{8}$. It is evident that the *larger the denominator*, the greater is the number of parts into which anything is divided and the less the value of the fraction for the same number of parts taken. In other words, $\frac{7}{9}$, for example, is smaller than $\frac{7}{8}$, because if an object be divided into 9 parts, the parts are smaller than if the same object had been divided into 8 parts; and, since $\frac{1}{9}$ is smaller than $\frac{1}{8}$,

it is clear that 7 one-ninths is a smaller amount than 7 one-eighths. Hence, also, $\frac{3}{8}$ is less than $\frac{3}{4}$.

75. The **value** of a fraction is the numerator divided by the denominator; as, $\frac{4}{2} = 2$, $\frac{6}{2} = 3$.

76. The line between the numerator and denominator means *divided by*, or $\div$.

$\frac{3}{4}$ is equivalent to $3 \div 4$.
$\frac{5}{8}$ is equivalent to $5 \div 8$.

77. The numerator and denominator of a fraction, when considered together, are called the **terms** of a fraction.

78. The *value* of a fraction whose numerator and denominator are equal is 1.

$\frac{4}{4}$, or four-fourths, $= 1$.
$\frac{8}{8}$, or eight-eighths, $= 1$.
$\frac{64}{64}$, or sixty-four-sixty-fourths, $= 1$.

79. A **proper fraction** is a fraction whose numerator is *less* than its denominator. Its value is *less* than 1; as, $\frac{3}{4}$, $\frac{5}{8}$, $\frac{1}{16}$.

80. An **improper fraction** is a fraction whose numerator *equals* or is *greater* than the denominator. Its value is 1 or *more* than 1; as, $\frac{4}{4}$, $\frac{9}{8}$, $\frac{42}{32}$.

81. A **mixed number** is a whole number and a fraction united. $4\frac{2}{3}$ is a mixed number, and is equivalent to $4 + \frac{2}{3}$. It is read *four and two-thirds*.

REDUCTION OF FRACTIONS.

82. **Reduction of fractions** is the process of changing their form without changing their value.

83. A *fraction is reduced to higher terms by multiplying both terms of the fraction by the same number.* Thus, $\frac{3}{4}$ is reduced to $\frac{6}{8}$ by multiplying both terms by 2.

$$\frac{3 \times 2}{4 \times 2} = \frac{6}{8}.$$

The value is not changed, since $\frac{3}{4} = \frac{6}{8}$. For, suppose that an object, say an apple, is divided into 8 equal parts. If

these parts be arranged into 4 piles, each containing 2 parts, it is evident that each pile will be composed of the same amount of the entire apple as would have been the case had the apple been originally cut into 4 equal parts. Now, if one of these piles (containing 2 parts) be removed, there will be 3 piles left, each containing 2 equal parts, or 6 equal parts in all, i. e., six-eighths. But, since one pile, or one quarter, was removed, there are three-quarters left. Hence, $\frac{3}{4} = \frac{6}{8}$. The same course of reasoning may be applied to any similar case Therefore, multiplying both terms of a fraction by the same number does not alter its value.

84. To reduce a fraction to an equal fraction having a given denominator:

EXAMPLE.—Reduce $\frac{7}{8}$ to an equal fraction having 96 for a denominator.

SOLUTION.—Both the numerator and the denominator must be multiplied by the same number in order not to change the value of the fraction. The denominator must be multiplied by some number which will, in this case, make the product 96; this number is evidently $96 \div 8 = 12$, since $8 \times 12 = 96$. Hence, $\frac{7 \times 12}{8 \times 12} = \frac{84}{96}$. Ans.

85. Rule.—*Divide the given denominator by the denominator of the given fraction, and multiply both terms of the fraction by the result.*

EXAMPLE.—Reduce $\frac{3}{4}$ to 100ths.

SOLUTION.— $100 \div 4 = 25$; hence, $\frac{3 \times 25}{4 \times 25} = \frac{75}{100}$. Ans.

86. A *fraction* is reduced to *lower terms* by *dividing both terms* by the *same number.* Thus, $\frac{8}{10}$ is reduced to $\frac{4}{5}$ by dividing both terms by 2.

$$\frac{8 \div 2}{10 \div 2} = \frac{4}{5}.$$

That $\frac{8}{10} = \frac{4}{5}$ is readily seen from the explanation given in Art. **83**; for, multiplying both terms of the fraction $\frac{4}{5}$ by 2, $\frac{4 \times 2}{5 \times 2} = \frac{8}{10}$, and, if $\frac{4}{5} = \frac{8}{10}$, $\frac{8}{10}$ must equal $\frac{4}{5}$. Hence, dividing both terms of a fraction by the same number does not alter its value.

87. A *fraction* is reduced to *lowest terms* when its *numerator and denominator* cannot both be *divided* by the *same*

number without a remainder; as, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{11}{24}$, $\frac{8}{15}$. In other words, the numerator and denominator are prime to each other.

EXAMPLES FOR PRACTICE.

88. Reduce the following:

(*a*) $\frac{7}{16}$ to 128ths.	Ans.	(*a*) $\frac{56}{128}$.
(*b*) $\frac{24}{132}$ to its lowest terms.		(*b*) $\frac{2}{11}$.
(*c*) $\frac{64}{1000}$ to its lowest terms.		(*c*) $\frac{8}{125}$.
(*d*) $\frac{5}{7}$ to 49ths.		(*d*) $\frac{35}{49}$.
(*e*) $\frac{13}{16}$ to 10,000ths.		(*e*) $\frac{8125}{10000}$.

---

89. To reduce a whole number or mixed number to an improper fraction:

EXAMPLE.—How many *fourths* in 5?

SOLUTION.—Since there are 4 *fourths* in 1 ($\frac{4}{4} = 1$), in 5 there will be 5×4 fourths, or 20 fourths; i. e., $5 \times \frac{4}{4} = \frac{20}{4}$. Ans.

EXAMPLE.—Reduce $8\frac{3}{4}$ to an improper fraction.

SOLUTION.—$8 \times \frac{4}{4} = \frac{32}{4}$. $\frac{32}{4} + \frac{3}{4} = \frac{35}{4}$. Ans.

90. Rule.—*Multiply the whole number by the denominator of the fraction, add the numerator to the product, and place the denominator under the result. If it is desired to reduce a whole number to a fraction, multiply the whole number by the denominator of the given fraction, and write the result over the denominator.*

EXAMPLES FOR PRACTICE.

91. Reduce to improper fractions:

(*a*) $4\frac{1}{8}$.	Ans.	(*a*) $\frac{33}{8}$.
(*b*) $5\frac{1}{9}$.		(*b*) $\frac{46}{9}$.
(*c*) $10\frac{2}{10}$.		(*c*) $\frac{102}{10}$.
(*d*) $37\frac{3}{4}$.		(*d*) $\frac{151}{4}$.
(*e*) $50\frac{4}{5}$.		(*e*) $\frac{254}{5}$.
(*f*) Reduce 7 to a fraction whose denominator is 16.		(*f*) $\frac{112}{16}$.

---

92. To reduce an improper fraction to a whole or mixed number:

EXAMPLE.—Reduce $\frac{21}{4}$ to a mixed number.

SOLUTION.—4 is contained in 21, 5 times and 1 remaining (see Art. **75**); as this is also divided by 4, its value is $\frac{1}{4}$. Therefore, $5 + \frac{1}{4}$, or $5\frac{1}{4}$, is the number. Ans.

93. Rule.—*Divide the numerator by the denominator, the quotient will be the whole number; the remainder, if there be any, will be the numerator of the fractional part of which the denominator is the same as the denominator of the improper fraction.*

EXAMPLES FOR PRACTICE.

94. Reduce to whole or mixed numbers:

(*a*) $\frac{145}{6}$.		(*a*) $24\frac{1}{6}$.
(*b*) $\frac{185}{3}$.		(*b*) $61\frac{2}{3}$.
(*c*) $\frac{701}{6}$.	Ans.	(*c*) $116\frac{5}{6}$.
(*d*) $\frac{149}{3}$.		(*d*) $49\frac{2}{3}$.
(*e*) $\frac{76}{19}$.		(*e*) 4.
(*f*) $\frac{125}{25}$.		(*f*) 5.

95. A **common denominator** of *two* or *more fractions* is a number which will contain (i. e., which may be divided by) all of the *denominators* of the *fractions* without a remainder. The **least common denominator** is the least number that will contain all of the denominators of the fractions without a remainder.

96. To find the least common denominator:

EXAMPLE.—Find the least common denominator of $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{9}$, and $\frac{1}{16}$.

SOLUTION.—We first place the denominators in a row, separated by commas.

$$\begin{array}{r|rrrr} 2 & 4, & 3, & 9, & 16 \\ \hline 2 & 2, & 3, & 9, & 8 \\ \hline 3 & 1, & 3, & 9, & 4 \\ \hline & 1, & 1, & 3, & 4 \end{array}$$

$2 \times 2 \times 3 \times 3 \times 4 = 144$, the least common denominator. Ans.

EXPLANATION.—Divide the numbers by some prime number that will divide at least two of them without a remainder (if possible), bringing down to the row below those denominators which will not contain the divisor without a remainder. Dividing each of the numbers by 2, the second row becomes 2, 3, 9, 8, since 2 will not divide 3 and 9 without a remainder. Dividing again by 2, the result is 1, 3, 9, 4.

Dividing the third row by 3, the result is 1, 1, 3, 4. Since the remaining numbers are prime to each other, we cease dividing further. The product of all the divisors and of the numbers prime to each other, is $2 \times 2 \times 3 \times 3 \times 4 = 144$, which is the required least common denominator.

97. Example.—Find the least common denominator of $\frac{1}{9}$, $\frac{5}{12}$, $\frac{7}{18}$.

Solution.—

$$\begin{array}{r|lll} 3 & 9, & 12, & 18 \\ \hline 3 & 3, & 4, & 6 \\ \hline 2 & 1, & 4, & 2 \\ \hline & 1, & 2, & 1 \end{array}$$

$3 \times 3 \times 2 \times 2 = 36$. Ans.

98. To reduce two or more fractions to fractions having a common denominator:

Example.—Reduce $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{1}{2}$ to fractions having a common denominator.

Solution.—The common denominator is any number which will contain 3, 4, and 2. The *least* common denominator is 12, because it is the smallest number which can be divided by 3, 4, and 2 without a remainder.

$$\frac{2}{3} = \frac{8}{12}, \quad \frac{3}{4} = \frac{9}{12}, \quad \frac{1}{2} = \frac{6}{12}.$$

Reducing $\frac{2}{3}$ (see Art. **84**), 3 is contained in 12, 4 times. By multiplying both numerator and denominator of $\frac{2}{3}$ by 4, we find

$\frac{2 \times 4}{3 \times 4} = \frac{8}{12}$. In the same way we find $\frac{3}{4} = \frac{9}{12}$ and $\frac{1}{2} = \frac{6}{12}$.

99. Rule.—*Divide the common denominator by the denominator of the given fraction, and multiply both terms of the fraction by the quotient.*

EXAMPLES FOR PRACTICE.

100. Reduce to fractions having a common denominator:

(*a*) $\frac{3}{4}$, $\frac{5}{8}$, $\frac{7}{8}$.		(*a*) $\frac{6}{8}$, $\frac{5}{8}$, $\frac{7}{8}$.
(*b*) $\frac{3}{16}$, $\frac{3}{4}$, $\frac{7}{32}$.		(*b*) $\frac{6}{32}$, $\frac{24}{32}$, $\frac{7}{32}$.
(*c*) $\frac{7}{8}$, $\frac{7}{88}$, $\frac{10}{11}$.	Ans.	(*c*) $\frac{77}{88}$, $\frac{7}{88}$, $\frac{80}{88}$.
(*d*) $\frac{3}{5}$, $\frac{5}{8}$, $\frac{11}{40}$.		(*d*) $\frac{24}{40}$, $\frac{25}{40}$, $\frac{11}{40}$.
(*e*) $\frac{4}{10}$, $\frac{6}{40}$, $\frac{9}{20}$.		(*e*) $\frac{16}{40}$, $\frac{6}{40}$, $\frac{18}{40}$.
(*f*) $\frac{7}{15}$, $\frac{17}{30}$, $\frac{21}{30}$.		(*f*) $\frac{14}{30}$, $\frac{17}{30}$, $\frac{21}{30}$.

ADDITION OF FRACTIONS.

101. *Fractions cannot be added unless they have a common denominator.* We cannot add $\frac{3}{4}$ to $\frac{7}{8}$ as they now stand, since the denominators represent parts of different sizes. Fourths cannot be added to eighths.

Suppose we divide an apple into 4 equal parts, and then divide 2 of these parts into two equal parts. It is evident that we shall have 2 one-fourths and 4 one-eighths. Now, if we add these parts, the result is $2 + 4 = 6$ something. But what is this something? It is not fourths, for six fourths are $1\frac{1}{2}$, and we had only 1 apple to begin with; neither is it eighths, for six eighths are $\frac{3}{4}$, which is less than 1 apple. By reducing the quarters to eighths, we have $\frac{2}{4} = \frac{4}{8}$, and adding the other 4 eighths, $4 + 4 = 8$ eighths. This result is correct, since $\frac{8}{8} = 1$. Or, we can, in this case, reduce the eighths to quarters. Thus, $\frac{4}{8} = \frac{2}{4}$; whence, adding $2 + 2 = 4$ quarters, a correct result since $\frac{4}{4} = 1$.

Before adding, fractions should be reduced to a common denominator, preferably the *least* common denominator.

102. Example.—Find the sum of $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{5}{8}$.

Solution.—The *least common denominator*, or the *least number* which will contain all the *denominators*, is 8.

$$\tfrac{1}{2} = \tfrac{4}{8}, \quad \tfrac{3}{4} = \tfrac{6}{8}, \quad \tfrac{5}{8} = \tfrac{5}{8}.$$

Explanation.—As the *denominator* tells or indicates the names of the *parts*, the *numerators* only are added in order to obtain the total number of *parts* indicated by the *denominator*. Thus, 4 one-eighths plus 6 one-eighths plus 5 one-eighths =

$$\tfrac{4}{8} + \tfrac{6}{8} + \tfrac{5}{8} = \tfrac{4+6+5}{8} = \tfrac{15}{8} = 1\tfrac{7}{8}. \quad \text{Ans.}$$

103. Example.—What is the sum of $12\frac{3}{4}$, $14\frac{5}{8}$, and $7\frac{5}{16}$?

Solution.—The least common denominator in this case is 16.

$$\begin{aligned} 12\tfrac{3}{4} &= 12\tfrac{12}{16} \\ 14\tfrac{5}{8} &= 14\tfrac{10}{16} \\ 7\tfrac{5}{16} &= \underline{\ \ 7\tfrac{5}{16}} \\ \textit{sum} &= 33 + \tfrac{27}{16} = 33 + 1\tfrac{11}{16} = 34\tfrac{11}{16}. \quad \text{Ans.} \end{aligned}$$

The sum of the fractions $= \frac{27}{16}$ or $1\frac{11}{16}$, which added to the sum of the whole numbers $= 34\frac{11}{16}$.

EXAMPLE.—What is the sum of 17, $13\frac{3}{16}$, $\frac{9}{32}$, and $3\frac{1}{4}$?

SOLUTION.—The least common denominator is 32. $13\frac{3}{16} = 13\frac{6}{32}$, $3\frac{1}{4} = 3\frac{8}{32}$.

$$\begin{array}{r} 17 \\ 13\frac{6}{32} \\ \frac{9}{32} \\ 3\frac{8}{32} \\ \hline \textit{sum}\ 33\frac{23}{32} \end{array} \text{ Ans.}$$

104. Rule.—I. *Reduce the given fractions to fractions having the least common denominator, and write the sum of the numerators over the common denominator.*

II. *When there are mixed numbers and whole numbers add the fractions first, and if their sum is an improper fraction, reduce it to a mixed number, and add the whole number with the other whole numbers.*

EXAMPLES FOR PRACTICE.

105. Find the sum of

(*a*) $\frac{4}{6}$, $\frac{7}{24}$, $\frac{5}{8}$.	(*a*) $1\frac{7}{12}$.
(*b*) $\frac{2}{3}$, $\frac{5}{15}$, $\frac{24}{45}$.	(*b*) $1\frac{8}{15}$.
(*c*) $\frac{1}{2}$, $\frac{3}{8}$, $\frac{5}{16}$.	(*c*) $1\frac{3}{16}$.
(*d*) $\frac{5}{6}$, $\frac{11}{72}$, $\frac{13}{18}$.	Ans. (*d*) $1\frac{51}{72}$.
(*e*) $\frac{10}{11}$, $\frac{6}{33}$, $\frac{23}{66}$.	(*e*) $1\frac{29}{66}$.
(*f*) $\frac{23}{45}$, $\frac{11}{15}$, $\frac{14}{45}$.	(*f*) $1\frac{5}{9}$.
(*g*) $\frac{4}{11}$, $\frac{7}{22}$, $\frac{14}{22}$.	(*g*) $1\frac{7}{22}$.
(*h*) $\frac{3}{7}$, $\frac{14}{49}$, $\frac{2}{7}$.	(*h*) 1.

1. A furnace charge consists of $\frac{1}{3}$ ton ore, $\frac{1}{5}$ ton coke, $\frac{1}{10}$ ton limestone, and the remainder slag. What is the total charge of ore, coke, and limestone? Ans. $\frac{19}{30}$ ton.

2. The burden a furnace carries is $\frac{23}{26}$ of a ton of ore, $\frac{11}{13}$ of a ton of coke, and $\frac{14}{39}$ of a ton of limestone. What is the burden in tons? Ans. $2\frac{7}{78}$ tons.

3. There are six lead furnaces running. No. 1 gets $\frac{1}{6}$ of the blast; No. 2, $\frac{1}{8}$ of the blast; No. 3 gets $\frac{1}{9}$; and No. 4, $\frac{1}{10}$ of it. What amount of the blast do these four furnaces receive? Ans. $\frac{181}{360}$.

SUBTRACTION OF FRACTIONS.

106. *Fractions can not be subtracted without first reducing them to a common denominator.* This can be shown in the same manner as in the case of addition of fractions.

EXAMPLE.—Subtract $\frac{3}{8}$ from $\frac{13}{16}$.

SOLUTION.—The common denominator is 16.

$$\tfrac{3}{8} = \tfrac{6}{16}. \quad \tfrac{13}{16} - \tfrac{6}{16} = \frac{13-6}{16} = \tfrac{7}{16}. \quad \text{Ans.}$$

107. EXAMPLE.—From 7 take $\frac{5}{8}$.

SOLUTION.—$1 = \frac{8}{8}$; therefore, since $7 = 6 + 1$, $7 = 6 + \frac{8}{8} = 6\frac{8}{8}$, or $6\frac{8}{8} - \frac{5}{8} = 6\frac{3}{8}$. Ans.

108. EXAMPLE.—What is the difference between $17\frac{9}{16}$ and $9\frac{15}{32}$?

SOLUTION.—The common denominator of the fractions is 32. $17\frac{9}{16} = 17\frac{18}{32}$.

minuend	$17\frac{18}{32}$
subtrahend	$9\frac{15}{32}$
difference	$8\frac{3}{32}$ Ans.

109. EXAMPLE.—From $9\frac{1}{4}$ take $4\frac{7}{16}$.

SOLUTION.—The common denominator of the fractions is 16. $9\frac{1}{4} = 9\frac{4}{16}$.

minuend	$9\frac{4}{16}$ or	$8\frac{20}{16}$
subtrahend	$4\frac{7}{16}$	$4\frac{7}{16}$
difference	$4\frac{13}{16}$	$4\frac{13}{16}$ Ans.

EXPLANATION.—As the fraction in the subtrahend is greater than the fraction in the minuend, it can not be subtracted; therefore, *borrow* 1, or $\frac{16}{16}$, from the 9 in the minuend and add it to the $\frac{4}{16}$; $\frac{4}{16} + \frac{16}{16} = \frac{20}{16}$. $\frac{7}{16}$ from $\frac{20}{16} = \frac{13}{16}$. Since 1 was borrowed from 9, 8 remains; 4 from $8 = 4$; $4 + \frac{13}{16} = 4\frac{13}{16}$.

110. EXAMPLE.—From 9 take $8\frac{3}{16}$.

SOLUTION.—

minuend	9 or	$8\frac{16}{16}$
subtrahend	$8\frac{3}{16}$	$8\frac{3}{16}$
difference	$\frac{13}{16}$	$\frac{13}{16}$ Ans.

EXPLANATION.—As there is no fraction in the minuend from which to take the fraction in the subtrahend, borrow 1, or $\frac{16}{16}$, from 9. $\frac{3}{16}$ from $\frac{16}{16} = \frac{13}{16}$. Since 1 was borrowed from 9, only 8 is left. 8 from $8 = 0$.

111. Rule.—I. *Reduce the fractions to fractions having a common denominator. Subtract one numerator from the other and place the remainder over the common denominator.*

II. *When there are mixed numbers, subtract the fractions and whole numbers separately, and place the remainders side by side.*

III. *When the fraction in the subtrahend is greater than the fraction in the minuend, borrow 1 from the whole number in the minuend and add it to the fraction in the minuend, from which subtract the fraction in the subtrahend.*

IV. *When the minuend is a whole number, borrow 1; reduce it to a fraction whose denominator is the same as the denominator of the fraction in the subtrahend, and place it over that fraction for subtraction.*

EXAMPLES FOR PRACTICE.

112. Subtract

(*a*) $\frac{10}{24}$ from $\frac{11}{12}$.		(*a*) $\frac{1}{2}$.
(*b*) $\frac{7}{14}$ from $\frac{17}{28}$.		(*b*) $\frac{3}{28}$.
(*c*) $\frac{4}{50}$ from $\frac{3}{10}$.		(*c*) $\frac{11}{50}$.
(*d*) $\frac{15}{35}$ from $\frac{45}{70}$.	Ans.	(*d*) $\frac{3}{14}$.
(*e*) $\frac{15}{16}$ from $\frac{57}{48}$.		(*e*) $\frac{1}{4}$.
(*f*) $13\frac{1}{4}$ from $30\frac{1}{2}$.		(*f*) $17\frac{1}{4}$.
(*g*) $12\frac{1}{8}$ from 27.		(*g*) $14\frac{7}{8}$.
(*h*) $5\frac{1}{4}$ from 30.		(*h*) $24\frac{3}{4}$.

1. A teamster was paid \$2.17$\frac{1}{2}$ a day; his helper, \$1.66$\frac{7}{8}$ a day. How much more wages does the teamster receive than his helper?

Ans. \$.50$\frac{5}{8}$.

2. A contractor receives 10$\frac{2}{3}$ cents a ton for unloading material and 15 cents a ton for loading same material. What is the difference in price?

Ans. 4$\frac{1}{3}$ cents per ton.

3. A furnace laborer's expenses one month for rent, groceries, etc. amount to \$23$\frac{3}{4}$. His rent was \$7$\frac{1}{2}$; what did his living cost?

Ans. \$16$\frac{1}{4}$.

MULTIPLICATION OF FRACTIONS.

113. *In multiplication of fractions it is not necessary to reduce the fractions to fractions having a common denominator.*

114. *Multiplying the numerator or dividing the denominator multiplies the fraction.*

EXAMPLE.—Multiply $\frac{3}{4}$ by 4.

SOLUTION.— $\frac{3}{4} \times 4 = \frac{3 \times 4}{4} = \frac{12}{4} = 3$. Ans.

or $\frac{3}{4} \times 4 = \frac{3}{4 \div 4} = \frac{3}{1} = 3$. Ans.

The word "of" when placed between two fractions, or between a fraction and a whole number, means the same as $\times$, or times. Thus,

$$\tfrac{3}{4} \text{ of } 4 = \tfrac{3}{4} \times 4 = 3.$$

$$\tfrac{1}{8} \text{ of } \tfrac{5}{16} = \tfrac{1}{8} \times \tfrac{5}{16} = \tfrac{5}{128}.$$

EXAMPLE.—Multiply $\frac{3}{8}$ by 2.

SOLUTION.— $2 \times \frac{3}{8} = \frac{3 \times 2}{8} = \frac{6}{8} = \frac{3}{4}$, Ans.

or $2 \times \frac{3}{8} = \frac{3}{8 \div 2} = \frac{3}{4}$. Ans.

115. EXAMPLE.—What is the product of $\frac{4}{16}$ and $\frac{7}{8}$?

SOLUTION.— $\frac{4}{16} \times \frac{7}{8} = \frac{4 \times 7}{16 \times 8} = \frac{28}{128} = \frac{7}{32}$, Ans.

or, by cancelation, $\frac{\cancel{4} \times 7}{\underset{4}{\cancel{16}} \times 8} = \frac{7}{4 \times 8} = \frac{7}{32}$. Ans.

116. EXAMPLE.—What is $\frac{4}{8}$ of $\frac{3}{4}$ of $\frac{16}{32}$?

SOLUTION.— $\frac{\cancel{4} \times 3 \times \cancel{16}}{8 \times \cancel{4} \times \underset{2}{\cancel{32}}} = \frac{3}{8 \times 2} = \frac{3}{16}$. Ans.

117. EXAMPLE.—What is the product of $9\frac{3}{4}$ and $5\frac{5}{8}$?

SOLUTION.—$9\frac{3}{4} = \frac{39}{4}$; $5\frac{5}{8} = \frac{45}{8}$.

$\frac{39}{4} \times \frac{45}{8} = \frac{39}{4} \times \frac{45}{8} = \frac{1755}{32} = 54\frac{27}{32}$. Ans.

118. EXAMPLE.—Multiply $15\frac{7}{8}$ by 3.

SOLUTION.—

$$\begin{array}{r} 15\frac{7}{8} \\ 3 \\ \hline 47\frac{5}{8} \end{array} \quad \text{or} \quad \begin{array}{r} 15\frac{7}{8} \\ 3 \\ \hline 45 \end{array} + \tfrac{21}{8} = 45 + 2\tfrac{5}{8} = 47\tfrac{5}{8}. \text{ Ans.}$$

119. Rule.—I. *Divide the product of the numerators by the product of the denominators. All factors common to the numerators and denominators should first be cast out by cancelation.*

II. *To multiply one mixed number by another, reduce them both to improper fractions.*

III. ***To multiply a mixed number by a whole number, first multiply the fractional part by the multiplier, and if the product is an improper fraction, reduce it to a mixed number, and add the whole number part to the product of the multiplier and whole number.***

EXAMPLES FOR PRACTICE.

120. Find the product of

(*a*) $7 \times \frac{3}{19}$.			(*a*) $1\frac{2}{19}$.
(*b*) $14 \times \frac{5}{16}$.			(*b*) $4\frac{3}{8}$.
(*c*) $\frac{21}{32} \times \frac{5}{14}$.			(*c*) $\frac{15}{64}$.
(*d*) $\frac{16}{27} \times 4$.			(*d*) $2\frac{10}{27}$.
(*e*) $\frac{19}{18} \times 7$.	Ans.		(*e*) $7\frac{7}{18}$.
(*f*) $17\frac{18}{21} \times 7$.			(*f*) 125.
(*g*) $\frac{105}{224} \times 32$.			(*g*) 15.
(*h*) $\frac{15}{28} \times 14$.			(*h*) $7\frac{1}{2}$.

1. A laborer is paid $13\frac{1}{2}$ cents per hour. What are his daily wages when he works 10 hours? Ans. 135 cents, or \$1.35.

2. A contractor who loaded and unloaded material at a furnace received \$$594\frac{16}{56}$. His expenses were \$$442\frac{1}{8}$, what were his earnings? Ans. \$$152\frac{9}{56}$.

3. A slag and matte cart cost \$15. The matte cart cost \$$9\frac{3}{8}$, what did the slag pot cost? Ans. \$$5\frac{5}{8}$.

DIVISION OF FRACTIONS.

121. *In division of fractions it is not necessary to reduce the fractions to fractions having a common denominator.*

122. *Dividing the numerator or multiplying the denominator, divides the fraction.*

EXAMPLE.—Divide $\frac{6}{8}$ by 3.

SOLUTION.—When *dividing* the *numerator*, we have

$$\frac{6}{8} \div 3 = \frac{6 \div 3}{8} = \frac{2}{8} = \frac{1}{4}. \quad \text{Ans.}$$

When *multiplying* the *denominator*, we have

$$\frac{6}{8} \div 3 = \frac{6}{8 \times 3} = \frac{6}{24} = \frac{1}{4}. \quad \text{Ans.}$$

EXAMPLE.—Divide $\frac{3}{16}$ by 2.

SOLUTION.— $\frac{3}{16} \div 2 = \frac{3}{16 \times 2} = \frac{3}{32}$. Ans.

EXAMPLE.—Divide $\frac{14}{32}$ by 7.

SOLUTION.— $\frac{14}{32} \div 7 = \frac{14 \div 7}{32} = \frac{2}{32} = \frac{1}{16}$. Ans.

123. To **invert** a fraction is to *turn it upside down;* that is, make the numerator and denominator change places.

Invert $\frac{3}{4}$ and it becomes $\frac{4}{3}$.

124. EXAMPLE.—Divide $\frac{9}{16}$ by $\frac{3}{16}$.

SOLUTION.—1. The fraction $\frac{3}{16}$ is contained in $\frac{9}{16}$ 3 times, for the denominators are the same, and one numerator is contained in the other 3 times. 2. If we now *invert* the *divisor* $\frac{3}{16}$, and multiply, the solution is

$$\frac{9}{16} \times \frac{16}{3} = \frac{\overset{3}{\cancel{9}} \times \cancel{16}}{\cancel{16} \times \cancel{3}} = 3. \quad \text{Ans.}$$

This brings the same quotient as in the first case.

125. EXAMPLE.—Divide $\frac{3}{8}$ by $\frac{1}{4}$.

SOLUTION.—We can not divide $\frac{3}{8}$ by $\frac{1}{4}$, as in the first case above, for the *denominators* are *not* the same; therefore, we must solve as in the second case.

$$\frac{3}{8} \div \frac{1}{4} = \frac{3}{8} \times \frac{4}{1} = \frac{3 \times \cancel{4}}{\underset{2}{\cancel{8}} \times 1} = \frac{3}{2} \text{ or } 1\tfrac{1}{2}. \quad \text{Ans.}$$

126. EXAMPLE.—Divide 5 by $\frac{10}{16}$.

SOLUTION.—$\frac{10}{16}$ inverted becomes $\frac{16}{10}$.

$$5 \times \frac{16}{10} = \frac{\cancel{5} \times \overset{8}{\cancel{16}}}{\underset{2}{\cancel{10}}} = 8. \quad \text{Ans.}$$

127. EXAMPLE.—How many times is $3\frac{3}{4}$ contained in $7\frac{7}{16}$?

SOLUTION.— $3\frac{3}{4} = \frac{15}{4}$; $7\frac{7}{16} = \frac{119}{16}$.

$\frac{15}{4}$ inverted equals $\frac{4}{15}$.

$$\frac{119}{16} \times \frac{4}{15} = \frac{119 \times \cancel{4}}{\underset{4}{\cancel{16}} \times 15} = \frac{119}{60} = 1\tfrac{59}{60}. \quad \text{Ans.}$$

128. **Rule.**—*Invert the divisor, and proceed as in multiplication.*

129. We have learned that a line placed between two numbers indicates that the number above the line is to be divided by the number below it. Thus, $\frac{18}{3}$ shows that 18 is to be divided by 3. This is also true if a fraction or a fractional expression be placed above or below a line.

$\dfrac{9}{\frac{3}{8}}$ means that 9 is to be divided by $\frac{3}{8}$; $\dfrac{3 \times 7}{\frac{8+4}{16}}$ means that

3×7 is to be divided by the value of $\dfrac{8+4}{16}$.

$\dfrac{\frac{1}{4}}{\frac{3}{8}}$ is the same as $\frac{1}{4} \div \frac{3}{8}$.

It will be noticed that there is a heavy line between the 9 and the $\frac{3}{8}$. This is necessary, since otherwise there would be nothing to show as to whether 9 was to be divided by $\frac{3}{8}$, or $\frac{9}{3}$ was to be divided by 8. Whenever a heavy line is used, as shown here, it indicates that *all above the line* is to be divided by *all below it.*

EXAMPLES FOR PRACTICE.

130. Divide

(*a*)	15 by $6\frac{3}{7}$.	(*a*)	$2\frac{1}{3}$.
(*b*)	30 by $\frac{6}{8}$.	(*b*)	40.
(*c*)	172 by $\frac{4}{5}$.	(*c*)	215.
(*d*)	$\frac{14}{18}$ by $1\frac{7}{16}$.	(*d*)	$\frac{112}{207}$.
(*e*)	$\frac{103}{6}$ by $14\frac{2}{3}$.	(*e*)	$1\frac{15}{88}$.
(*f*)	$\frac{142}{27}$ by $17\frac{1}{9}$.	(*f*)	$\frac{71}{231}$.
(*g*)	$\frac{14}{18}$ by $\frac{145}{72}$.	(*g*)	$\frac{56}{145}$.
(*h*)	$\frac{128}{18}$ by $72\frac{1}{3}$.	(*h*)	$\frac{64}{651}$.

Ans.

1. If a track-layer laid 47 yards of track in $9\frac{7}{8}$ hours, how many yards of track did he average per hour? Ans. $4\frac{60}{79}$ yards.

2. A blacksmith can sharpen a drill in 5 minutes. If he works for 4 hours, how many drills can he sharpen? Ans. 48 drills.

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131. Whenever an expression like one of the three following is obtained, it may always be simplified by transposing the denominator from *above* to *below* the line, or from *below* to *above*, as the case may be, taking care, however, to indicate that the denominator when so transferred is a multiplier.

1. $\dfrac{\frac{3}{4}}{9} = \dfrac{3}{9 \times 4} = \frac{3}{36} = \frac{1}{12}$; for, regarding the fraction above the heavy line as the numerator of a fraction whose denominator is 9, $\dfrac{\frac{3}{4} \times 4}{9 \times 4} = \dfrac{3}{9 \times 4}$, as before.

2. $\dfrac{9}{\frac{3}{4}} = \dfrac{9 \times 4}{3} = 12.$ The proof is the same as in the first case.

3. $\dfrac{\frac{5}{9}}{\frac{3}{4}} = \dfrac{5 \times 4}{3 \times 9} = \frac{20}{27}$; for, regarding $\frac{5}{9}$ as the numerator of a fraction whose denominator is $\frac{3}{4}$, $\dfrac{\frac{5}{9} \times 9}{\frac{3}{4} \times 9} = \dfrac{5}{\frac{3 \times 9}{4}}$; and

$$\dfrac{5 \times 4}{\frac{3 \times 9}{4} \times 4} = \dfrac{5 \times 4}{3 \times 9} = \tfrac{20}{27}, \text{ as above.}$$

This principle may be used to great advantage in cases like $\dfrac{\frac{1}{4} \times 310 \times \frac{27}{12} \times 72}{40 \times 4\frac{1}{2} \times 5\frac{1}{6}}$. Reducing the mixed numbers to fractions, the expression becomes $\dfrac{\frac{1}{4} \times 310 \times \frac{27}{12} \times 72}{40 \times \frac{9}{2} \times \frac{31}{6}}$. Now transferring the denominators of the fractions and canceling,

$$\frac{1 \times 310 \times 27 \times 72 \times 2 \times 6}{40 \times 9 \times 31 \times 4 \times 12} = \frac{1 \times \overset{\cancel{10}}{\cancel{310}} \times \overset{3}{\cancel{27}} \times \overset{\overset{3}{\cancel{6}}}{\cancel{72}} \times \cancel{2} \times \overset{3}{\cancel{6}}}{\underset{\underset{\cancel{2}}{\cancel{4}}}{\cancel{40}} \times \cancel{9} \times \cancel{31} \times \underset{2}{\cancel{4}} \times \cancel{12}} =$$

$\dfrac{27}{2} = 13\frac{1}{2}.$

Greater exactness in results can usually be obtained by using this principle than can be obtained by reducing the fractions to decimals. The principle, however, should not be employed *if a sign of addition or subtraction occurs either above or below the dividing line.*

ARITHMETIC.

(PART 3.)

DECIMALS.

132. **Decimals** are *tenth* fractions; that is, the parts of a unit are expressed on the scale of ten, as *tenths*, *hundredths*, *thousandths*, etc.

133. The *denominator*, which is always 10, 100, 1,000, etc. is not expressed, as it would be in fractions, by writing it under the *numerator* with a line between them, as $\frac{3}{10}$, $\frac{3}{100}$, $\frac{3}{1000}$, but is expressed by placing a *period* (.), which is called a **decimal point**, to the *left* of the *figures of the numerator*, to indicate that the figures on the right form the numerator of a fraction whose denominator is *ten*, *one hundred*, *one thousand*, etc.

134. The *reading* of a decimal number depends upon the number of decimal places in it, i. e., the number of figures to the right of the decimal point.

One decimal place expresses *tenths*.
Two decimal places express *hundredths*.
Three decimal places express *thousandths*.
Four decimal places express *ten-thousandths*.
Five decimal places express *hundred-thousandths*.
Six decimal places express *millionths*.

Thus:

$.3 = \frac{3}{10}$ = 3 tenths.
$.03 = \frac{3}{100}$ = 3 hundredths.
$.003 = \frac{3}{1000}$ = 3 thousandths.
$.0003 = \frac{3}{10000}$ = 3 ten-thousandths.
$.00003 = \frac{3}{100000}$ = 3 hundred-thousandths.
$.000003 = \frac{3}{1000000}$ = 3 millionths.

We see in the above that the *number of decimal places in a decimal equals the number of ciphers to the right of the figure 1 in the denominator of its equivalent fraction.* This fact kept in mind will be of much assistance in reading and writing decimals.

Whatever may be written to the *left* of a decimal point is a whole number. *The decimal point affects only the figures to its right.*

When a whole number and decimal are written together, the expression is a *mixed number.* Thus, 8.12 and 17.25 are mixed numbers.

The relation of decimals and whole numbers to each other is clearly shown by the following table:

hundreds of millions.	tens of millions.	millions.	hundreds of thousands.	tens of thousands.	thousands.	hundreds.	tens.	units.	decimal point.	tenths.	hundredths.	thousandths.	ten-thousandths.	hundred-thousandths.	millionths.	ten-millionths.	hundred-millionths.
9	8	7	6	5	4	3	2	1	.	2	3	4	5	6	7	8	9

The figures to the *left* of the decimal point represent *whole numbers;* those to the *right* are *decimals.*

135. In both decimals and whole numbers, the *units* place is made the starting point of notation and numeration. The decimals *decrease* on the scale of *ten* to the *right*, and the whole numbers *increase* on the scale of *ten* to the *left*. The *first* figure to the *left* of units is *tens*, and the *first* figure to the *right* of units is *tenths*. The *second* figure to the *left* of units is *hundreds*, and the *second* figure to the *right* is *hundredths*. The *third* figure to the *left* is *thousands*, and the *third* to the *right* is *thousandths*, and so on, the *whole* numbers on the *left* and the *decimals* on the *right*. The figures equally distant from units place correspond in name.

The *decimals* have the ending *ths*, which distinguish them from *whole* numbers. The following is the numeration of the number in the above table: nine hundred eighty-seven million six hundred fifty-four thousand three hundred twenty-one and twenty-three million four hundred fifty-six thousand seven hundred eighty-nine hundred-millionths.

The decimals increase to the *left* on the scale of *ten*, the same as whole numbers; for, beginning at, say, 4-*thousandths* in the table, the next figure to the left is *hundredths*, which is ten times as great, and the next *tenths*, or ten times the *hundredths*, and so on through both decimals and whole numbers.

136. *Annexing or taking away a cipher at the right of a decimal does not affect its value.*

.5 is $\frac{5}{10}$; .50 is $\frac{50}{100}$, but $\frac{5}{10} = \frac{50}{100}$; therefore, $.5 = .50$.

137. *Inserting a cipher between a decimal and the decimal point divides the decimal by 10.*

$$.5 = \frac{5}{10};\ \frac{5}{10} \div 10 = \frac{5}{100} = .05.$$

138. *Taking away a cipher from the left of a decimal multiplies the decimal by 10.*

$$.05 = \frac{5}{100};\ \frac{5}{100} \times 10 = \frac{5}{10} = .5.$$

ADDITION OF DECIMALS.

139. Addition of decimals is similar in all respects to addition of whole numbers—units are placed under units, tens under tens, etc.; this, of course, brings the decimal points in line directly under one another. Hence, in placing the numbers to be added, it is only necessary to take care that the *decimal points are in line.* In adding both whole numbers and decimals the right-hand unit figures are always in line; but in adding decimals, the right-hand unit figures will not be in line unless each decimal contains the same number of figures.

whole numbers	*decimals*	*mixed numbers*
342	.342	342.032
4234	.4234	4234.5
26	.26	26.6782
3	.03	3.06
sum 4605 Ans.	*sum* 1.0554 Ans.	*sum* 4606.2702 Ans.

140. A decimal, as .342, ought really to be expressed as 0.342, but it is quite customary to omit the cipher on the left of the decimal point, though many authors use it.

EXAMPLE.—What is the sum of 242, .36, 118.725, 1.005, 6, and 100.1?

SOLUTION.—

```
     242.
        .36
     118.725
        1.005
        6.
     100.1
     -------
sum  468.190  Ans.
```

141. Rule.—*Place the numbers to be added so that the decimal points will be directly under each other. Add as in whole numbers, and place the decimal point in the sum directly under the decimal points above.*

EXAMPLES FOR PRACTICE.

142. Find the sum of

(*a*)	.2143, .105, 2.3042, and 1.1417.	Ans. (*a*)	3.7652.
(*b*)	783.5, 21.473, .2101, and .7816.	(*b*)	805.9647.
(*c*)	21.781, 138.72, 41.8738, .72, and 1.413.	(*c*)	204.5078.
(*d*)	.3724, 104.15, 21.417, and 100.042.	(*d*)	225.9814.
(*e*)	200.172, 14.105, 12.1465, .705, and 7.2.	(*e*)	234.3285.
(*f*)	1,427.16, .244, .32, .032, and 10.0041.	(*f*)	1,437.7601.
(*g*)	2,473.1, 41.65, .7243, 104.067, and 21.073.	(*g*)	2,640.6143.
(*h*)	4,107.2, .00375, 21.716, 410.072, and .0345.	(*h*)	4,539.02625.

1. By carefully measuring the six sides of a tract of land it was found that the first side measured 537.683 feet, the second 87.36 feet, the third 836.391 feet, the fourth 732.129 feet, the fifth 237.261 feet, and the sixth 523.689 feet. What is the exact distance around the property? Ans. 2,954.513 feet.

2. The area of a circle is 3.1416 square feet, the area of a square

74.326 square feet, and the area of a triangle 83.56 square feet. What is the total area of the three figures? Ans. 161.0276 square feet.

3. A certain flue is made up of three sections. One section requires 9.786 pounds per square foot to pass the desired quantity of air through the first section, 7.86 pounds to pass it through the second section, and 5.63 pounds to pass it through the third section. What is the total pressure for the flue? Ans. 23.276 pounds per square foot.

4. The exact weight of a barrow is .3915 ton, and the weight of the coke which it contains is .492 ton. What is the total weight of the loaded barrow? Ans. .8835 ton.

SUBTRACTION OF DECIMALS.

143. As in subtraction of whole numbers, units are placed under units, tens under tens, etc., bringing the decimal points under each other, as in addition of decimals.

EXAMPLE.—Subtract .132 from .3063.

SOLUTION.—

minuend	.3063
subtrahend	.132
difference	.1743 Ans.

144. EXAMPLE.—What is the difference between 7.895 and .725?

SOLUTION.—

minuend	7.895
subtrahend	.725
difference	7.170 or 7.17 Ans.

145. EXAMPLE.—Subtract .625 from 11.

SOLUTION.—

minuend	11.000
subtrahend	.625
difference	10.375. Ans.

146. Rule.—*Place the subtrahend under the minuend, so that the decimal points will be directly under each other. Subtract as in whole numbers, and place the decimal point in the remainder, directly under the decimal points above.*

When the figures in the decimal part of the subtrahend extend beyond those in the minuend, place ciphers in the minuend above them, and subtract as before.

EXAMPLES FOR PRACTICE.

147. From

(*a*) 407.385 take 235.0004.	Ans.	(*a*) 172.3846.
(*b*) 22.718 take 1.7042.		(*b*) 21.0138.
(*c*) 1,368.17 take 13.6817.		(*c*) 1,354.4883.
(*d*) 70.00017 take 7.000017.		(*d*) 63.000153.
(*e*) 630.630 take .6304.		(*e*) 629.9996.
(*f*) 421.73 take 217.162.		(*f*) 204.568.
(*g*) 1.000014 take .00001.		(*g*) 1.000004.
(*h*) .783652 take .542314.		(*h*) .241338.

1. The weight of a cubic foot of Connellsville coke is 26.3 pounds, and of ore, 166.6 pounds. What is the difference in the weights of a cubic foot of such coke and ore? Ans. 140.3 pounds.

2. A 2-foot bar composed of 1 foot of iron and 1 foot of steel was heated until its entire length became 2.00234799 feet. What was the expansion of the steel, if the iron expanded .001258 of a foot? Ans. .00108999 foot.

3. A meter is 39.370432 inches long and a decimeter is 3.9370432 inches long. What is the difference in the lengths of the meter and decimeter? Ans. 35.4333888 in.

MULTIPLICATION OF DECIMALS.

148. In multiplication of decimals, we do not place the decimal points directly under each other, as in addition and subtraction. We pay no attention for the time being to the decimal points. Place the multiplier under the multiplicand, so that the *right-hand* figure of the one is under the *right-hand* figure of the other, and proceed exactly as in multiplication of whole numbers. After multiplying, *count the number of decimal places in both multiplicand and multiplier, and point off the same number in the product.*

EXAMPLE.—Multiply .825 by 13.

SOLUTION.—

multiplicand	.825
multiplier	13
	2475
	825
product	10.725 Ans.

In this example there are three decimal places in the multiplicand and none in the multiplier; therefore, 3 decimal places are pointed off in the product.

149. EXAMPLE.—What is the product of 426 and the decimal .005?

SOLUTION.— *multiplicand* 426
multiplier .005
product 2.130 or 2.13 Ans.

In this example there are 3 decimal places in the multiplier and none in the multiplicand; therefore, 3 decimal places are pointed off in the product.

150. It is *not* necessary to multiply by the ciphers on the *left* of a decimal; they merely determine the number of decimal places. Ciphers to the *right* of a decimal should be omitted, as they only make more figures to deal with, and do not change the value.

151. EXAMPLE.—Multiply 1.205 by 1.15.

SOLUTION.— *multiplicand* 1.205
multiplier 1.15
6025
1205
1205
product 1.38575 Ans.

In this example there are 3 decimal places in the multiplicand, and 2 in the multiplier; therefore, 3 + 2, or 5, decimal places must be pointed off in the product.

152. EXAMPLE.—Multiply .232 by .001.

SOLUTION.— *multiplicand* .232
multiplier .001
product .000232 Ans.

In this example we multiply the multiplicand by the digit in the multiplier, which makes 232 in the product, but since there are 3 decimal places in each, the multiplier and the

multiplicand, we must prefix 3 ciphers to the 232, to make 3 + 3, or 6, decimal places in the product.

153. Rule.—*Place the multiplier under the multiplicand, disregarding the position of the decimal points. Multiply as in whole numbers, and in the product point off as many decimal places as there are decimal places in both multiplier and multiplicand, prefixing ciphers if necessary.*

EXAMPLES FOR PRACTICE.

154. Find the product of

(*a*) .000492 × 4.1418.			(*a*) .0020377656.
(*b*) 4,003.2 × 1.2.			(*b*) 4,803.84.
(*c*) 78.6531 × 1.03.			(*c*) 81.012693.
(*d*) .3685 × .042.			(*d*) .015477.
(*e*) 178,352 × .01.		Ans.	(*e*) 1,783.52.
(*f*) .00045 × .0045.			(*f*) .000002025.
(*g*) .714 × .00002.			(*g*) .00001428.
(*h*) .00004 × .008.			(*h*) .00000032.

1. If it costs .743 of a dollar to ship one ton of ore from the mines to the mill, what will it cost to ship 4,376.58 tons?
Ans. 3,251.79894 dollars.

2. A contractor pays 7.563 cents per ton to his workmen. Supposing they handle 1,853.65 tons of ore a month, what do the workmen receive per month? Ans. 14,019.15495 cents.

3. A meter is 3.2808992 feet long. What is the length of 8.31 meters? Ans. 27.264272352 feet.

4. If a steam pump delivers 2.39 gallons of water per stroke and runs at 51 strokes a minute, how many gallons of water would it pump in 58.5 minutes? Ans. 7,130.565 gallons.

DIVISION OF DECIMALS.

155. In division of decimals we pay no attention to the decimal point until after the division is performed. The *number of decimal places in the dividend must equal, or be made to equal by annexing ciphers, the number of decimal places in the divisor. Divide exactly as in whole numbers. Subtract the number of decimal places in the divisor from the number of decimal places in the dividend, and point off as many decimal*

places in the quotient as there are units in the remainder thus found.

EXAMPLE.—Divide .625 by 25.

SOLUTION.—

```
        divisor dividend quotient
           25 ) .625 ( .025  Ans.
                 50
                 ---
                 125
                 125
                 ---
      remainder    0
```

In this example there are no decimal places in the divisor, and 3 decimal places in the dividend; therefore, there are 3 minus 0, or 3, decimal places in the quotient. One cipher has to be prefixed to the 25, to make the 3 decimal places.

156. EXAMPLE.—Divide 6.035 by .05.

SOLUTION.—

```
        divisor dividend quotient
          .05 ) 6.035 ( 120.7  Ans.
                5
                --
                10
                10
                ---
                  35
                  35
                  --
      remainder    0
```

In this example we divide by 5, as if the cipher were not before it. There is one more decimal place in the dividend than in the divisor; therefore, one decimal place is pointed off in the quotient.

157. EXAMPLE.—Divide .125 by .005.

SOLUTION.—

```
        divisor dividend quotient
         .005 ) .125 ( 25  Ans.
                 10
                 --
                  25
                  25
                  --
      remainder    0
```

In this example there are the same number of decimal places in the dividend as in the divisor; therefore, the quotient has no decimal places, and is a whole number.

158. Example.—Divide 326 by .25.

Solution.—

```
 divisor  dividend  quotient
   .25 ) 326.00 ( 1304   Ans.
         25
         --
          76
          75
          ---
           100
           100
           ---
 remainder   0
```

In this problem two ciphers were annexed to the dividend, to make the number of decimal places equal to the number in the divisor. The quotient is a whole number.

159. Example.—Divide .0025 by 1.25.

Solution.—

```
   1.25 ) .00250 ( .002   Ans.
             250
             ---
 remainder     0
```

Explanation.—In this example we are to divide .0025 by 1.25. Consider the dividend as a whole number, or 25 (disregarding the two ciphers at its left, for the present); also, consider the divisor as a whole number, or 125. It is clearly evident that the dividend 25 will not contain the divisor 125; we must, therefore, annex one cipher to the 25, thus making the dividend 250. 125 is contained twice in 250, so we place the figure 2 in the quotient. In pointing off the decimal places in the quotient, it must be remembered that there were only four decimal places in the dividend; but one cipher was annexed, thereby making $4 + 1$, or 5, decimal places. Since there are 5 decimal places in the dividend and 2 decimal places in the divisor, we must point off $5 - 2$, or 3, decimal places in the quotient. In order to point off 3 decimal places, two ciphers must be prefixed to the figure 2, thereby making .002 the quotient. It is not necessary to consider the ciphers at the left of a decimal when dividing, except when determining the position of the decimal point in the quotient.

160. **Rule.—I.** *Place the divisor to the left of the dividend, and proceed as in division of whole numbers; in the*

quotient, point off as many decimal places as the number of decimal places in the dividend exceed those in the divisor, prefixing ciphers to the quotient, if necessary.

II. *If in dividing one number by another there be a remainder, the remainder can be placed over the divisor, as a fractional part of the quotient, but it is generally better to annex ciphers to the remainder, and continue dividing until there are 3 or 4 decimal places in the quotient, and then if there still be a remainder, terminate the quotient by the plus sign (+), which shows that it can be carried further.*

161. EXAMPLE.—What is the quotient of 199 divided by 15?

SOLUTION.—

```
          15)199(13 + 4/15  Ans.
             15
             --
              49
              45
              --
  remainder    4
Or,  15)199.000(13.266+  Ans.
        15
        --
         49
         45
         --
          40
          30
          --
          100
           90
          ---
           100
            90
           ---
  remainder 10
```

$13\frac{4}{15} = 13.266+$

$\frac{4}{15} = .266+$

162. It frequently happens, as in the above example, that the division will never terminate. In such cases, decide to how many decimal places the division is to be carried, and carry the work one place further. If the last figure of the quotient thus obtained is 5 or a greater number, increase the preceding figure by 1, and write after it the minus sign (—), thus indicating that the quotient is not quite as large as indicated; if the figure thus obtained is less than 5, write the plus sign (+) after the quotient, thus indicating that

the number is slightly greater than as indicated. In the last example, had it been desired to obtain the answer correct to four decimal places, the work would have been carried to five places, obtaining 13.26666, and the answer would have been given as 13.2667−. This remark applies to any other calculation involving decimals, when it is desired to omit some of the figures in the decimal. Thus, if it is desired to retain three decimal places in the number .2471253, it would be expressed as .247 +; if it was desired to retain five decimal places, it would be expressed as .24713−. Both the + and − signs are frequently omitted; they are seldom used outside of Arithmetic, except in exact calculations, when it is desired to call particular attention to the fact that the result obtained is not *quite* exact.

EXAMPLES FOR PRACTICE.

163. Divide

	Ans.
(*a*) 101.6688 by 2.36.	(*a*) 43.08
(*b*) 187.12264 by 123.107.	(*b*) 1.52.
(*c*) .08 by .008.	(*c*) 10.
(*d*) .0003 by 3.75.	(*d*) .00008.
(*e*) .0144 by .024.	(*e*) .6.
(*f*) .00375 by 1.25.	(*f*) .003.
(*g*) .004 by 400.	(*g*) .00001.
(*h*) .4 by .008.	(*h*) 50.
(*i*) 177.6 by 2.4.	(*i*) 74.
(*j*) .98 by .7.	(*j*) 1.4.

1. A man received $3.95 for unloading and bedding 19.75 yards of ore. What did he receive per yard? Ans. $.20 per yard.

2. A cistern has 52,845 pounds of water in it. How many cubic feet of water does it contain if one cubic foot of water weighs 62.5 pounds? Ans. 846 cubic feet.

3. An atmosphere is equal to 14.7 pounds per square inch. Under how many atmospheres are men working in a caisson where the pressure is 30.87 pounds per square inch? Ans. 2.1 atmospheres.

4. A pump delivers 12.13 gallons at each stroke. How many strokes must it make in order to deliver 1,285.78 gallons?

Ans. 106 strokes.

TO REDUCE A FRACTION TO A DECIMAL.

164. EXAMPLE.—$\frac{3}{4}$ equals what decimal?

SOLUTION.— $\dfrac{4)3.00}{.75}$, or $\frac{3}{4} = .75$. Ans.

EXAMPLE.—What decimal is equivalent to $\frac{7}{8}$?

SOLUTION.—

```
8)7.000(.875
  64
  --
   60
   56      or 7/8 = .875.  Ans.
   --
    40
    40
    --
     0
```

165. Rule.—*Annex ciphers to the numerator and divide by the denominator. Point off as many decimal places in the quotient as there are ciphers annexed.*

EXAMPLES FOR PRACTICE.

166. Reduce the following common fractions to decimals:

(*a*) $\frac{15}{32}$.	Ans.	(*a*) .46875.
(*b*) $\frac{7}{8}$.		(*b*) .875.
(*c*) $\frac{21}{32}$.		(*c*) .65625.
(*d*) $\frac{51}{64}$.		(*d*) .796875.
(*e*) $\frac{4}{25}$.		(*e*) .16.
(*f*) $\frac{5}{8}$.		(*f*) .625.
(*g*) $\frac{10}{200}$.		(*g*) .05.
(*h*) $\frac{4}{1000}$.		(*h*) .004.

167. To reduce inches to decimal parts of a foot:

EXAMPLE.—What decimal part of a foot is 9 inches?

SOLUTION.—Since there are 12 inches in one foot, 1 inch is $\frac{1}{12}$ of a foot, and 9 inches is $9 \times \frac{1}{12}$ or $\frac{9}{12}$ of a foot. This, reduced to a decimal by the above rule, shows what decimal part of a foot 9 inches is.

```
12)9.00(.75 of a foot.  Ans.
   84
   --
    60
    60
    --
     0
```

168. Rule.—I. *To reduce inches to decimal parts of a foot, divide the number of inches by 12.*

II. *Should the resulting decimal be an unending one and it is desired to terminate the division at some point, say, the fourth decimal place, carry the division one place further, and if the fifth figure is 5 or greater, increase the fourth figure by 1. Omit the signs + and –.*

EXAMPLES FOR PRACTICE.

169. Reduce to the decimal part of a foot:

(*a*)	3 in.	Ans. (*a*)	.25.
(*b*)	$4\frac{1}{2}$ in.	(*b*)	.375.
(*c*)	5 in.	(*c*)	.4167.
(*d*)	$6\frac{5}{8}$ in.	(*d*)	.5521.
(*e*)	11 in.	(*e*)	.9167.

TO REDUCE A DECIMAL TO A FRACTION.

170. EXAMPLE.—Reduce .125 to a fraction.

SOLUTION.— $.125 = \frac{125}{1000} = \frac{5}{40} = \frac{1}{8}$. Ans.

EXAMPLE.—Reduce .875 to a fraction.

SOLUTION.— $.875 = \frac{875}{1000} = \frac{35}{40} = \frac{7}{8}$. Ans.

171. Rule.—*Under the figures of the decimal place 1 with as many ciphers at its right as there are decimal places in the decimal, and reduce the resulting fraction to its lowest terms by dividing both numerator and denominator by the same number.*

EXAMPLES FOR PRACTICE.

172. Reduce the following to common fractions:

(*a*)	.125.	Ans. (*a*)	$\frac{1}{8}$.
(*b*)	.625.	(*b*)	$\frac{5}{8}$.
(*c*)	.3125.	(*c*)	$\frac{5}{16}$.
(*d*)	.04.	(*d*)	$\frac{1}{25}$.
(*e*)	.06.	(*e*)	$\frac{3}{50}$.
(*f*)	.75.	(*f*)	$\frac{3}{4}$.
(*g*)	.15625.	(*g*)	$\frac{5}{32}$.
(*h*)	.875.	(*h*)	$\frac{7}{8}$.

---

1. A car holds .987 of a ton of coal. Determine the number of hundredweights and pounds it holds. Ans. 19 cwt. 74 lb.

2. The outside diameter of a pipe is 6.382 inches, and the thickness of the iron of which it is made is .121 of an inch. What is the inside diameter in inches and fraction of an inch? Ans. $6\frac{7}{50}$ inches.

3. A spring is made up of four flat steel bars. The thicknesses of the bars are, respectively, .5 inch, .25 inch, .125 inch, and .0625 inch. What is the entire thickness of the spring, expressed as a fraction of an inch? Ans. $\frac{15}{16}$ inch.

173. To express a decimal approximately as a fraction having a given denominator:

174. Example.—Express .5827 in 64ths.

Solution.— $.5827 \times \frac{64}{64} = \frac{37.2928}{64}$, say $\frac{37}{64}$.

Hence, $.5827 = \frac{37}{64}$, nearly. Ans.

Example.—Express .3917 in 12ths.

Solution.— $.3917 \times \frac{12}{12} = \frac{4.7004}{12}$, say $\frac{5}{12}$.

Hence, $.3917 = \frac{5}{12}$, nearly. Ans.

175. Rule.—*Reduce 1 to a fraction having the given denominator. Multiply the given decimal by the fraction so obtained, and the result will be the fraction required.*

EXAMPLES FOR PRACTICE.

176. Express

(*a*) .625 in 8ths.	Ans.	(*a*) $\frac{5}{8}$.
(*b*) .3125 in 16ths.		(*b*) $\frac{5}{16}$.
(*c*) .15625 in 32ds.		(*c*) $\frac{5}{32}$.
(*d*) .77 in 64ths.		(*d*) $\frac{49}{64}$.
(*e*) .81 in 48ths.		(*e*) $\frac{39}{48}$.
(*f*) .928 in 96ths.		(*f*) $\frac{89}{96}$.

---

177. The sign for dollars is $. It is read *dollars.* $25 is read *25 dollars.*

Since there are 100 cents in a dollar, one cent is 1-one-hundredth of a dollar; the first two figures of a decimal part of a dollar represent *cents.* Since a mill is $\frac{1}{10}$ of a cent, or $\frac{1}{1000}$ of a dollar, the third figure represents *mills.*

Thus, $25.16 is read *twenty-five dollars and sixteen cents;* $25.168 is read *twenty-five dollars, sixteen cents, and eight mills.*

178. The **vinculum** ——, **parenthesis** ( ), **brackets** [ ], and **brace** { } are called **symbols of aggregation,** and are used to include numbers which are to be considered

together; thus, $13 \times \overline{8 - 3}$, or $13 \times (8 - 3)$, shows that 3 is to be taken from 8 before multiplying by 13.

$$13 \times (8 - 3) = 13 \times 5 = 65. \quad \text{Ans.}$$
$$13 \times \overline{8 - 3} = 13 \times 5 = 65. \quad \text{Ans.}$$

When the vinculum or parenthesis is not used, we have

$$13 \times 8 - 3 = 104 - 3 = 101. \quad \text{Ans.}$$

179. In any series of numbers connected by the signs $+$, $-$, $\times$, and $\div$, the operations indicated by the signs must be performed in order from left to right, *except* that no addition or subtraction may be performed if a sign of multiplication or division *follows* the number on the *right* of a sign of addition or subtraction, until the indicated multiplication or division has been performed. In all cases the sign of multiplication takes the precedence, the reason being that when two or more numbers or expressions are connected by the sign of multiplication, the numbers thus connected are regarded as factors of the product indicated, and not as separate numbers.

EXAMPLE.—What is the value of $4 \times 24 - 8 + 17$?

SOLUTION.—Performing the operations in order from left to right, $4 \times 24 = 96$; $96 - 8 = 88$; $88 + 17 = 105$. Ans.

180. EXAMPLE.—What is the value of the following expression: $1{,}296 \div 12 + 160 - 22 \times 3\frac{1}{2} = ?$

SOLUTION.—$1{,}296 \div 12 = 108$; $108 + 160 = 268$; here we cannot subtract 22 from 268 because the sign of multiplication *follows* 22; hence, multiplying 22 by $3\frac{1}{2}$, we get 77, and $268 - 77 = 191$. Ans.

Had the above expression been written $1{,}296 \div 12 + 160 - 22 \times 3\frac{1}{2} \div 7 + 25$, it would have been necessary to have divided $22 \times 3\frac{1}{2}$ by 7 before subtracting, and the final result would have been $22 \times 3\frac{1}{2} = 77$; $77 \div 7 = 11$; $268 - 11 = 257$; $257 + 25 = 282$. Ans. In other words, it is necessary to perform *all* of the multiplication or division included between the signs $+$ and $-$, or $-$ and $+$, before adding or subtracting. Also, had the expression been written $1{,}296 \div 12 + 160 - 24\frac{1}{2} \div 7 \times 3\frac{1}{2} + 25$, it would have been necessary to have multiplied $3\frac{1}{2}$ by 7 before dividing $24\frac{1}{2}$, since the sign of multiplication takes the precedence, and the final result would

have been $3\frac{1}{2} \times 7 = 24\frac{1}{2}$; $24\frac{1}{2} \div 24\frac{1}{2} = 1$; $268 - 1 = 267$; $267 + 25 = 292$. Ans.

It likewise follows that if a succession of multiplication and division signs occurs, the indicated operations must not be performed in order, from left to right—the multiplication must be performed first. Thus, $24 \times 3 \div 4 \times 2 \div 9 \times 5 = \frac{1}{5}$. Ans. In order to obtain the same result that would be obtained by performing the indicated operations in order, from left to right, symbols of aggregation must be used. Thus, by using two vinculums, the last expression becomes $24 \times \overline{3 \div 4} \times \overline{2 \div 9} \times 5 = 20$, the same result that would be obtained by performing the indicated operations in order, from left to right.

EXAMPLES FOR PRACTICE.

181. Find the values of the following expressions.

			Ans.	
(*a*)	$(8 + 5 - 1) \div 4$.		(*a*)	3.
(*b*)	$5 \times 24 - 32$.		(*b*)	88.
(*c*)	$5 \times 24 \div 15$.		(*c*)	8.
(*d*)	$144 - 5 \times 24$.		(*d*)	24.
(*e*)	$(1,691 - 540 + 559) \div 3 \times 57$.		(*e*)	10.
(*f*)	$2,080 + 120 - 80 \times 4 - 1,670$.		(*f*)	210.
(*g*)	$(\overline{90 + 60} \div 25) \times 5 - 29$.		(*g*)	1.
(*h*)	$\overline{90 + 60} \div 25 \times 5$.		(*h*)	1.2.

ARITHMETIC.

(PART 4.)

PERCENTAGE.

182. Percentage is the process of calculating by *hundredths.*

183. The *term* **per cent.** is an abbreviation of the Latin words *per centum*, which mean *by the hundred.* A certain per cent. of a number is the number of hundredths of that number which is indicated by the number of units in the per cent. Thus, 6 per cent. of 125 is $125 \times \frac{6}{100} = 7.5$; 25 per cent. of 80 is $80 \times \frac{25}{100} = 20$; 43 per cent. of 432 pounds is $432 \times \frac{43}{100} = 185.76$ pounds.

184. The **sign** of per cent. is %, and is read *per cent.* Thus, 6% is read *six per cent.;* 12½% is read *twelve and one-half per cent.*, etc.

When expressing the per cent. of a number to use in calculations, it is customary to express it decimally instead of fractionally. Thus, instead of expressing 6%, 25%, and 43% as $\frac{6}{100}$, $\frac{25}{100}$, and $\frac{43}{100}$, it is usual to express them as .06, .25, and .43.

The following table will show how any per cent. can be expressed either as a decimal or as a fraction:

Per Cent.	Decimal.	Fraction.	Per Cent.	Decimal.	Fraction.
1%.....	.01	$\frac{1}{100}$	150 %....	1.50	$\frac{150}{100}$ or $1\frac{1}{2}$
2%.....	.02	$\frac{2}{100}$ or $\frac{1}{50}$	500 %....	5.00	$\frac{500}{100}$ or 5
5%.....	.05	$\frac{5}{100}$ or $\frac{1}{20}$	$\frac{1}{4}$%....	.0025	$\frac{\frac{1}{4}}{100}$ or $\frac{1}{400}$
10%.....	.10	$\frac{10}{100}$ or $\frac{1}{10}$	$\frac{1}{2}$%....	.005	$\frac{\frac{1}{2}}{100}$ or $\frac{1}{200}$
25%.....	.25	$\frac{25}{100}$ or $\frac{1}{4}$	$1\frac{1}{2}$%....	.015	$\frac{1\frac{1}{2}}{100}$ or $\frac{3}{200}$
50%.....	.50	$\frac{50}{100}$ or $\frac{1}{2}$	$8\frac{1}{3}$%....	$.08\frac{1}{3}$	$\frac{8\frac{1}{3}}{100}$ or $\frac{1}{12}$
75%.....	.75	$\frac{75}{100}$ or $\frac{3}{4}$	$12\frac{1}{2}$%....	.125	$\frac{12\frac{1}{2}}{100}$ or $\frac{1}{8}$
100%.....	1.00	$\frac{100}{100}$ or 1	$16\frac{2}{3}$%....	$.16\frac{2}{3}$	$\frac{16\frac{2}{3}}{100}$ or $\frac{1}{6}$
125%.....	1.25	$\frac{125}{100}$ or $1\frac{1}{4}$	$62\frac{1}{2}$%....	.625	$\frac{62\frac{1}{2}}{100}$ or $\frac{5}{8}$

The preceding table will show how any per cent. can be expressed either as a decimal or as a fraction.

185. The names of the different elements used in percentage are: the *base*, the *rate per cent.*, the *percentage*, the *amount*, and the *difference*.

186. The **base** is the number on which the per cent. is computed.

187. The **rate** is the number of hundredths of the base to be taken.

188. The **percentage** is the part, or number of *hundredths*, of the base indicated by the rate; or, the percentage is the result obtained by multiplying the base by the rate.

Thus, when it is stated that 7% of $25 is $1.75, $25 is the base, 7% is the rate, and $1.75 is the percentage.

189. The **amount** is the sum of the base and percentage.

190. The **difference** is the remainder obtained by subtracting the percentage from the base.

Thus, if a man has $180, and he earns 6% more, he will have, altogether, $\$180 + \$180 \times .06$, or $\$180 + \$10.80 = \$190.80$. Here $180 is the base; 6%, the rate; $10.80, the percentage, and $190.80, the *amount*.

Again, if an engine of 125 horsepower uses 16% of it in overcoming friction and other resistances, the amount left for obtaining useful work is $125 - 125 \times .16 = 125 - 20 = 105$ horsepower. Here 125 is the base; 16%, the rate; 20, the percentage, and 105, the *difference*.

191. From the foregoing it is evident that to find the percentage, the base must be multiplied by the rate. Hence,

Rule.—*To find the percentage, multiply the base by the rate expressed decimally.*

EXAMPLE.—Out of a lot of 300 tons of coal, 76% were sold. How many tons were sold?

SOLUTION.—76%, the rate, expressed decimally, is .76; the base is 300; hence, the number of tons sold, or the percentage, is, by the above rule, $300 \times .76 = 228$ tons. Ans.

Expressing the rule as a formula,

$$percentage = base \times rate.$$

192. When the percentage and rate are given, the base may be found by dividing the percentage by the rate. For, suppose that 12 is 6%, or $\frac{6}{100}$, of some number; then, 1%, or $\frac{1}{100}$, of the number, is $12 \div 6$, or 2. Consequently, if $2 = 1\%$, or $\frac{1}{100}$, 100%, or $\frac{100}{100}$, $= 2 \times 100 = 200$. But, since the same result may be arrived at by dividing 12 by .06, for $12 \div .06 = 200$, it follows that

Rule.—*When the percentage and rate are given, to find the base, divide the percentage by the rate, expressed decimally.*

Formula, *base = percentage ÷ rate.*

EXAMPLE.—Bought a certain number of tons of coal and sold 76% of it. If I sold 228 tons, how many tons did I buy?

SOLUTION.—Here 228 is the percentage, and 76%, or .76, is the rate; hence, applying the rule,

$$228 \div .76 = 300 \text{ tons. Ans.}$$

193. When the base and percentage are given, to find the rate, the rate may be found, expressed decimally, by dividing the percentage by the base. For, suppose that it is desired to find what per cent. 12 is of 200. 1% of 200 is $200 \times .01 = 2$. Now, if 1% is 2, 12 is evidently as many per cent. as the number of times that 2 is contained in 12, or $12 \div 2 = 6\%$. But the same result may be obtained by dividing 12, the percentage, by 200, the base, since $12 \div 200 = .06 = 6\%$. Hence,

Rule.—*When the percentage and base are given, to find the rate, divide the percentage by the base, and the result will be the rate, expressed decimally.*

Formula, *rate = percentage ÷ base.*

EXAMPLE.—Bought 300 tons of coal and sold 228 tons. What per cent. of the total number of tons was sold?

SOLUTION.—Here 300 is the base and 228 is the percentage; hence, applying rule, $\text{rate} = 228 \div 300 = .76 = 76\%$. Ans.

EXAMPLE.—What per cent. of 875 is 25?

SOLUTION.—Here 875 is the base and 25 is the percentage; hence, applying rule, $25 \div 875 = .02\frac{6}{7} = 2\frac{6}{7}\%$. Ans.

PROOF.—$875 \times .02\frac{6}{7} = 25$.

EXAMPLES FOR PRACTICE.

194. What per cent. of

(*a*) 360 is 90?		(*a*) 25%.	
(*b*) 900 is 360?		(*b*) 40%.	
(*c*) 125 is 25?		(*c*) 20%.	
(*d*) 150 is 750?	Ans.	(*d*) 500%.	
(*e*) 280 is 112?		(*e*) 40%.	
(*f*) 400 is 200?		(*f*) 50%.	
(*g*) 47 is 94?		(*g*) 200%.	
(*h*) 500 is 250?		(*h*) 50%.	

195. The amount may be found, when the base and rate are given, by multiplying the base by 1 plus the rate, expressed decimally. For, suppose that it is desired to find the amount when 200 is the base and 6% is the rate. The percentage is $200 \times .06 = 12$, and, according to definition, Art. **189,** the amount is $200 + 12 = 212$. But the same result may be obtained by multiplying 200 by $1 + .06$, or 1.06, since $200 \times 1.06 = 212$. Hence,

Rule.—*When the base and rate are given, to find the amount, multiply the base by 1 plus the rate, expressed decimally.*

Formula, $amount = base \times (1 + rate)$.

EXAMPLE.—If a man earned $725 in a year, and the next year 10% more, how much did he earn the second year?

SOLUTION.—Here 725 is the base and 10% is the rate, and the amount is required. Hence, applying the rule,

$725 \times 1.10 = \$797.50$. Ans.

196. When the base and rate are given, the difference may be found by multiplying the base by 1 minus the rate, expressed decimally. For, suppose that it is desired to find the difference when the base is 200 and the rate is 6%. The percentage is $200 \times .06 = 12$; and, according to definition, Art. **190,** the difference $= 200 - 12 = 188$. But the same result may be obtained by multiplying 200 by $1 - .06$, or .94, since $200 \times .94 = 188$. Hence,

Rule.—*When the base and rate are given, to find the difference, multiply the base by 1 minus the rate, expressed decimally.*

Formula, $difference = base \times (1 - rate)$.

EXAMPLE.—Bought 300 tons of coal, and sold all but 24% of it. How many tons were sold?

SOLUTION.—Here 300 is the base, 24% is the rate, and it is desired to find the difference. Hence, applying the rule,

$$300 \times (1 - .24) = 228 \text{ tons. Ans.}$$

197. When the amount and rate are given, the base may be found by dividing the amount by 1 plus the rate. For, suppose that it is known that 212 equals some number increased by 6% of itself. Then it is evident that 212 equals 106% of the number (base) that it is desired to find. Consequently, if $212 = 106\%$, $1\% = \frac{212}{106} = 2$, and $100\% = 2 \times 100 = 200 =$ the base. But the same result may be obtained by dividing 212 by $1 + .06$, or 1.06, since $212 \div 1.06 = 200$. Hence,

Rule.—*When the amount and rate are given, to find the base, divide the amount by 1 plus the rate, expressed decimally.*

Formula, *base* = *amount* ÷ (1 + *rate*).

EXAMPLE.—The theoretical discharge of a certain pump, when running at a piston speed of 100 feet per minute, is 278,910 gallons per day of 10 hours. Owing to leakage and other defects, this value is 25% greater than the actual discharge. What is the actual discharge?

SOLUTION.—Here 278,910 equals the actual discharge (base) increased by 25% of itself. Consequently, 278,910 is the amount; 25% is the rate, and, applying rule,

$$\text{actual discharge} = 278,910 \div 1.25 = 223,128 \text{ gallons. Ans.}$$

198. When the difference and rate are given, the base may be found by dividing the difference by 1 minus the rate. For, suppose that 188 equals some number less 6% of itself. Then, 188 evidently equals $100 - 6 = 94\%$ of some number. Consequently, if $188 = 94\%$, $1\% = 188 \div 94 = 2$, and $100\% = 2 \times 100 = 200$. But the same result may be obtained by dividing 188 by $1 - .06$, or .94, since $188 \div .94 = 200$. Hence,

Rule.—*When the difference and rate are given, to find the base, divide the difference by 1 minus the rate, expressed decimally.*

Formula, *base* = *difference* ÷ (1 — *rate*).

EXAMPLE.—Bought a certain number of tons of coal and sold 76% of it. If there were 72 tons left unsold, how many tons did I buy?

SOLUTION.—Here 72 is the difference and 76% is the rate. Applying rule,

$$72 \div (1 - .76) = 300 \text{ tons. Ans.}$$

EXAMPLE.—The theoretical number of foot-pounds of work per minute required to operate a boiler feed-pump is 127,344. If 30% of the total number actually required be allowed for friction, leakage, etc., how many foot-pounds are actually required to work the pump?

SOLUTION.—Here the number actually required is the base; hence, 127,344 is the difference, and 30% is the rate. Applying the rule,

$$127{,}344 \div (1 - .30) = 181{,}920 \text{ foot-pounds. Ans.}$$

199. EXAMPLE.—A certain air stack produces a ventilating pressure of 2.76 inches of water. By increasing the height 20 feet, the pressure was increased to 3 inches of water. What was the gain per cent.?

SOLUTION.—Here it is evident that 3 inches is the amount and that 2.76 inches is the base. Consequently, $3 - 2.76 = .24$ inch is the percentage, and it is required to find the rate. Hence, applying the rule given in Art. **193**,

$$\text{gain per cent.} = .24 \div 2.76 = .087 = 8.7\%. \text{ Ans.}$$

200. EXAMPLE.—A certain air stack produces a ventilating pressure of 3 inches of water. The stack being injured by a storm, the pressure was reduced to 1.2 inches of water. What was the loss per cent.?

SOLUTION.—Here it is evident that 1.2 inches is the difference (since it equals 3 inches diminished by a certain per cent., loss of itself) and 3 inches is the base. Consequently, $3 - 1.2 = 1.8$ inches is the percentage. Hence, applying the rule given in Art. **193**,

$$\text{loss per cent.} = 1.8 \div 3 = .60 = 60\%. \text{ Ans.}$$

201. To find the gain or loss per cent.:

Rule.—*Find the difference between the initial and final values; divide this difference by the initial value.*

EXAMPLE.—If a man buys a house for $1,860, and some time afterwards builds a barn for 25% of the cost of the house, does he gain or lose, and how much per cent., if he sells both house and barn for $2,100?

SOLUTION.—The cost of the barn was $\$1{,}860 \times .25 = \$465$; consequently, the initial value, or cost, was $\$1{,}860 + \$465 = \$2{,}325$. Since he sold them for $2,100, he lost $\$2{,}325 - \$2{,}100 = \$225$. Hence, applying rule,

$$225 \div 2{,}325 = .0968 = 9.68\% \text{ loss. Ans.}$$

EXAMPLES FOR PRACTICE.

202. Solve the following:

(*a*)	What is $12\frac{1}{2}$% of \$900?	Ans.	(*a*)	\$112.50.
(*b*)	What is $\frac{4}{5}$% of 627?		(*b*)	5.016.
(*c*)	What is $33\frac{1}{3}$% of 54?		(*c*)	18.
(*d*)	101 is $68\frac{3}{4}$% of what number?		(*d*)	$146\frac{10}{11}$.
(*e*)	784 is $83\frac{1}{3}$% of what number?		(*e*)	940.8.
(*f*)	What % of 960 is 160?		(*f*)	$16\frac{2}{3}$%.
(*g*)	What % of \$3,606 is \$$450\frac{3}{4}$?		(*g*)	$12\frac{1}{2}$%.
(*h*)	What % of 280 is 112?		(*h*)	40%.

1. A steam plant consumed an average of 3,640 pounds of coal per day. The engineer made certain alterations which resulted in a saving of 250 pounds per day. What was the per cent. of coal saved? Ans. 7%, nearly.

2. If the speed of an engine running at 126 revolutions per minute should be increased $6\frac{1}{2}$%, how many revolutions per minute would it then make? Ans. 134.19 revolutions.

3. A hydraulic ram, when the valves were in perfect condition, discharged 190.4 gallons of water per hour. A little sand got under the valve and reduced the discharge 15%. What amount of water did it then discharge per hour? Ans. 161.84 gal.

4. If I lend a man \$1,100, and this is $18\frac{1}{2}$% of the amount that I have on interest, how much money have I on interest? Ans. \$5,945.95.

5. A test showed that an engine developed 190.4 horsepower, 15% of which was consumed in friction. How much power was available for use? Ans. 161.84 H.P.

6. By adding a condenser to a steam-engine, the power was increased 14%, and the consumption of coal per horsepower per hour was decreased 20%. If the engine could originally develop 50 horsepower, and required $3\frac{1}{2}$ pounds of coal per horsepower per hour, what would be the total weight of coal used in an hour, with the condenser, assuming the engine to run full power? Ans. 159.6 pounds.

DENOMINATE NUMBERS.

203 A **denominate number** is a concrete number, and may be either simple or compound, as 8 quarts, 5 feet, 10 inches, etc.

204. A **simple denominate number** consists of units of but one denomination, as 16 cents, 10 hours, 5 dollars, etc.

205. A **compound denominate number** consists of units of two or more denominations of a similar kind, as 3 yards 2 feet 1 inch ; 34 square feet 57 square inches.

206. In **whole numbers** and in **decimals** the *law* of increase and decrease is on the scale of 10, but in **compound** or **denominate numbers** the scale varies.

MEASURES.

207. A **measure** is a *standard unit*, established by *law* or *custom*, by which *quantity* of any kind is measured. The *standard unit* of **dry measure** is the Winchester bushel ; of **weight,** the pound ; of **liquid measure,** the gallon, etc.

208. Measures are of six kinds :

1. Extension.
2. Weight.
3. Capacity.
4. Time.
5. Angles.
6. Money or value.

MEASURES OF EXTENSION.

209. **Measures of extension** are used in measuring lengths, distances, surfaces, and solids.

LINEAR MEASURE.

TABLE 3.

			Abbreviation.	in.	ft.	yd.	rd.	fur.	mi.
12	inches (in.)	= 1 foot . .	ft.						
3	feet	= 1 yard . .	yd.	36 =	3 =	1			
5.5	yards	= 1 rod . . .	rd.	198 =	$16\frac{1}{2}$ =	5.5 =	1		
40	rods	= 1 furlong	fur.	7,920 =	660 =	220 =	40 =	1	
8	furlongs	= 1 mile . .	mi.	63,360 =	5,280 =	1,760 =	320 =	8 =	1

SURVEYOR'S LINEAR MEASURE.

TABLE 4.

7.92 inches	= 1 link	li.
25 links	= 1 rod	rd.
4 rods / 100 links	= 1 chain	ch.
80 chains	= 1 mile	mi.

mi.		ch.		rd.		li.		in.
1	=	80	=	320	=	8,000	=	63,360

210. The linear unit, generally used by surveyors, is **Gunter's chain,** which is equal to 4 rods, or 66 feet.

211. An **engineer's chain,** used by civil engineers, is 100 feet long, and consists of 100 links. In computations, the links are written as so many hundredths of a chain.

SQUARE MEASURE.

TABLE 5.

144	square inches (sq. in.)	=	1 square foot	sq.ft.
9	square feet	=	1 square yard	sq.yd.
$30\frac{1}{4}$	square yards	=	1 square rod	sq.rd.
160	square rods	=	1 acre	A.
640	acres	=	1 square mile	sq.mi.

sq.mi.		A.		sq.rd.		sq.yd.		sq.ft.		sq.in.
1	=	640	=	102,400	=	3,097,600	=	27,878,400	=	4,014,489,600

SURVEYOR'S SQUARE MEASURE.

TABLE 6.

625	square links	=	1 square rod	sq.rd.
16	square rods	=	1 square chain	sq.ch.
10	square chains	=	1 acre	A.
640	acres	=	1 square mile	sq.mi.
36	square miles (6 mi. square)	=	1 township	Tp.

sq.mi.		A.		sq.ch.		sq.rd.		sq.li.
1	=	640	=	6,400	=	102,400	=	64,000,000

CUBIC MEASURE.

TABLE 7.

1728	cubic inches (cu. in.)	=	1 cubic foot	cu.ft.
27	cubic feet	=	1 cubic yard	cu.yd.
128	cubic feet	=	1 cord	cd.
$24\frac{3}{4}$	cubic feet	=	1 perch	P.

cu.yd.		cu.ft.		cu.in.
1	=	27	=	46,656

MEASURES OF WEIGHT.

AVOIRDUPOIS WEIGHT.

TABLE 8.

16	ounces (oz.)	=	1 pound	lb.
100	pounds	=	1 hundredweight	cwt.
20	cwt., or 2,000 lb.	=	1 ton	T.

T.		cwt.		lb.		oz.
1	=	20	=	2,000	=	32,000

212. The ounce is divided into halves, quarters, etc. Avoirdupois weight is used for weighing coarse and heavy articles. One avoirdupois pound contains 7,000 grains.

LONG TON TABLE.

TABLE 9.

16 ounces	=	1 pound	lb.
112 pounds	=	1 hundredweight	cwt.
20 cwt., or 2,240 lb.	=	1 ton	T.

213. In all the calculations throughout this and the succeeding volumes, 2,000 pounds will be considered one ton, unless the long ton (2,240 pounds) is especially mentioned.

TROY WEIGHT.

TABLE 10.

24 grains (gr.)	=	1 pennyweight	pwt.
20 pennyweights	=	1 ounce	oz.
12 ounces	=	1 pound	lb.

lb.		oz.		pwt.		gr.
1	=	12	=	240	=	5,760

214. Troy weight is used in weighing gold and silverware, jewels, etc. It is used by jewelers.

MEASURES OF CAPACITY.

LIQUID MEASURE.

TABLE 11.

4 gills (gi.)	=	1 pint	pt.
2 pints	=	1 quart	qt.
4 quarts	=	1 gallon	gal.
31½ gallons	=	1 barrel	bbl.
2 barrels, or 63 gallons	=	1 hogshead	hhd.

hhd.		bbl.		gal.		qt.		pt.		gi.
1	=	2	=	63	=	252	=	504	=	2,016

DRY MEASURE.

TABLE 12.

2 pints (pt.)	=	1 quart	qt.
8 quarts	=	1 peck	pk.
4 pecks	=	1 bushel	bu.

bu.		pk.		qt.		pt.
1	=	4	=	32	=	64

MEASURE OF TIME.

TABLE 13.

60 seconds (sec.)	=	1 minute	min.
60 minutes	=	1 hour	hr.
24 hours	=	1 day	da.
7 days	=	1 week	wk.
365 days / 12 months	=	1 common year	yr.
366 days	=	1 leap year.	
100 years	=	1 century.	

NOTE.—It is customary to consider one month as 30 days.

MEASURE OF ANGLES OR ARCS.

TABLE 14.

60 seconds (″)	=	1 minute	′.
60 minutes	=	1 degree	°.
90 degrees	=	1 right angle or quadrant	∟.
360 degrees	=	1 circle	cir.

cir.						
1	=	360°	=	21,600′	=	1,296,000″

MEASURE OF MONEY.

UNITED STATES MONEY.

TABLE 15.

10 mills (m.)	=	1 cent	ct.
10 cents	=	1 dime	d.
10 dimes	=	1 dollar	$.
10 dollars	=	1 eagle	E.

E.		$		d.		ct.		m.
1	=	10	=	100	=	1,000	=	10,000

MISCELLANEOUS TABLE.

TABLE 16.

12 things are 1 dozen.
12 dozen are 1 gross.
12 gross are 1 great gross.
2 things are 1 pair.
20 things are 1 score.
1 league is 3 miles.
1 fathom is 6 feet.
1 meter is nearly 39.37 inches.
1 hand is 4 inches.
1 palm is 3 inches.
1 span is 9 inches.
24 sheets are 1 quire.
20 quires, or 480 sheets, are 1 ream.
1 bushel contains 2,150.4 cubic in.
1 U. S. standard gallon (also called a wine gallon) contains 231 cubic in.
1 U. S. standard gallon of water weighs 8.355 pounds, nearly.
1 cubic foot of water contains 7.481 U. S. standard gallons, nearly.
1 British imperial gallon weighs 10 pounds.

It will be of great advantage to the student to carefully memorize all of the above tables.

REDUCTION OF DENOMINATE NUMBERS.

215. Reduction of denominate numbers is the process of changing their denomination without changing their value. They may be changed from a higher to a lower denomination or from a lower to a higher—either is reduction. As,

2 hours = 120 minutes.
32 ounces = 2 pounds.

216. Principle.—Denominate numbers are changed to *lower* denominations by *multiplying*, and to *higher* denominations by *dividing*.

To reduce denominate numbers to lower denominations:

217. EXAMPLE.—Reduce 5 yd. 2 ft. 7 in. to inches.

SOLUTION.—

```
 yd.   ft.   in.
   5     2     7
   3
 ---
  15 ft.
   2 ft.
 ---
  17 ft.
  12
 ---
  34
 17
 ---
 204 in.
   7 in.
 ---
 211 inches.  Ans.
```

EXPLANATION.—Since there are 3 feet in 1 yard, in 5 yards there are 5×3, or 15 feet, and 15 feet plus 2 feet = 17 feet. There are 12 inches in a foot; therefore, $12 \times 17 = 204$ inches, and 204 inches plus 7 inches = 211 inches = number of inches in 5 yards 2 feet and 7 inches. Ans.

218. EXAMPLE.—Reduce 6 hours to seconds.

SOLUTION.—

```
     6   hours.
    60
   ---
   360   minutes.
    60
 -----
 21600 seconds.  Ans.
```

EXPLANATION.—As there are 60 minutes in one hour, in six hours there are 6 × 60, or 360 minutes; as there are no minutes to add, we multiply 360 minutes by 60, to get the number of seconds.

219. In order to avoid mistakes, if any denomination be omitted, represent it by a cipher. Thus, before reducing 3 rods 6 inches to inches, insert a cipher for yards and a cipher for feet; as,

rd.	yd.	ft.	in.
3	0	0	6

220. Rule.—*Multiply the number representing the highest denomination by the number of units in the next lower required to make one of the higher denomination, and to the product add the number of given units of that lower denomination. Proceed in this manner until the number is reduced to the required denomination.*

EXAMPLES FOR PRACTICE.

221. Reduce

(*a*)	4 rd. 2 yd. 2 ft. to ft.	Ans.	(*a*)	74 ft.
(*b*)	4 bu. 3 pk. 2 qt. to qt.		(*b*)	154 qt.
(*c*)	13 rd. 5 yd. 2 ft. to ft.		(*c*)	231.5 ft.
(*d*)	5 mi. 100 rd. 10 ft. to ft.		(*d*)	28,060 ft.
(*e*)	8 lb. 4 oz. 6 pwt. to gr.		(*e*)	48,144 gr.
(*f*)	52 hhd. 24 gal. 1 pt. to pt.		(*f*)	26,401 pt.
(*g*)	5 cir. 16° 20′ to minutes.		(*g*)	108,980′.
(*h*)	14 bu. to qt.		(*h*)	448 qt.

To reduce lower to higher denominations:

222. EXAMPLE.—Reduce 211 in. to higher denominations.

SOLUTION.—

$$\begin{array}{r} 12\,\underline{)\,211\text{ in.}} \\ 3\,\underline{)\,17\text{ ft.} + 7\text{ in.}} \\ 5\text{ yd.} + 2\text{ ft.} \quad \text{Ans.} \end{array}$$

EXPLANATION.—There are 12 inches in 1 foot; therefore, 211 divided by 12 = 17 feet and 7 inches over. There

are 3 feet in 1 yard ; therefore, 17 feet divided by $3 = 5$ yards and 2 feet over. The last quotient and the two remainders constitute the answer, 5 yards 2 feet 7 inches.

223. EXAMPLE.—Reduce 15,735 grains Troy weight to higher denominations.

SOLUTION.—

```
24)15735 gr.(655 pwt.
   144
   ---
    133
    120
    ---
     135
     120
     ---
      15 gr.

20)655 pwt.(32 oz.
   60
   --
    55
    40
    --
    15 pwt.

12)32 oz.(2 lb.
   24
   --
    8 oz.
```

EXPLANATION.—There are 24 grains in 1 pennyweight, and in 15,735 grains there are as many pennyweights as 24 is contained in 15,735, or 655 pennyweights and 15 grains remaining. There are 20 pennyweights in 1 ounce, and in 655 pennyweights there are 32 ounces and 15 pennyweights remaining. There are 12 ounces in 1 pound, and in 32 ounces there are 2 pounds and 8 ounces remaining. The last quotient and the three remainders constitute the answer, 2 pounds 8 ounces 15 pennyweights 15 grains.

The above problem is worked out by long division, because the numbers are too large to solve easily by short division. The student may use either method.

224. Rule.—*Divide the number representing the denomination given by the number of units of this denomination required to make one unit of the next higher denomination. The remainder will be of the same denomination, but the quotient will be of the next higher. Divide this quotient by*

the number of units of its denomination required to make one unit of the next higher. Continue until the highest denomination is reached, or until there is not enough of a denomination left to make one of the next higher. The last quotient and the remainders constitute the required result.

EXAMPLES FOR PRACTICE.

225. Reduce to units of higher denominations :

(*a*) 7,460 sq. in.; (*b*) 7,580 sq. yd.; (*c*) 148,760 cu. in.; (*d*) 7,896 cu. ft. to cd.; (*e*) 17,651″; (*f*) 1,120 cu. ft. to cd.; (*g*) 8,000 gi.; (*h*) 36,450 lb.

Ans.
- (*a*) 5 sq. yd. 6 sq. ft. 116 sq. in.
- (*b*) 1 A. 90 sq. rd. 17 sq. yd. 4 sq. ft. 72 sq. in.
- (*c*) 3 cu. yd. 5 cu. ft. 152 cu. in.
- (*d*) 61 cd. 88 cu. ft.
- (*e*) 4° 54′ 11″.
- (*f*) 8 cd. 96 cu. ft.
- (*g*) 3 hhd. 61 gal.
- (*h*) 18 T. 4 cwt. 50 lb.

ADDITION OF DENOMINATE NUMBERS.

226. EXAMPLE.—Find the sum of 3 cwt. 46 lb. 12 oz.; 8 cwt. 12 lb. 13 oz.; 12 cwt. 50 lb. 13 oz.; 27 lb. 4 oz.

SOLUTION.—

T.	cwt.	lb.	oz.	
0	3	46	12	
0	8	12	13	
0	12	50	13	
0	0	27	4	
1	4	37	10	Ans.

EXPLANATION.—Begin to add at the right-hand column : $4 + 13 + 13 + 12 = 42$ ounces ; as 16 ounces make 1 pound, 42 ounces $\div 16 = 2$ and a remainder of 10 ounces, or 2 pounds and 10 ounces. Place 10 ounces under ounce column, and add 2 pounds to the next or pound column. Then, $2 + 27 + 50 + 12 + 46 = 137$ pounds ; as 100 pounds make a hundredweight, $137 \div 100 = 1$ hundredweight and a remainder of 37 pounds. Place the 37 under the pounds column, and add 1 hundredweight to the next or hundredweight column. Next, $1 + 12 + 8 + 3 = 24$ hundredweight.

20 hundredweight make a ton ; therefore $24 \div 20 = 1$ ton and 4 hundredweight remaining. Hence, the sum is 1 ton 4 hundredweight 37 pounds 10 ounces. Ans.

227. EXAMPLE.—What is the sum of 2 rd. 3 yd. 2 ft. 5 in.; 6 rd. 1 ft. 10 in.; 17 rd. 11 in.; 4 yd. 1 ft.?

SOLUTION.—

	rd.	yd.	ft.	in.	
	2	3	2	5	
	6	0	1	10	
	17	0	0	11	
	0	4	1	0	
	26	$3\frac{1}{2}$	0	2	
or	26	3	1	8	Ans.

EXPLANATION.—The sum of the numbers in the first column = 26 inches, or 2 feet and 2 inches remaining. The sum of the numbers in the next column plus 2 feet = 6 feet, or 2 yards and 0 feet remaining. The sum of the next column plus 2 yards = 9 yards, or $9 \div 5\frac{1}{2} = 1$ rod and $3\frac{1}{2}$ yards remaining. The sum of the next column plus 1 rod = 26 rods. To avoid fractions in the sum, the $\frac{1}{2}$ yard is reduced to 1 foot and 6 inches, which added to 26 rods 3 yards 0 feet and 2 inches = 26 rods 3 yards 1 foot 8 inches. Ans.

228. EXAMPLE.—What is the sum of 47 ft. and 3 rd. 2 yd. 2 ft. 10 in.?

SOLUTION.—When 47 ft. is reduced it equals 2 rd. 4 yd. 2 ft., which can be added to 3 rd. 2 yd. 2 ft. 10 in. Thus,

	rd.	yd.	ft.	in.	
	3	2	2	10	
	2	4	2	0	
	6	$1\frac{1}{2}$	1	10	
or	6	2	0	4	Ans.

229. Rule.—*Place the numbers so that like denominations are under each other. Begin at the right-hand column, and add. Divide the sum by the number of units of this denomination required to make one unit of the next higher. Place the remainder under the column added, and carry the quotient to the next column. Continue in this manner until the highest denomination given is reached.*

EXAMPLES FOR PRACTICE.

230. What is the sum of

(*a*) 25 lb. 7 oz. 15 pwt. 23 gr.; 17 lb. 16 pwt.; 15 lb. 4 oz. 12 pwt.; 18 lb. 16 gr.; 10 lb. 2 oz. 11 pwt. 16 gr.?

(*b*) 9 mi. 13 rd. 4 yd. 2 ft.; 16 rd. 5 yd. 1 ft. 5 in.; 16 mi. 2 rd. 3 in.; 14 rd. 1 yd. 9 in.?

(*c*) 3 cwt. 46 lb. 12 oz.; 12 cwt. $9\frac{1}{4}$ lb.; $2\frac{1}{4}$ cwt. $21\frac{5}{8}$ lb.?

(*d*) 10 yr. 8 mo. 5 wk. 3 da.; 42 yr. 6 mo. 7 da.; 7 yr. 5 mo. 18 wk. 4 da.; 17 yr. 17 da.?

(*e*) 17 tons 11 cwt. 49 lb. 14 oz.; 16 tons 47 lb. 13 oz.; 20 tons 13 cwt. 14 lb. 6 oz.; 11 tons 4 cwt. 16 lb. 12 oz.?

(*f*) 14 sq. yd. 8 sq. ft. 19 sq. in.; 105 sq. yd. 16 sq. ft. 240 sq. in.; 42 sq. yd. 28 sq. ft. 165 sq. in.?

Ans. (*a*) 86 lb. 3 oz. 16 pwt. 7 gr.
(*b*) 25 mi. 47 rd. 1 ft. 5 in.
(*c*) 18 cwt. 2 lb. 14 oz.
(*d*) 78 yr. 1 mo. 3 wk. 3 da.
(*e*) 65 tons 9 cwt. 28 lb. 13 oz.
(*f*) 167 sq. yd. 136 sq. in.

SUBTRACTION OF DENOMINATE NUMBERS.

231. Example.—From 21 rd. 2 yd. 2 ft. $6\frac{1}{4}$ in., take 9 rd. 4 yd. $10\frac{1}{4}$ in.

Solution.—

	rd.	yd.	ft.	in.	
	21	2	2	$6\frac{1}{4}$	
	9	4	0	$10\frac{1}{4}$	
	11	$3\frac{1}{2}$	1	$8\frac{1}{4}$	Ans.

Explanation.—Since $10\frac{1}{4}$ inches cannot be taken from $6\frac{1}{4}$ inches, we must borrow 1 foot, or 12 inches, from the 2 feet in the next column and add it to the $6\frac{1}{4}$. $6\frac{1}{4} + 12 = 18\frac{1}{4}$. $18\frac{1}{4}$ inches $-$ $10\frac{1}{4}$ inches $= 8\frac{1}{4}$ inches. Then, 0 foot from the 1 remaining foot $=$ 1 foot. 4 yards cannot be taken from 2 yards; therefore, we borrow 1 rod, or $5\frac{1}{2}$ yards, from 21 rods and add it to 2. $2 + 5\frac{1}{2} = 7\frac{1}{2}$; $7\frac{1}{2} - 4 = 3\frac{1}{2}$ yards. 9 rods from 20 rods $=$ 11 rods. Hence, the remainder is 11 rods $3\frac{1}{2}$ yards 1 foot $8\frac{1}{4}$ inches. Ans.

To avoid fractions as much as possible, we reduce the $\frac{1}{2}$ yard to inches, obtaining 18 inches; this added to $8\frac{1}{4}$ inches, gives $26\frac{1}{4}$ inches, which equals 2 feet $2\frac{1}{4}$ inches. Then, 2 feet $+$ 1 foot $=$ 3 feet $=$ 1 yard, and 3 yards $+$ 1 yard $=$ 4 yards. Hence, the above answer becomes 11 rods 4 yards 0 feet $2\frac{1}{4}$ inches.

232. Example.—What is the difference between 8 rd. 2 yd 2 ft. 10 in. and 47 ft. ?

Solution.—47 ft. = 2 rd. 4 yd. 2 ft.

	rd.	yd.	ft.	in.	
	8	2	2	10	
	2	4	2	0	
	0	$3\frac{1}{2}$	0	10	
or		3	2	4	Ans.

To find (approximately) the interval of time between two dates:

233. Example.—How many years, months, days, and hours between 4 o'clock P.M. of June 15, 1868, and 10 o'clock A.M., September 28, 1891 ?

Solution.—

yr.	mo.	da.	hr.	
1891	8	28	10	
1868	5	15	16	
23	3	12	18	Ans.

Explanation.—Counting 24 hours in 1 day, 4 o'clock P.M. is the 16th hour from the beginning of the day, or midnight. On September 28, 8 months and 28 days have elapsed, and on June 15, 5 months and 15 days. After placing the earlier date under the later date, subtract as in the previous problems. Count 30 days as 1 month.

234. Rule.—*Place the smaller quantity under the larger quantity, with like denominations under each other. Beginning at the right, subtract successively the number in the subtrahend in each denomination from the one above, and place the differences underneath. If the number in the minuend of any denomination is less than the number under it in the subtrahend, one must be borrowed from the minuend of the next higher denomination, reduced and added to it.*

EXAMPLES FOR PRACTICE.

235. From

(*a*) 125 lb. 8 oz. 14 pwt. 18 gr. take 96 lb. 9 oz. 10 pwt. 4 gr.
(*b*) 126 hhd. 27 gal. take 104 hhd. 14 gal. 1 qt. 1 pt.
(*c*) 65 T. 14 cwt. 64 lb. 10 oz. take 16 T. 11 cwt. 14 oz.
(*d*) 148 sq. yd. 16 sq. ft. 142 sq. in. take 132 sq. yd. 136 sq. in.

(*e*) 100 bu. take 28 bu. 2 pk. 5 qt. 1 pt.

(*f*) 14 mi. 34 rd. 16 yd. 13 ft. 11 in. take 3 mi. 27 rd. 11 yd. 4 ft. 10 in.

Ans.
- (*a*) 28 lb. 11 oz. 4 pwt. 14 gr.
- (*b*) 22 hhd. 12 gal. 2 qt. 1 pt.
- (*c*) 49 T. 3 cwt. 63 lb. 12 oz.
- (*d*) 16 sq. yd. 16 sq. ft. 6 sq. in.
- (*e*) 71 bu. 1 pk. 2 qt. 1 pt.
- (*f*) 11 mi. 7 rd. 5 yd. 9 ft. 1 in.

MULTIPLICATION OF DENOMINATE NUMBERS.

236. Example.—Multiply 7 lb. 5 oz. 13 pwt. 15 gr. by 12.

Solution.—

lb.	oz.	pwt.	gr.	
7	5	13	15	
			12	
89	8	3	12	Ans.

Explanation.—15 grains $\times$ 12 = 180 grains. 180 $\div$ 24 = 7 pennyweights and 12 grains remaining. Place the 12 in the grain column and carry the 7 pennyweights to the next. Now, $13 \times 12 + 7 = 163$ pennyweights; $163 \div 20 = 8$ ounces and 3 pennyweights remaining. Then, $5 \times 12 + 8 = 68$ ounces; $68 \div 12 = 5$ pounds and 8 ounces remaining. Then, $7 \times 12 + 5 = 89$ pounds. The entire product is 89 pounds 8 ounces 3 pennyweights 12 grains. Ans.

237. Rule.—*Multiply the number representing each denomination by the multiplier, and reduce each product to the next higher denomination, writing the remainders under each denomination, and carrying the quotient to the next, as in Addition of Denominate Numbers.*

238. Note.—In multiplication and division of denominate numbers, it is sometimes easier to reduce the number to the lowest denomination given before multiplying or dividing, especially if the multiplier or divisor is a decimal. Thus, in the above example, had the multiplier been 1.2, the easiest way to multiply would have been to reduce the number to grains; then, multiply by 1.2, and reduce the product to higher denominations. For example, 7 lb. 5 oz. 13 pwt. 15 gr. = 43,047 gr. 43,047 $\times$ 1.2 = 51,656.4 gr. = 8 lb. 11 oz. 12 pwt. 8.4 gr. Also, 43,047 $\times$ 12 = 516,564 gr. = 89 lb. 8 oz. 3 pwt. 12 gr., as above. The student may use either method.

EXAMPLES FOR PRACTICE.

239. Multiply

(*a*) 15 cwt. 90 lb. by 5; (*b*) 12 yr. 10 mo. 4 wk. 3 da. by 14; (*c*) 11 mi. 145 rd. by 20; (*d*) 12 gal. 4 pt. by 9; (*e*) 8 cd. 76 cu. ft. by 15; (*f*) 4 hhd. 3 gal. 1 qt. 1 pt. by 12.

Ans. (*a*) 79 cwt. 50 lb.
(*b*) 180 yr. 11 mo. 2 wk.
(*c*) 229 mi. 20 rd.
(*d*) 112 gal. 2 qt.
(*e*) 128 cd. 116 cu. ft.
(*f*) 48 hhd. 40 gal. 2 qt.

DIVISION OF DENOMINATE NUMBERS.

240. Example.—Divide 48 lb. 11 oz. 6 pwt. by 8.

Solution.—

	lb.	oz.	pwt.	gr.	
8)	48	11	6	0	
	6 lb.	1 oz.	8 pwt.	6 gr.	Ans.

Explanation.—After placing the quantities as above, proceed as follows: 8 is contained in 48 six times without a remainder. 8 is contained in 11 ounces once with 3 ounces remaining. $3 \times 20 = 60$; $60 + 6 = 66$ pennyweights; 66 pennyweights $\div 8 = 8$ pennyweights and 2 remaining; 2×24 grains $= 48$ grains; 48 grains $\div 8 = 6$ grains. Therefore, the entire quotient is 6 pounds 1 ounce 8 pennyweights 6 grains. Ans.

Example.—A silversmith melted up 2 lb. 8 oz. 10 pwt. of silver which he made into 6 spoons; what was the weight of each spoon?

Solution.—

	lb.	oz.	pwt.		
6)	2	8	10		
		5 oz.	8 pwt.	8 gr.	Ans.

Explanation.—Since we cannot divide 2 pounds by 6, we reduce it to ounces. 2 pounds $= 24$ ounces, and 24 ounces $+ 8$ ounces $= 32$ ounces; 32 ounces $\div 6 = 5$ ounces and 2 ounces over. 2 ounces $= 40$ pennyweights. 40 pennyweights $+ 10$ pennyweights $= 50$ pennyweights, and 50 pennyweights $\div 6 = 8$ pennyweights and 2 pennyweights over. 2 pennyweights $= 48$ grains, and 48 grains $\div 6 = 8$ grains. Hence, each spoon contains 5 ounces 8 pennyweights 8 grains. Ans.

241. Example.—Divide 820 rd. 4 yd. 2 ft. by 112.

```
                     rd.  yd. ft.  rd. yd. ft.   in.
Solution.—      112)820   4  2 ( 7   1   2  5.143  Ans.
                    784
                    ---
                     36 rd. rem.
                    5.5
                    ---
                    180
                   180
                   ----
                   198.0 yd.
                     4
                   ---
               112)202 yd. ( 1 yd.
                   112
                   ---
                    90 yd. rem.
                     3
                   ---
                   270 ft.
                     2 ft.
                   ---
               112)272 ft. ( 2 ft.
                   224
                   ---
                    48 ft. rem.
                    12
                    --
                    96
                   48
                   ---
               112)576 in. ( 5.1428 + in., or 5.143 in.
                   560
                   ---
                    160
                    112
                    ---
                     480
                     448
                     ---
                      320
                      224
                      ---
                       960
                       896
                       ---
                        64
```

Explanation.—The first quotient is 7 rods with 36 rods remaining. $5.5 \times 36 = 198$ yards; 198 yards $+ 4$ yards $=$ 202 yards; 202 yards $\div 112 = 1$ yard and 90 yards remaining. $90 \times 3 = 270$ feet; 270 feet $+ 2$ feet $= 272$ feet; 272 feet $\div 112 = 2$ feet and 48 feet remaining; $48 \times 12 = 576$ inches; 576 inches $\div 112 = 5.143$ inches, nearly. Ans.

The preceding example is solved by long division, because the numbers are too large to deal with mentally. Instead of expressing the last result as a decimal, it might have been expressed as a common fraction. Thus, $576 \div 112 = 5\frac{16}{112} = 5\frac{1}{7}$ inches. The chief advantage of using a common fraction is that if the quotient be multiplied by the divisor, the result will always be the same as the original dividend.

242. Rule.—*Find how many times the divisor is contained in the first or highest denomination of the dividend. Reduce the remainder (if any) to the next lower denomination, and add to it the number in the given dividend expressing that denomination. Divide this new dividend by the divisor. The quotient will be the next denomination in the quotient required. Continue in this manner until the lowest denomination is reached. The successive quotients will constitute the entire quotient.*

EXAMPLES FOR PRACTICE.

243. Divide

(*a*) 376 mi. 276 rd. by 22; (*b*) 1,137 bu. 3 pk. 4 qt. 1 pt. by 10; (*c*) 84 cwt. 48 lb. 49 oz. by 16; (*d*) 78 sq. yd. 18 sq. ft. 41 sq. in. by 18; (*e*) 148 mi. 64 rd. 24 yd. by 12; (*f*) 100 tons 16 cwt. 18 lb. 11 oz. by 15; (*g*) 36 lb. 18 oz. 18 pwt. 14 gr. by 8; (*h*) 112 mi. 48 rd. by 100.

Ans.
- (*a*) 17 mi. $41\frac{7}{11}$ rd.
- (*b*) 113 bu. 3 pk. 1 qt. $\frac{1}{2}$ pt.
- (*c*) 5 cwt. 28 lb. $3\frac{1}{16}$ oz.
- (*d*) 4 sq. yd. 4 sq. ft. $2\frac{5}{18}$ sq. in.
- (*e*) 12 mi. 112 rd. 2 yd.
- (*f*) 6 tons 14 cwt. 41 lb. $3\frac{14}{15}$ oz.
- (*g*) 4 lb. 8 oz. 7 pwt. $7\frac{3}{4}$ gr.
- (*h*) 1 mi. $38\frac{22}{25}$ rd.

1. If 12 mine cars were dumped to load a R. R. car with coal, what was the average weight of coal in each, if the total weight of coal in the R. R. car was 28 tons 17 cwt. 32 lb. Ans. 2 tons 8 cwt. 11 lb.

2. A turnout 65 yd. 0 ft. 6 in. long will hold exactly 23 mine cars. What is the length of each car? Ans. 2 yd. 2 ft. 6 in.

3. A shaft is 286 ft. 3 in. deep, and it is timbered down to the bed rock, a distance equal to $\frac{2}{7}$ the depth of the shaft. How much of the shaft is timbered? Ans. 81 ft. $9\frac{3}{7}$ in.

4. A boiler shell which is 16 ft. $3\frac{1}{16}$ in. long is made up of 3 sheets.

If the lap at each of the two middle seams is $2\frac{1}{4}$ in., what is the length of each sheet? Ans. 5 ft. $6\frac{25}{48}$ in.

5. In a return tubular boiler the heating surface is divided as follows: Outside shell, 98 sq. ft. 9.8 sq. in.; heads, 5 sq. ft. $4\frac{1}{2}$ sq. in.; tubes, 683 sq. ft. 10.75 sq. in. What is the total area of the heating surface in square feet and square inches? Ans. 786 sq. ft. 25.05 sq. in.

6. In a mine having an endless-rope haulage system 1 mi. 2 rd. 4 yd. 1 ft. long, 5 trips are attached to the main or outgoing band. What is the distance the trips are apart? Ans. 64 rd. 3 yd. $\frac{1}{5}$ ft.

ARITHMETIC.

(PART 5.)

INVOLUTION.

244. If a product consists of equal factors, it is called a **power** of one of those equal factors, and one of the equal factors is called a **root** of the product. The power and the root are named according to the number of equal factors in the product. Thus, 3×3, or 9, is the *second power*, or **square,** of 3; $3 \times 3 \times 3$, or 27, is the *third power*, or **cube,** of 3; $3 \times 3 \times 3 \times 3$, or 81, is the **fourth power** of 3. Also, 3 is the **second root,** or **square root,** of 9; 3 is the **third root,** or **cube root,** of 27; 3 is the **fourth root** of 81.

245. For the sake of brevity,

3×3 is written 3^2, and read **three square,** or *three exponent two;*

$3 \times 3 \times 3$ is written 3^3, and read **three cube,** or *three exponent three;*

$3 \times 3 \times 3 \times 3$ is written 3^4, and read **three fourth,** or *three exponent four;*

and so on.

A number written above and to the right of another number, to show how often the latter number is used as a factor, is called an **exponent.** Thus, in 3^{12}, the number ^{12} is the exponent, and shows that 3 is to be used as a factor twelve times ; so that 3^{12} is a contraction for

$$3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3.$$

In an expression like 3^5, the exponent ^{5} shows how often 3 is used as a factor. Hence, if the exponent of a number is unity, the number is used once as a factor; thus, $3^1 = 3$, $4^1 = 4$, $5^1 = 5$.

246. If the side of a square contains 5 inches, the area of the square contains 5×5, or 5^2, square inches. If the edge of a cube contains 5 inches, the volume of the cube contains $5 \times 5 \times 5$, or 5^3, cubic inches. It is for this reason that 5^2 and 5^3 are called the square and cube of 5, respectively.

247. To find any power of a number:

EXAMPLE 1.—What is the third power, or cube, of 35?

SOLUTION.— $35 \times 35 \times 35$

$$\begin{array}{rr}
\text{or} & 35 \\
 & 35 \\
\hline
 & 175 \\
 & 105 \\
\hline
 & 1225 \\
 & 35 \\
\hline
 & 6125 \\
 & 3675 \\
\hline
\textit{cube} = & 42875 \quad \text{Ans.}
\end{array}$$

EXAMPLE 2.—What is the fourth power of 15?

SOLUTION.— $15 \times 15 \times 15 \times 15$

$$\begin{array}{rr}
\text{or} & 15 \\
 & 15 \\
\hline
 & 75 \\
 & 15 \\
\hline
 & 225 \\
 & 15 \\
\hline
 & 1125 \\
 & 225 \\
\hline
 & 3375 \\
 & 15 \\
\hline
 & 16875 \\
 & 3375 \\
\hline
\textit{fourth power} = & 50625 \quad \text{Ans.}
\end{array}$$

EXAMPLE 3.— $1.2^3 =$ what ?

SOLUTION.— $1.2 \times 1.2 \times 1.2$

$$\begin{array}{rr}
\text{or} & 1.2 \\
 & 1.2 \\
\hline
 & 1.44 \\
 & 1.2 \\
\hline
 & 288 \\
 & 144 \\
\hline
\textit{cube} = & 1.728
\end{array}$$
Ans.

EXAMPLE 4.—What is the third power, or cube, of $\frac{3}{8}$?

SOLUTION.— $\left(\frac{3}{8}\right)^3 = \frac{3^3}{8^3} = \frac{3}{8} \times \frac{3}{8} \times \frac{3}{8} = \frac{3 \times 3 \times 3}{8 \times 8 \times 8} = \frac{27}{512}$. Ans.

248. Rule.—I. *To raise a whole number or a decimal to any power, use it as a factor as many times as there are units in the exponent.*

II. *To raise a fraction to any power, raise both the numerator and denominator to the power indicated by the exponent.*

EXAMPLES FOR PRACTICE.

Raise the following to the powers indicated:

(*a*) 85^2.
(*b*) $\left(\frac{12}{13}\right)^2$.
(*c*) 6.5^2.
(*d*) 14^4.
(*e*) $\left(\frac{3}{4}\right)^3$.
(*f*) $\left(\frac{5}{6}\right)^3$.
(*g*) $\left(\frac{7}{2}\right)^3$.
(*h*) 1.4^5.

Ans.
(*a*) 7,225.
(*b*) $\frac{144}{169}$.
(*c*) 42.25.
(*d*) 38,416.
(*e*) $\frac{27}{64}$.
(*f*) $\frac{125}{216}$.
(*g*) $\frac{343}{8}$.
(*h*) 5.37824.

EVOLUTION.

249. Evolution is the reverse of involution. It is the process of finding the root of a number that is considered as a power.

250. The **square root** of a number is that number which when used twice as a factor produces the number.

Thus, 2 is the square root of 4, since 2×2, or $2^2 = 4$.

251. The **cube root** of a number is that number which when used three times as a factor produces the number.

Thus, 3 is the cube root of 27, since $3 \times 3 \times 3$, or $3^3 = 27$.

252. The **radical sign** $\sqrt{}$ when placed before a number indicates that some root of that number is to be found. The vinculum is almost always used in connection with the radical sign, as shown in Art. **253.**

253. The **index** of the root is a *small figure* placed *over* and to the *left* of the *radical sign*, to show what root is to be found.

Thus, $\sqrt[2]{100}$ denotes the *square root* of 100.
$\sqrt[3]{125}$ denotes the *cube root* of 125.
$\sqrt[4]{256}$ denotes the *fourth root* of 256, and so on.

254. When the square root is to be extracted, the index is generally omitted. Thus, $\sqrt{100}$ indicates the square root of 100. Also, $\sqrt{225}$ indicates the square root of 225.

255. In any number, the figures beginning with the first digit* at the left and ending with the last digit at the right, are called the **significant figures** of the number. Thus, the number 405,800 has the four significant figures 4, 0, 5, 8; and the number .000090067 has the five significant figures 9, 0, 0, 6, and 7.

The part of a number consisting of its significant figures is called the **significant part** of the number. Thus, in the number 28,070, the significant part is 2807; in the number .00812, the significant part is 812; and in the number 170.3, the significant part is 1703.

In speaking of the significant figures or of the significant part of a number, we consider the figures, in their proper order, from the first digit at the left to the last digit at the right, but we pay no attention to the position of the decimal point. Hence, *all numbers that differ only in the position of the decimal point have the same significant part.* For example, .002103, 21.03, 21,030, and 210,300 have the same significant figures 2, 1, 0, and 3 and the same significant part 2103.

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* A cipher is not a digit.

SQUARE ROOT.

256. The *largest* number that can be written with *one* figure is 9, and $9^2 = 81$; the *largest* number that can be written with *two* figures is 99, and $99^2 = 9,801$; with *three* figures 999, and $999^2 = 998,001$; with *four* figures 9,999, and $9,999^2 = 99,980,001$, etc.

In *each* of the above it will be noticed that the square of the number contains just *twice* as many figures as the number.

In order to find the square root of a number, the first step is to find how many figures there will be in the root. This is done by pointing off the number into *periods* of *two* figures each, *beginning at the right.* The number of periods will indicate the number of figures in the root.

Thus, the square root of 83,740,801 must contain 4 figures, since, pointing off the periods, we get 83'74'08'01, or 4 periods; consequently, there must be 4 figures in the root. In like manner, the square root of 50,625 must contain 3 figures, since there are (5'06'25) 3 periods. The extreme left-hand period may contain either one or two figures, according to the size of the number squared.

257. The square of any number wholly decimal always contains twice as many figures as the number squared. For example, $.1^2 = .01$, $.13^2 = .0169$, $.751^2 = .564001$, etc.

258. It will also be noticed that the square of a decimal is always less than the decimal. Hence, the square root of a number wholly decimal is greater than the number itself. If it be required to find the square root of a decimal, and the decimal has not an even number of figures in it, annex a cipher. The best way to point off a decimal is to begin at the decimal point, and, going towards the *right*, point off the decimal into periods of two figures each. Then, if the last period contains but one figure, annex a cipher to complete the period.

259. There are comparatively few numbers that can be separated into exactly equal factors; these numbers are called

perfect powers, and the factors are called *rational factors.* Numbers that cannot be separated into exactly equal factors are called **surds,** and the factors are called *irrational factors.* In the numbers from 1 to 1,000, inclusive, there are only 42 perfect powers, not counting 1; and of these only 30 are perfect squares and 9 perfect cubes.

The root of any number that cannot be divided into as many equal factors as there are units in the index of the root contains an interminable decimal. For example, the number 20 lies between 16 ($= 4^2$) and 25 ($= 5^2$); hence, the square root of 20, or $\sqrt{20}$, is greater than 4 and less than 5, and is therefore equal to 4 plus an interminable decimal. In other words, no matter to how many figures the square root of 20 may be calculated, the root will never be found exactly.

260. Although the root of a surd cannot be found exactly, as close an approximation may be obtained as is desired. In practice, five significant figures are all that are likely to be required, and four are generally sufficient. In the following examples, all roots will be calculated to five figures, unless the given number is a perfect power whose root contains less than five figures.

261. The student will find the following principles of value, both in connection with the extraction of roots and in other arithmetical calculations:

a. In general, if any two numbers are multiplied together —no matter how many significant figures they contain—the first five significant figures of the product will be the same as the first five significant figures of the product obtained by multiplying the same two numbers when limited to five significant figures.

For example, the product of 4,562,357 and 6,421,849 is 29,298,767,738,093; limiting the numbers to five significant figures, the product of 45,624 and 64,218 is 2,929,882,032; and the value of both these products to five significant figures is 29,299. In other words, if only five significant figures are required in the product, it is not

necessary to use more than five significant figures in the multiplier and multiplicand, the remaining figures, if any, being replaced by ciphers, and the fifth figures being increased by 1 if the sixth figure is 5 or a larger digit. In some cases, however, the fifth figure may be one unit too large or one unit too small; hence, if it is necessary that the fifth figure be absolutely exact, it is better to limit the multiplier and multiplicand to six figures instead of five.

For example, $4,562,347 \times 6,421,849 = 29,298,703,519,603$, or 29,299,000,000,000 to five significant figures; $4,562,300 \times 6,421,800 = 29,298,178,140,000 = 29,298,000,000,000$ to five significant figures, the fifth figure being 1 less than it should be; but $4,562,350 \times 6,421,850 = 29,298,727,347,500 = 29,299,000,000,000$ to five significant figures.

b. If the divisor and dividend are limited to six significant figures, the quotient will always be correct to five (usually to six) significant figures, regardless of how many significant figures there may have been in the dividend and divisor.

For example, $6,421,849 \div 4,562,357 = 1.407572+ = 1.4076$ to five significant figures; also, $642,185 \div 456,236 = 1.407571+ = 1.4076$ to five significant figures.

c. If the number whose root is to be extracted be limited to six significant figures, the root will be correct to five (usually to six) significant figures.

262. These principles may all be summed up in the following general statement: *In any series of arithmetical operations —addition, subtraction, multiplication, division, involution, and evolution—if it be desired to have the final result limited to a certain number of significant figures, it is unnecessary to use more significant figures in any of the numbers operated on than the desired number in the result plus 1.* For example, if only four significant figures are desired in the final result, all the numbers used in the various operations may be limited to $4 + 1 = 5$ significant figures, the fifth figure being increased by 1 in all cases if the sixth figure is 5 or a greater digit.

From the foregoing, it follows that any method that will give five significant figures of the root correctly will be

sufficiently exact for all practical purposes. Such a method will now be explained for extracting square root.

263. Suppose it is desired to find the square root of 20; that is, $\sqrt{20} = ?$ The problem is to divide 20 into two equal factors, or into two factors, the first five significant figures of which shall be equal. Since 20 is not a perfect square, inspection shows that one of the equal factors is 4 plus an interminable decimal, since 20 lies between $4^2 = 16$ and $5^2 = 25$. Dividing 20 by 4, the result is 5; i. e., $4 \times 5 = 20$. Now, by taking the average of these unequal factors, a new factor will be obtained, which will be nearer the correct value of the root than either of the two unequal factors, viz., $\frac{4+5}{2} = 4.5$, the square of which is $4.5^2 = 20.25$.

Assuming 4.5 for a new factor and dividing 20 by it, the result is $20 \div 4.5 = 4.444+$; that is, $4.444 \times 4.5 = 20$, nearly, the product not being exactly equal to 20 because 4.444 was used as one factor, instead of $4\frac{4}{9}$, the exact value. Again, taking the average of the two factors, $\frac{4.444 + 4.5}{2} = 4.472$, which is the root correct to at least three figures.

Assuming 4.47 to be one of the factors and dividing 20 by it to obtain the other, the result is $20 \div 4.47 = 4.474272+$; that is, $4.47 \times 4.474273 = 20$, very nearly. The average of these two factors is $\frac{4.47 + 4.474272}{2} = 4.472136+ = 4.4721$ to five significant figures. The exact root to 13 figures is 4.472135954999+.

That 4.4721 is the square root of 20 correct to five figures may easily be proved by squaring it; thus, $4.4721^2 = 19.99967841$, or 20.000 to five figures. Since the square agrees with the given number to five figures, the root is correct to five figures.

264. A close examination of the foregoing results reveals some remarkable facts. (1) The value of the first average 4.5 is correct to two figures of the root. (2) The value of the second average 4.472+ is correct to four figures of the root. (3) The value of the third average 4.472136 is correct

to seven figures of the root. (4) All these averages are somewhat greater than the correct value of the root. (5) Of the two factors used in finding the average, one is a little greater and the other a little less than the correct value of the root. (6) Each step of the process gives a result approaching more and more nearly to the correct value of the root.

265. Calling the first average value the **first approximation,** the second average value the **second approximation,** and the third average value the **third approximation,** the following general method of procedure may be adopted: *Calculate the first approximation to two significant figures; the second approximation to three significant figures; and the third approximation to five significant figures. It is not safe to calculate the second approximation to more than three significant figures, because the fourth figure cannot, as a rule, be depended on. If the second significant figure of the first approximation be determined correctly, the third approximation will always be correct to at least five significant figures.*

266. The method will now be applied to numbers in general, and the best manner of explaining it is by means of examples.

EXAMPLE.— $\sqrt{714,627} = ?$

SOLUTION.—The first step is to point off the number into periods of two figures each, obtaining 71'46'27. To find the first approximation, only the first two significant figures are necessary; in this case, the first period, 71. The first figure of the root is evidently 8, since $8^2 = 64$ and $9^2 = 81$. The two factors then are 8 and $71 \div 8 = 8.87+$. The first approximation is $\frac{8 + 8.87}{2} = 8.43+ = 8.4$ to two figures.

To find the second approximation, use the first two periods, or 7146, and drop the decimal point in the first approximation. One factor is then 84 and the other $7146 \div 84 = 85.07+$. The second approximation is therefore $\frac{84 + 85.07}{2} = 84.53+$, or 84.5 to three figures.

To find the third approximation, use the first three periods, or 714627, and drop the decimal point in the second approximation. One factor is then 845 and the other is $714627 \div 845 = 845.712+$. The third approximation is therefore $\frac{845 + 845.712}{2} = 845.356$, or 845.36 to five significant figures. Ans.

REMARK.—It will be noticed in the last example, and also in those that follow, that when finding the unknown factor to be used in determining the value of the first, second, or third approximation, the division is carried one place farther than the number of figures desired in the approximation and that no attention is paid to the succeeding figures. Thus, in the last example, $71 \div 8 = 8.875$, or 8.88, correct to three figures, while the number used was 8.87. The reason for this is that the value obtained for the approximation would be the same in either case, and it saves time to calculate as here shown. For instance, using 8.88 for the second factor, the first approximation is $\frac{8 + 8.88}{2} = 8.44$, or 8.4 to two figures.

267. The decimal point is located by employing the following principle: *There must be as many figures in the integral* part of the root as there are periods in the integral part of the given number whose root is to be found. If the given number is wholly decimal and there are two or more ciphers between the decimal point and the first significant figure, there will be as many ciphers between the decimal point and the first significant figure of the root as there are entirely cipher periods between the decimal point and the first significant figure of the given number.* Had the number in the last example been 71.4627, the root would have been 8.4536; had it been .714627, the root would have been .84536; had it been .0000714627, the root would have been .0084536. In the latter case, the number would have been pointed off thus, .00'00'71'46'27.

268. In all cases, numbers having the same significant parts and the same number of significant figures in the first (or left-hand) period of the significant part of the number, have the same significant figures in the root, the roots differing only in the position of the decimal point.

EXAMPLE.— $\sqrt{714.627} = ?$

SOLUTION.—Pointing off into periods, we have 7'14.62'70, adding a cipher to complete the last period. In all cases when pointing off the decimal part of numbers, begin at the decimal point and point off to the right, and add ciphers to the last period when it does not contain enough figures to make up a period. Since the first period contains but one figure and it is necessary to have two figures at least in order that the first approximation may be correct to two figures, regard the

* The *integral* part of a number is the part to the left of the decimal point. Thus, the integral part of 1,726.943 is 1,726.

decimal point as situated between 7 and 1 instead of between 4 and 6, thus obtaining 7.1 for the first two figures of the given number. The first two figures of the square root of 7.1 will be the same as the first two figures of the square root of 714.

It is evident that the first figure of the root is 2, since $2^2 = 4$ and $3^2 = 9$. Using 2 as one factor, the other is $7.1 \div 2 = 3.55$, and the first approximation is $\frac{2 + 3.55}{2} = 2.77+$, or 2.8 to two figures. Had 3 been used as one factor, the other would have been $7.1 \div 3 = 2.36+$, and the first approximation would have been $\frac{3 + 2.36}{2} = 2.68$, or 2.7 to two figures. In the first case, the difference between the two factors is $3.55 - 2 = 1.55$; in the second case, the difference is $3 - 2.36 = .64$. As the factors are more nearly equal in the second case than in the first, it is evident that 2.7 is more nearly equal to the correct value of the root than 2.8 is; hence, 2.7 will be used for the first approximation.

For the second approximation, use the first two periods and 27 for one factor, the other factor being $715 \div 27 = 26.48+$; hence, the second approximation $= \frac{27 + 26.48}{2} = 26.74$, or 26.7 to three figures. We used 715 for the first three figures of the given number, instead of 714, because the fourth figure was 6 and the number correct to three figures was 715. In finding the third approximation, the first three periods may be used or all the figures; the result will be the same in either case. Since it is better to use six figures than five, move the decimal point two places to the right, obtaining 71462.7; one factor is 267 and the other $71462.7 \div 267 = 267.650+$. The third approximation is $\frac{267 + 267.650}{2} = 267.325$, or 267.33 to five figures. Since there are two periods in the integral part of the given number, there are two figures in the integral part of the root, and $\sqrt{714.627} = 26.733$. Ans.

269. When determining the first approximation, that number should always be used for the first factor which will make the less difference between it and the second factor, as was done in the last example. Thus, for 2.5, the factors would be 2 and 1.25, the difference between them being .75. If 1 were selected for the first factor, the other would be $2.5 \div 1 = 2.5$, and the difference between them 1.5. In one case, the first approximation would be $\frac{2 + 1.25}{2} = 1.6+$, and in the other case, $\frac{1 + 2.5}{2} = 1.8-$. Since $1.6^2 = 2.56$ and

$1.8^2 = 3.24$, it is evident 1.6 is very much nearer the correct value of the root than 1.8.

270. If the given number is a perfect square and contains not more than ten significant figures, the exact root will be obtained in all cases. That the number is a perfect square may be suspected by the fact that there are one or more 9's or 0's following the fifth figure of the number expressing the third approximation, and that when the third approximation is expressed correct to five figures, the square of its last digit (or the second figure of this square when the square contains more than one figure) will be the same as the last digit of the given number. This will be illustrated by two examples.

EXAMPLE.— $\sqrt{3,749,602,756} = ?$

SOLUTION.—Pointing off, we obtain 37'49'60'27'56. The first two factors are evidently 6 and $37 \div 6 = 6.16+$, and the first approximation $\frac{6 + 6.16}{2} = 6.08$, or 6.1 to two figures.

$3749 \div 61 = 61.45+$; $\frac{61 + 61.45}{2} = 61.23-$, or 61.2 to three figures.

$$374960 \div 612 = 612.67973+; \quad \frac{612 + 612.67973}{2} = 612.33986+,$$

or 612.34 to five figures. But $4^2 = 16$, and as the last figure of the given number is also 6, and as the sixth and seventh figures are 9 and 8, respectively, we suspect that the given number is a perfect power. It may not be, however, for the reason that the figures 5, 7, and 2, preceding 6, may be different from the ones given without changing the value of the root to five figures. Hence, the only way to ascertain the fact beyond possibility of doubt is to square the root; doing so, it is found that $61,234^2 = 3,749,602,756$, which is therefore a perfect square.

Had all the figures of the given number been used in finding the third approximation, the result would have been as follows: $3,749,602,756 \div 612 = 612.6801+$, and $\frac{612 + 612.6801}{2} = 612.34005+$, or 612.34 to five figures, as before, or 61,234 after locating the decimal point. Ans.

271. If the given number contains not more than three periods of significant figures—that is, if it contains not more than five or six significant figures—and is a perfect power, the fact will be revealed when finding the second factor in

the third approximation, for the two factors will then be exactly equal.

EXAMPLE.— $\sqrt{.00095481} = ?$

SOLUTION.— $.00095481 = .00'09'54'81$ when pointed off into periods of two figures each. The first two significant figures are 9.5. The first factor is evidently 3 and the second factor $9.5 \div 3 = 3.16+$. The first approximation is $\frac{3 + 3.16}{2} = 3.08$, or 3.1 to two figures.

$954 \div 31 = 30.77+$; the second approximation is $\frac{31 + 30.77}{2} = 30.88+$ $= 30.9$ to three figures.

$95481 \div 309 = 309$; hence, .00095481 is a perfect power and the significant figures of the root are 309. There being one full cipher period following the decimal point, the root is .0309. Ans.

272. One more example will be given to show the student how to arrange his work when solving examples in square root by this method.

EXAMPLE.— $\sqrt{3,265.47} = ?$

SOLUTION.— $3,265.47 = 32'65.47.$

$33 \div 5 = 6.6$; $33 \div 6 = 5.5$; $6.6 - 5 = 1.6$; $6 - 5.5 = .5$; hence, use 6 for first factor.

$$\frac{6 + 5.5}{2} = 5.75, \text{ or } 5.8.$$

$$3,265 \div 58 = 56.29; \frac{58 + 56.29}{2} = 57.14, \text{ or } 57.1.$$

$326,547 \div 571 = 571.886$; $\frac{571 + 571.886}{2} = 571.443$, or 571.44 to five figures. Therefore, $\sqrt{3,265.47} = 57.144$. Ans.

EXAMPLES FOR PRACTICE

Find the square root of:

(*a*) 186,624.		(*a*) 432.	
(*b*) 2,050,624.		(*b*) 1,432.	
(*c*) 29,855,296.		(*c*) 5,464.	
(*d*) .0116964.		(*d*) .10815—.	
(*e*) 198.1369.		(*e*) 14.076+.	
(*f*) 994,009.	Ans.	(*f*) 997.	
(*g*) 2.375.		(*g*) 1.5411+.	
(*h*) 1.625.		(*h*) 1.2748—.	
(*i*) .3025.		(*i*) .55.	
(*j*) .571428.		(*j*) .75593—.	
(*k*) .78125.		(*k*) .88388+.	

CUBE ROOT.

273. Cube root may be extracted in a manner similar to that just described for square root, the only essential differences being that the given number must be pointed off into periods of *three* figures each; the first period, if integral, may contain one, two, or three figures; and the number must be divided into three equal factors.

274. As might be expected, cube root is a longer operation than square root, but the method is similar and is no more difficult to remember or apply. As in the case of square root, it is unnecessary to use more than six significant figures in order to obtain five significant figures of the root. The method is best illustrated by an example. The student is advised to make a little table, containing the cubes of numbers from 1 to 9, similar to that here given.

$1^3 =$	1
$2^3 =$	8
$3^3 =$	27
$4^3 =$	64
$5^3 =$	125
$6^3 =$	216
$7^3 =$	343
$8^3 =$	512
$9^3 =$	729

EXAMPLE.— $\sqrt[3]{389{,}247} = ?$

SOLUTION.— 389,247 = 389'247 when pointed off into periods of three figures each. As in the case of square root, consider the first period only when finding the first approximation. In other words, divide 389 into three factors as nearly equal as possible. It is readily seen that 389 lies between $7^3 = 343$ and $8^3 = 512$; hence, the first figure of the root is 7. Now, assume that two of the equal factors are each equal to 7 and divide 389 by their product to obtain the third factor; that is, divide 389 by $7^2 = 49$. The result is $389 \div 49 = 7.93+$. Hence, $7 \times 7 \times 7.93+ = 389$, nearly. The average of these factors is $\frac{7 + 7 + 7.93}{3} = \frac{2 \times 7 + 7.93}{3} = 7.31$, or 7.3 to two figures, the first approximation.

Assuming 73 to be the value of two of the three equal factors, divide the first two periods of the given number by their product 73×73: that is, by 73^2, or 5,329. The result is $389{,}247 \div 5{,}329 = 73.04+$; that is, $7.3 \times 7.3 \times 73.04 = 389{,}247$, nearly. The average of the three factors is $\frac{2 \times 73 + 73.04}{3} = 73.01$, or 73 0 to three figures, the second approximation.

Assuming 730 to be two of the three equal factors, divide 389,247 by 730^2, or by 73^2, since the cipher at the right is not a significant figure and will not affect the result, obtaining for the third factor 73.0431.

The average of these three factors is $\frac{2 \times 73 + 73.0431}{3} = 73.0143$, or 73.014 to five figures. Ans.

NOTE.—The decimal point is located by applying the principle of Art. **267;** viz., there must be as many figures in the integral part of the root as there are periods in the integral part of the given number.

275. An inspection of the foregoing example shows that about the only respect in which the work of extracting cube root exceeds the work of extracting square root consists in squaring one number of two figures and one number of three figures. The work of division in finding the third factor is a little harder on account of the divisors being a little larger than when finding the second factor in square root.

276. If the given number contains an integral part, it is better to locate the decimal point as soon as possible, in order to prevent confusion, instead of waiting until the third approximation has been found.

EXAMPLE.— $\sqrt[3]{3.274} = ?$

SOLUTION.—The first period contains but one figure; therefore, we operate on three figures in order to have the first approximation correct to two figures (see **c,** Art. **261**). If 1 be chosen for one of the two equal factors, the third factor will be $3.27 \div 1^2 = 3.27$, and the difference between one of the equal factors and the third factor will be $3.27 - 1 = 2.27$. If 2 be chosen for one of the equal factors, the third one will be $3.27 \div 2^2 = .817+$, and the difference between this and one of the equal factors is $2 - .817 = 1.183$. Since 1.18 is less than 2.27, use 2 for one of the two equal factors. The first approximation is $\frac{2 \times 2 + .817}{3} = 1.60+$, or 1.6 to two figures. Since there is but one period in the integral part of the given number, the root is equal to 1 plus an interminable decimal, as the given number is not a perfect cube. Therefore, retain the decimal point in its present position through all the subsequent operations.

Assuming 1.6 to be one of the two equal factors, the third factor is $3.274 \div 1.6^2 = 3.274 \div 2.56 = 1.278+$, and the second approximation is $\frac{2 \times 1.6 + 1.278}{3} = 1.492+$, or 1.49 to three figures.

Assuming 1.49 to be one of the two equal factors, the other factor is $3.274 \div 1.49^2 = 3.274 \div 2.2201 = 1.474708+$; hence, the third approximation is $\frac{2 \times 1.49 + 1.474708}{3} = 1.484902+$, or 1.4849 to five figures. Ans.

The exact root to seven figures is $1.484886-$.

277. The only case in cube root that will give any trouble in determining the fifth significant figure correctly is when the difference between the numbers representing the first and second approximations, expressed to two figures, is greater than one unit in the second figure. In the last example, the first approximation was 1.6 and the second 1.49, or 1.5 to two figures; the difference is .1, or one unit in the second figure. For numbers the significant part of whose first period is 2, the difference between the first and second approximations may differ by more than one unit in the second figure; in such cases, recalculate the second approximation, using for one of the equal factors the value of the second approximation to two figures as first determined. An example will illustrate this.

EXAMPLE.— $\sqrt[3]{.0027} = ?$

SOLUTION.— $.0027 = .002'700$ when pointed off into periods. But, the significant figures in the cube root of 2.7 will be the same as in the cube root of $.002'700$; therefore, find the cube root of 2.7 and locate the decimal point after the operation is finished.

If 1 be chosen as one of the equal factors, the third factor will be $2.7 \div 1^2 = 2.7$, and the first approximation is $\frac{2 \times 1 + 2.7}{3} = 1.56+$, or 1.6 to two figures. If 2 be chosen for one of the equal factors, the third factor is $2.7 \div 2^2 = .675$, and the first approximation is $\frac{2 \times 2 + .675}{3} = 1.55+$, or 1.6 to two figures.

Using 1.6 for one of the equal factors, the third factor is $2.7 \div 1.6^2 = 2.7 \div 2.56 = 1.054+$, and the second approximation is $\frac{2 \times 1.6 + 1.054}{3} = 1.418$, or 1.42 to three figures, or 1.4 to two figures. The difference between the first and second approximations is $1.6 - 1.4 = .2$, or two units in the second figure. Therefore, recalculate the second approximation, using 1.4 for one of the equal factors. The third factor is then equal to $2.7 \div 1.4^2 = 2.7 \div 1.96 = 1.377$, and the second approximation is $\frac{2 \times 1.4 + 1.377}{3} = 1.392+$.

Using 1.39 for one of the equal factors, the third factor is $2.7 \div 1.39^2 = 2.7 \div 1.9321 = 1.39744+$, and the third approximation is $\frac{2 \times 1.39 + 1.39744}{3} = 1.39248$, or 1.3925 to five figures. The root correct to nine figures is 1.39247665. Since the given number is wholly decimal and has no period composed entirely of ciphers, $\sqrt[3]{.0027} = .13925$. Ans.

Had 1.42 been used for one of the equal factors, the third approximation would have been 1.3930.

278. The remarks made in Art. **270** regarding the square root of perfect squares apply, with slight modifications, to the cube root of perfect cubes. If the given number is a perfect cube and contains not more than five periods, i. e., not more than $5 \times 3 = 15$ significant figures, the exact root can always be found. That the given number is a perfect cube will be suspected from the fact that the root ends in a string of 9's or 0's; that in one of the approximations the three factors become exactly equal; and that the last digit in the cube of the last figure of the root is the same as the last digit of the given number. An example will illustrate this.

EXAMPLE.— $\sqrt[3]{106,294,343.553} = ?$

SOLUTION.—The number when pointed off becomes 106'294'343.553; hence, there are three figures in the integral part of the root. The first period 106 lies between $4^3 = 64$ and $5^3 = 125$. Trying 4 for one of the equal factors, the third factor is $106 \div 4^2 = 6.62+$. Trying 5, the third factor is $106 \div 5^2 = 4.24$; hence, use 5, and obtain for the first approximation $\frac{2 \times 5 + 4.24}{3} = 4.74+$. Using two periods and 47 for one of the equal factors, the third factor is $106,294 \div 47^2 = 106,294 \div 2,209 = 48.11+$, and the second approximation is $\frac{2 \times 47 + 48.11}{3} = 47.37$.

To find the third approximation, two, or three, or all four periods may be used, since the first two periods contain six significant figures, and hence will give the root correct to five figures (see **c**, Art. **261**). Using the first three periods, to avoid the decimal point, and 474 for one of the equal factors, the third factor is $106,294,343 \div 474^2 = 106,294,343 \div 224,676 = 473.1005+$, and the third approximation is $\frac{2 \times 474 + 473.1005}{3} = 473.7001+$, or 473.70 to five figures. It will be noticed that the results obtained for the second and third approximations are alike and the last digit in $7^3 = 343$ is the same as the last significant figure of the given number; hence, it is at once suspected that the given number is a perfect power, and this is proved by cubing the root. Therefore, $\sqrt[3]{106,294,343.553} = 473.7$. Ans.

279. Square and cube root are two of the most important operations described in Arithmetic, and the student is earnestly advised to thoroughly familiarize himself with the process. Few practical problems involving mensuration arise that do not require the extraction of the square

or cube root. For instance, to find the diameter of a circle that will contain a given area requires the extraction of square root; to find the diameter of a sphere that will contain a given volume requires the extraction of cube root.

EXAMPLES FOR PRACTICE.

Find the cube root of:

		Ans.	
(*a*)	78,347.809639.	(*a*)	42.79.
(*b*)	2.	(*b*)	1.2599+.
(*c*)	4,180,769,192.462.	(*c*)	1,611.0−.
(*d*)	.696.	(*d*)	.88621−.
(*e*)	.375.	(*e*)	.72112+.
(*f*)	513,229.783302144.	(*f*)	80.064.

TABLE METHOD OF EXTRACTING SQUARE AND CUBE ROOT.

280. By means of the table of Squares, Cubes, Fourth, and Fifth Powers, which contains the squares and cubes of numbers from 1 to 10, varying by tenths, and the first five figures of the fourth and fifth powers of the same numbers, the first three, and frequently the first four, significant figures of the square root or cube root of any number can be readily determined. The remaining figures can then be easily determined in the same manner as the third approximation in the preceding pages.

The student is advised to use the table in all cases, as it will greatly shorten his work.

281. By the aid of this table the first two significant figures of the root can be obtained directly and one more by a slight calculation. For example, suppose it is desired to find the first three significant figures of $\sqrt{5,269.73}$. Pointing off into periods and moving the decimal point so that it falls between the first and second periods, the number becomes 52.69′73; in other words, the significant figures of $\sqrt{5,269.73}$ are the same as for $\sqrt{52.6973}$. Since four figures only are given in the table, reduce the given number to four figures. The problem then becomes: find the first three figures of $\sqrt{52.70}$. Referring to the table, 52.70 lies between

SQUARES, CUBES, FOURTH, AND FIFTH POWERS.

No.	Square.	Cube.	4th Power.	5th Power.	No.	Square.	Cube.	4th Power.	5th Power.
1.0	1.00	1.000	1 0000	1.0000	5.5	30.25	166.375	915.06	5 032.8
1.1	1.21	1.331	1.4641	1.6105	5.6	31.36	175 616	983.45	5 507.3
1.2	1.44	1.728	2.0736	2.4883	5.7	32.49	185.193	1,055.6	6,016.9
1.3	1.69	2.197	2.8561	3.7129	5.8	33.64	195.112	1,131.6	6,563 6
1.4	1.96	2.744	3.8416	5.3782	5.9	34.81	205.379	1,211.7	7,149.2
1.5	2.25	3.375	5.0625	7.5938	6.0	36.00	216.000	1,296.0	7,776.0
1.6	2.56	4.096	6.5536	10.486	6.1	37.21	226.981	1,384.6	8,446.0
1.7	2.89	4.913	8.3521	14.199	6 2	38.44	238.328	1,477.6	9,161.3
1.8	3.24	5.832	10.498	18.896	6 3	39.69	250.047	1,575.3	9,924.4
1.9	3.61	6.859	13.032	24.761	6.4	40.96	262.144	1,677.7	10,737
2.0	4.00	8.000	16.000	32.000	6.5	42 25	274.625	1,785.1	11,603
2.1	4.41	9.261	19.448	40.841	6.6	43.56	287.496	1,897.5	12,523
2.2	4.84	10.648	23.426	51.536	6 7	44.89	300.763	2,015.1	13,501
2.3	5.29	12.167	27.984	64.363	6.8	46.24	314.432	2,138.1	14,539
2.4	5.76	13.824	33.178	79.626	6.9	47.61	328 509	2,266.7	15,640
2.5	6.25	15 625	39.063	97.656	7.0	49 00	343.000	2,401.0	16,807
2.6	6 76	17.576	45.698	118.81	7.1	50.41	357.911	2,541.2	18,042
2.7	7.29	19.683	53.144	143.49	7.2	51.84	373.248	2,687.4	19,349
2.8	7.84	21.952	61.466	172.10	7.3	53.29	389.017	2,839.8	20,731
2.9	8.41	24.389	70.728	205.11	7.4	54.76	405.224	2,998.7	22,190
3.0	9.00	27.000	81.000	243.00	7.5	56.25	421.875	3,164.1	23.730
3.1	9 61	29.791	92.352	286.29	7.6	57.76	438.976	3,336.2	25,355
3.2	10.24	32.768	104.86	335.54	7.7	59.29	456.533	3,515.3	27,068
3.3	10.89	35.937	118.59	391.35	7.8	60.84	474.552	3,701.5	28,872
3.4	11.56	39.304	133.63	454.35	7.9	62.41	493.039	3,895.0	30,771
3.5	12.25	42.875	150.06	525.22	8.0	64.00	512.000	4,096.0	32 768
3.6	12.96	46.656	167.93	604.66	8.1	65.61	531.441	4,304.7	34,868
3.7	13.69	50.653	187.42	693.44	8.2	67.24	551.368	4,521.2	37,074
3.8	14.44	54.872	208 51	792.35	8.3	68.89	571.787	4,745.8	39,390
3.9	15.21	59.319	231.34	902.24	8.4	70.56	592 704	4,978.7	41,821
4.0	16.00	64.000	256.00	1,024.0	8.5	72.25	614.125	5,220.1	44,371
4.1	16.81	68.921	282.58	1,158.6	8.6	73.96	636.056	5,470.1	47,043
4.2	17.64	74.088	311.17	1 306.9	8.7	75.69	658.503	5,729.0	49,842
4.3	18.49	79.507	341.88	1,470.1	8.8	77 44	681.472	5,997.0	52,773
4.4	19.36	85.184	374 81	1,649 2	8.9	79.21	704.969	6,274.2	55,841
4.5	20.25	91.125	410.06	1,845.3	9.0	81.00	729.000	6,561.0	59,049
4.6	21.16	97.336	447.75	2,059 6	9.1	82.81	753.571	6,857.5	62,403
4.7	22.09	103.823	487 97	2,293.5	9.2	84.64	778.688	7,163.9	65,908
4.8	23.04	110.592	530.84	2 548.0	9.3	86.49	804.357	7,480.5	69,569
4.9	24.01	117.649	576.48	2,824.8	9.4	88.36	830.584	7,807.5	73.390
5.0	25 00	125.000	625 00	3,125.0	9 5	90.25	857.375	8,145.1	77,378
5.1	26.01	132.651	676.52	3,450.3	9.6	92.16	884.736	8,493.5	81,537
5 2	27.04	140.608	731.16	3,802.0	9.7	94.09	912.673	8,852.9	85,873
5.3	28 09	148.877	789.05	4,182.0	9.8	96.04	941.192	9,223.7	90,392
5.4	29.16	157.464	850.31	4,591.7	9.9	98.01	970.299	9,606.0	95.099

$51.84 = 7.2^2$ and $53.29 = 7.3^2$; hence, the first two figures of the root are 7.2. Find the difference between the two numbers in the table between which the given number falls and call it the **first difference**; thus, $53.29 - 51.84 = 1.45 =$ the first difference. Find the difference between the lower number in the table and the given number and call it the **second difference**; thus, $52.70 - 51.84 = .86 =$ the second difference. Divide the second difference by the first difference, and the first figure of the quotient, if the quotient is .05 or greater, will be the third figure of the root, when reduced to one figure. If the quotient is less than .05, the third figure of the root is a cipher. Thus, $.86 \div 1.45 = .59+$, or .6 when reduced to one figure. Therefore, the first three figures of $\sqrt{52.70}$ are 7.26. Since the integral part of the given number contains two periods, there are two figures in the integral part of the root; therefore, $\sqrt{5,269.73} = 72.6$ to three figures. Ans.

282. The cube root is found to three significant figures in exactly the same way, as shown in the following example:

EXAMPLE.—Find the first three figures of $\sqrt[3]{.0625}$.

SOLUTION.—Pointing off and placing the decimal point between the first and second significant periods, the result is 62.500. Referring to the table, the first two figures of the root are 3.9; the first difference is $64.000 - 59.319 = 4.681$; the second difference is $62.500 - 59.319 = 3.181$; $3.181 \div 4.681 = .67+$, or .7 to one figure. Therefore, $\sqrt[3]{62.5} = 3.97$, and $\sqrt[3]{.0625} = .397$ to three significant figures. Ans.

283. Having found the first three significant figures by means of the table, find the fourth and fifth figures in the usual manner by using the first three figures in finding the third approximation.

For example, find the cube root of 126.57 to five figures. Referring to the table, the first two figures are 5.0. The first difference is $132.651 - 125.000 = 7.651$; the second difference is $126.57 - 125.000 = 1.57$; $1.57 \div 7.651 = .20+$. Hence, the first three figures are 5.02. Using 5.02 for one of the equal factors, the third factor is $126.57 \div 5.02^2$

$= 5.02253+$, and the third approximation is $\frac{2 \times 5.02 + 5.02253}{3}$ $= 5.02084+$, or $\sqrt[3]{126.57} = 5.0208$ to five figures. Ans.

284. If more than five significant figures of the square or cube root are desired, use the five figures of the third approximation for one of the equal factors and calculate the unknown factor to as many figures as are desired plus one; the next approximation will be correct to at least nine figures, if the unknown factor has been calculated to ten figures.

ROOTS OF FRACTIONS.

285. If the given number is in the form of a fraction, and it is required to find some root of it, the simplest and most exact method is to reduce the fraction to a decimal and extract the required root of the decimal. If, however, the numerator and denominator of the fraction are perfect powers, extract the required root of each separately, and write the root of the numerator for a new numerator and the root of the denominator for a new denominator.

EXAMPLE 1.—What is the square root of $\frac{9}{64}$?

SOLUTION.— $\sqrt{\frac{9}{64}} = \frac{\sqrt{9}}{\sqrt{64}} = \frac{3}{8}$. Ans.

EXAMPLE 2.—What is the square root of $\frac{5}{8}$?

SOLUTION.— $\sqrt{\frac{5}{8}} = \sqrt{.625} = .79057-$, since $\frac{5}{8} = .625$. Ans.

EXAMPLE 3.—What is the cube root of $\frac{27}{64}$?

SOLUTION.— $\sqrt[3]{\frac{27}{64}} = \frac{\sqrt[3]{27}}{\sqrt[3]{64}} = \frac{3}{4}$. Ans.

EXAMPLE 4.—What is the cube root of $\frac{1}{4}$?

SOLUTION.—Since $\frac{1}{4} = .25$, $\sqrt[3]{\frac{1}{4}} = \sqrt[3]{.25} = .62996+$. Ans.

286. Rule.—*Extract the required root of the numerator and denominator separately; or, reduce the fraction to a decimal and extract the root of the decimal.*

EXAMPLES FOR PRACTICE.

(*a*) $\sqrt{\frac{9}{16}} = ?$

(*b*) $\sqrt[3]{\frac{27}{1728}} = ?$

(*c*) $\sqrt[3]{\frac{27}{375}} = ?$

(*d*) $\sqrt[3]{\frac{560}{128}} = ?$

Ans. $\begin{cases} (a) & \frac{3}{4}. \\ (b) & \frac{1}{4}. \\ (c) & .41602-. \\ (d) & 1.6355+. \end{cases}$

FOURTH ROOT.

287. The fourth root may be found by a method similar to that just described for extracting square and cube roots, dividing the given number into periods of four figures each and resolving the given number into four equal factors. It is generally easier and shorter, however, to extract the square root and then extract the square root of the result. For example, to extract the fourth root of 5,735,796,283.8016, which is a perfect fourth power (and consequently a perfect square, also), the square root would be extracted in the usual manner, obtaining 75,735.04. The square root of this result would then be extracted, obtaining 275.2. In other words $\sqrt[4]{5,735,796,283.8016} = \sqrt{\sqrt{5,735,796,283.8016}} = \sqrt{75,735.04} = 275.2$.

The fourth root is very seldom required and can always be found as just described.

FIFTH ROOT.

288. The fifth root is required oftener than the fourth root, but nevertheless it is seldom necessary to extract it. The method is the same in principle as that explained for cube root. The given number is divided into periods of five figures each and resolved into five equal factors; the first period may contain one, two, three, four, or five figures. As in the case of cube root, it is advisable to construct a little table giving the fifth powers of the nine digits, similar to that here given. An example will illustrate the process.

$1^5 =$	1
$2^5 =$	32
$3^5 =$	243
$4^5 =$	1,024
$5^5 =$	3,125
$6^5 =$	7,776
$7^5 =$	16,807
$8^5 =$	32,768
$9^5 =$	59,049

EXAMPLE.— $\sqrt[5]{5,186.42} = ?$

SOLUTION.—The first period 5186 lies between $5^5 = 3125$ and $6^5 = 7776$; hence, the root is 5 plus an interminable decimal. Trying 5 as one of the four equal factors and dividing the first period by their product to find the fifth factor, we have $\frac{5186}{5 \times 5 \times 5 \times 5} = 5186 \div 5^4 = 5186 \div 625 = 8.29+$. Trying 6 as one of the four equal factors, the fifth factor is $5186 \div 6^4 = 5186 \div 1296 = 4.00+$. Since the difference between 6 and 4 is less than the difference between 5 and 8.29, use 6 as one of the equal factors. Then, $5186 = 6 \times 6 \times 6 \times 6 \times 4$. The average of these factors is $\frac{6+6+6+6+4}{5} = \frac{4 \times 6 + 4}{5} = \frac{28}{5} = 5.6$, the first approximation.

Using 5.6 as one of the equal factors, the fifth factor is $5186.42 \div 5.6^4 = 5186 \div 983.4496 = 5186 \div 983.4000$ (using but four significant figures, since only three figures of the second approximation are required—see **b**, Art. **261**) $= 5.273$, and the second approximation is $\frac{4 \times 5.6 + 5.273}{5} = 5.534+$, or 5.53 to three figures.

Using 5.53 as one of the equal factors, the fifth factor is $5186.42 \div 5.53^4 = 5186.42 \div 935.191 = 5.54584+$, and the third approximation is $5.53316+$, or 5.5332 to five figures. Ans.

The exact root to seven figures is 5.533164.

289. In order that the fifth significant figure of the fifth root of a number may be correct, it is absolutely essential that the third significant figure of the second approximation be correct. In the last example, it will be noticed that the difference between the first and second approximations is $5.60 - 5.53 = .07$, which is less than one unit in the second figure, but very near to it. Had this difference been as much as 1, or had it exceeded one unit in the second figure, it would have been advisable to recalculate the second approximation.

290. The labor involved in extracting the fifth root is very much greater than that necessary to extract the cube root, chiefly on account of raising numbers to the fourth power. This labor may be shortened considerably in the following manner:

Consider any number, as 4. Now, $4^4 = 4 \times 4 \times 4 \times 4 = (4 \times 4) \times (4 \times 4) = 16 \times 16 = 256$. In other words, to

raise a number to the fourth power, square the number and then square the square. Now, the square of any number contains twice as many significant figures as the number or twice as many less 1; the cube of any number contains three times as many significant figures as the number or three times as many less one or two; the fourth power will contain four times as many or four times as many less one, two, or three; and so on. Hence, the fourth power of a number containing two figures will contain five, six, seven, or eight figures; and of one containing three figures, nine, ten, eleven, or twelve figures. In determining the fifth factor for the second approximation, only four figures of the fourth power are required, and in determining the fifth factor for the third approximation, only six figures of the fourth power are required. Therefore, any method that will enable us to dispense with unnecessary figures will lessen the work. The following method, which will also apply to any case of multiplication when only a certain definite number of figures are desired in the product, is the best we know of; it is best illustrated by an example.

EXAMPLE.—Multiply 467,295 by 634,137 and obtain six figures of the product correct.

SOLUTION.—

(a)		(b)
467295		4′6′7′2′9′5
634137		634137
2803770		2803770
140188	5	140188
18691	80	18691
467	295	467
140	1885	140
32	71065	32
2963290	49415	2963288, or 296329 to six figures.

EXPLANATION.—Instead of beginning with the last, or right-hand, digit, as in ordinary multiplication, begin with the first, or left-hand, digit of the multiplicand and multiply in the ordinary manner, the product being 2,803,770. Multiply by the second digit and write the first figure obtained

in the second partial product one place to the right of the last figure of the first partial product. So proceed until all the partial products have been found, adding them to find the entire product. The result is shown at (*a*). Now, in order to have six figures of the product correct, seven figures should be obtained (see **a**, Art. **261**). It is therefore evident that all figures to the right of the vertical line in (*a*) are unnecessary. Hence, proceed as shown in (*b*). The first partial product contains seven figures—all that are required; therefore, cut off the figure 5 in the multiplicand when finding the second partial product, but multiply it by 3 in order to determine how much to carry. Thus, say mentally "three times five is fifteen," and carry 1. Then say "three times nine is twenty-seven and one is twenty-eight," etc. When multiplying by the next digit 4, cut off the second figure from right of the multiplicand, but carry the 3 that is obtained by multiplying 9 by 4, and say "four times two is eight and three is eleven." Proceeding in this manner, no figure of any of the partial products will extend beyond the place occupied by the seventh figure of the entire product.

291. The operation of division may be shortened in a similar manner to that just described for multiplication. Perform the division in the usual manner until the number of significant figures in the quotient equals the number obtained by subtracting the number of significant figures in the divisor from the number desired in the quotient plus three; then cut off one figure from the right of the divisor before finding the next figure of the quotient; cut off the second figure from the right of the divisor before finding the succeeding figure of the quotient; and so on until the quotient contains one more than the required number of figures. It is here assumed that the dividend and divisor do not contain more than one significant figure in excess of the number required in the quotient. (See **a** and **b**, Art. **261**.)

EXAMPLE.—Divide 71,346.247 by 27,846.392 and obtain five significant figures of the quotient correct.

SOLUTION.—

```
7 1 3 4 6.2 0 | 2 7'8'4'6'.4
5 5 6 9 2 8     2.5 6 2 1 3, or 2.5621 to five figures.  Ans.
1 5 6 5 3 4 0
1 3 9 2 3 2 0
  1 7 3 0 2 0
  1 6 7 0 7 8
      5 9 4 2
      5 5 6 9
        3 7 3
        2 7 8
          9 5
          8 3
          1 2
```

EXPLANATION.—Since five significant figures are required in the quotient, the dividend and divisor are limited to six significant figures. The number of significant figures required in the quotient before beginning to cut off figures from the divisor is $5 + 3 - 6 = 8 - 6 = 2$; hence, before finding the third figure of the quotient, cut off the figure 4 from the right of the divisor, but multiply 4 by 6 in order to see how much to carry. Thus, say "six times four is twenty-four," and carry 2; then, say "six times six is thirty-six and two is thirty-eight," and write 8 and carry 3; and so on. Before finding the fourth figure of the quotient, cut off the next figure 6 of the divisor. The student will find it convenient to place the divisor at the right of the dividend with the quotient underneath, as shown. This arrangement saves space and brings each figure of the quotient directly under the divisor, making the multiplication easier.

292. To locate the decimal point in the quotient, the easiest way is to proceed as follows: Move the decimal point in the divisor to the right until it follows the right-hand figure; that is, make the divisor a whole number; move the decimal point in the dividend as many decimal places to the right as it was moved in the divisor, annexing ciphers if necessary. If the dividend will contain the divisor one or

more times, there will be as many figures in the integral part of the quotient as there are figures left in the dividend after finding the first remainder plus one. If the dividend will not contain the divisor, annex ciphers to follow the decimal point until the dividend contains the divisor, and the first significant figure of the quotient will then be located as many decimal places to the right of the decimal point as there were ciphers annexed. For instance, $.046 \div 21.76 = 4.6 \div 2176. = 4.600 \div 2,176. = .002+$; $4.6 \div 21.76 = 460. \div 2,176. = 460.0 \div 2,176 = .2+$; $460 \div 21.76 = 46,000. \div 2,176 = 21.+$. In the last example, $71,346.2 \div 27,846.4 = 713,462. \div 278,464. = 2+$.

293. Having shown how the work of calculation may be greatly reduced, an example is given on the following page, showing all the figures used in extracting the fifth root, each operation being numbered in the order in which it is performed.

294. When the sixth significant figure of the third approximation is 5, it is not always advisable to increase the fifth figure by one. To ascertain whether or not the fifth figure should be increased, recalculate the third approximation, using for one of the equal factors the third approximation first found, correct to *four* figures; if the sixth significant figure is then 5+, increase the fifth figure by 1.

EXAMPLE 1.— $\sqrt[5]{3.056} = ?$

SOLUTION.—Using the first two significant figures and trying 1 for the equal factors, the fifth factor is $3.1 \div 1^4 = 3.1$, and the first approximation is $\frac{4 \times 1 + 3.1}{5} = 1.42$. Since $3.1 - 1 = 2.1$ is less than 2.5 (see Remark in example referred to in Art. **293**), it is not necessary to try 2 for one of the equal factors.

$$3.056 \div 1.4^4 = .795+. \quad \frac{4 \times 1.4 + .795}{5} = 1.279+.$$

Since the difference between 1.42, the first approximation, and 1.279, the second approximation, is greater than one unit in the second figure, try 1.3 for one of the equal factors and recalculate the second approximation.

$$3.056 \div 1.3^4 = 1.069+. \quad \frac{4 \times 1.3 + 1.069}{5} = 1.253+.$$

EXAMPLE.— $\sqrt[5]{8.269} = ?$

SOLUTION.—

(1)

```
    6
    6
 ----
   36
   36
 ----
  108
  216
 ----
 1296
```

(2)

```
8269 | 1296
7776   6.38+
----
 493
 388
----
 105
 103
----
   2
```

(3)

$$\frac{4 \times 6 + 6.38}{5} = 6.07+$$

(4)

```
    6.1
    6.1
  -----
   366
    61
  -----
   37.21
   37.21
  -----
  11163
   2604
     74
      3
  -----
  1384.4
```

(5)

```
8269.0 | 1384
6920     5.974
------
13490
12456
------
 1034
  968
------
   66
   55
------
   11
```

(6)

$$\frac{4 \times 6.1 + 5.974}{5} = \frac{30.374}{5} = 6.074+$$

(7)

```
   6.07
   6.07
  -----
  3642
    4249
  --------
  36.8449
  36.8'4'5
  36.845
 ----------
 110535
  221070
   29476
    1473
     184
 ----------
 1357.553
```

(8)

```
8269.00 | 1357.55
814530    6.09112
-------
 123700
 122179
  ------
   1521
   1357
   ----
    164
    135
    ---
     29
     27
     --
      2
```

(9)

$$\frac{4 \times 6.07 + 6.09112}{5} =$$

$$\frac{30.37112}{5} = 6.07422$$

$= 6.0742$ to five figures. Ans.

REMARK.—It was unnecessary to try 7 for one of the equal factors, because there was but a very little difference between the fifth factor 6.38 and one of the equal factors 6; in fact, when this difference is not greater than 2.5 units, it is unnecessary to try the next higher number for one of the equal factors. In this case the difference is $6.38 - 6 = .38$, or less than 1 unit.

Since the difference, $1.3 - 1.253 = .047$, is less than one unit in the second figure, use 1.25 for one of the equal factors in finding the third approximation.

$$3.056 \div 1.25^4 = 1.25173+. \quad \frac{4 \times 1.25 + 1.25173}{5} = 1.25034+,$$

or 1.2503 to five figures. Ans.

The exact root to seven figures is 1.250347.

EXAMPLE 2.— $\sqrt[5]{3} = ?$

SOLUTION.—Trying 1.3 for one of the equal factors (see last example), $3 \div 1.3^4 = 1.050+$. $\frac{4 \times 1.3 + 1.050}{5} = 1.25.$

Using 1.25 for one of the equal factors, the fifth factor is $3 \div 1.25^4 = 1.22879+$, and the third approximation is $\frac{4 \times 1.25 + 1.22879}{5} = 1.24575+$. Since the sixth figure is 5, it will be well to recalculate the third approximation, using 1.246 for one of the equal factors. Hence, $3 \div 1.246^4 = 1.244656+$, and the third approximation is

$$\frac{4 \times 1.246 + 1.244656}{5} = 1.245731+, \text{ or } 1.2457 \text{ to five figures. Ans.}$$

The exact root to seven figures is 1.245731.

295. The fifth root is very seldom required; the most prominent case in practice arises in connection with finding the diameter of a pipe that will discharge a required amount of water, the head and length of pipe being known. It is also required in connection with certain problems in mine ventilation. The fourth root is used even less frequently than the fifth root. Roots higher than the fifth are never required.

TABLE METHOD OF EXTRACTING THE FIFTH ROOT.

296. In exactly the same way as in the case of square and cube roots, the first three significant figures of the fifth root may be found by means of the table of the powers of numbers.

EXAMPLE.— $\sqrt[5]{238.75} = ?$

SOLUTION.—Referring to the table, the first two figures are 2.9; the first difference is $243.00 - 205.11 = 37.89$; the second difference is $238.75 - 205.11 = 33.64$; $33.64 \div 37.89 = .88+$, or .9 to one figure. Therefore, $\sqrt[5]{238.75} = 2.99$ to three figures.

297. When finding the fifth root of numbers whose first period contains but one significant figure, carry the quotient obtained by dividing the second difference by the first difference to three decimal places, and if the third figure is 5 or a greater digit, increase the second figure by 1 and add these two figures of the quotient to those previously found for the third and fourth figures of the root. Then use all four figures when finding the third approximation.

EXAMPLE.— $\sqrt[5]{3} = ?$

SOLUTION.—Referring to the table, the first two figures are 1.2; the first difference is $3.7129 - 2.4883 = 1.2246$; the second difference is $3 - 2.4883 = .5117$; $.5117 \div 1.2246 = .417+$, or .42 to two figures. Hence, assume that one of the equal factors is 1.242; the fifth factor is $3 \div 1.242^4 = 1.26076$, and the third approximation is $\frac{4 \times 1.242 + 1.26076}{5}$ $= 1.245752$. Since the sixth figure is 5, recalculate the third approximation, using the result just found correct to four figures for one of the equal factors (see Example 2, Art. **294**). The result is 1.2457. Ans.

ARITHMETIC.

(PART 6.)

RATIO.

298. Suppose that it is desired to compare two numbers, say 20 and 4. If we wish to know how many times larger 20 is than 4, we divide 20 by 4 and obtain 5 for the quotient; thus, $20 \div 4 = 5$. Hence, we say that 20 is 5 times as large as 4, i. e., 20 contains 5 times as many units as 4. Again, suppose we desire to know what part of 20 is 4. We then divide 4 by 20 and obtain $\frac{1}{5}$; thus, $4 \div 20 = \frac{1}{5}$, or .2. Hence, 4 is $\frac{1}{5}$ or .2 of 20. This operation of comparing two numbers is termed *finding the ratio* of the two numbers. Ratio, then, is a comparison. It is evident that the two numbers to be compared must be expressed in the same unit; in other words, the two numbers must both be abstract numbers or concrete numbers of the same kind. It would be absurd to compare 20 horses with 4 birds, or 20 horses with 4. Hence, **ratio** may be defined as a comparison between two numbers of the same kind.

299. A ratio is *expressed* in one of two ways; thus, if it is desired to compare 20 and 4, and express this comparison as a ratio, it is written $20 : 4$, or $\frac{20}{4}$. Both are read: *the ratio of 20 to 4*. The ratio of 4 to 20 would be expressed: $4 : 20$, or $\frac{4}{20}$. The first form is the one oftenest met with, while the second form, called the *fractional* form, is rapidly growing in favor, and is likely to supersede the first. The first form seems to be better adapted to arithmetical subjects, and is one we shall ordinarily adopt.

300. The **terms** of a ratio are the two numbers to be compared; thus, in the above ratio, 20 and 4 are the terms. When both terms are considered together, they are called a **couplet;** when considered separately, the first term is called the **antecedent,** and the second term the **consequent.** Thus, in the ratio 20 : 4, 20 and 4 form a couplet, and 20 is the antecedent, and 4 the consequent.

301. A ratio may be **direct** or **inverse.** The *direct ratio* of 20 to 4 is 20 : 4, while the *inverse ratio* of 20 to 4 is 4 : 20. The direct ratio of 4 to 20 is 4 : 20, and the inverse ratio is 20 : 4. An inverse ratio is sometimes called a **reciprocal** ratio. The **reciprocal** of a number is 1 divided by the number. Thus, the reciprocal of 17 is $\frac{1}{17}$; of $\frac{3}{8}$ is $1 \div \frac{3}{8} = \frac{8}{3}$; i.e., the reciprocal of a fraction is the fraction inverted. Hence, the inverse ratio of 20 to 4 may be expressed as 4 : 20, or as $\frac{1}{20} : \frac{1}{4}$. Both have equal values; for, $4 \div 20 = \frac{1}{5}$, and $\frac{1}{20} \div \frac{1}{4} = \frac{1}{20} \times \frac{4}{1} = \frac{1}{5}$.

302. The term **vary** implies a ratio. When we say that two numbers vary as some other two numbers, we mean that the ratio between the first two numbers is the same as the ratio between the other two numbers.

303. The **value** of a ratio is the result obtained by performing the division indicated. Thus, the value of the ratio 20:4 is 5, it is the quotient obtained by dividing the antecedent by the consequent.

304. By expressing the ratio in the fractional form, for example, the ratio of 20 to 4 as $\frac{20}{4}$, it is easy to see, from the laws of fractions, that if both terms be multiplied, or both divided by the same number, it will not alter the value of the ratio. Thus,

$$\frac{20}{4} = \frac{20 \times 5}{4 \times 5} = \frac{100}{20}; \text{ and } \frac{20}{4} = \frac{20 \div 4}{4 \div 4} = \frac{5}{1}.$$

305. It is also evident, from the laws of fractions, that multiplying the antecedent or dividing the consequent multiplies the ratio; and dividing the antecedent or multiplying the consequent divides the ratio.

306. When a ratio is expressed in words, as the ratio of 20 to 4, the first number named is always regarded as the antecedent and the second as the consequent, without regard to whether the ratio itself is direct or inverse. *When not otherwise specified, all ratios are understood to be direct.* To express an inverse ratio, the simplest way of doing it is to express it as if it were a direct ratio, with the first number named as the antecedent, and then transpose the antecedent to the place occupied by the consequent and the consequent to the place occupied by the antecedent; or if expressed in the fractional form, invert the fraction. Thus, to express the inverse ratio of 20 to 4, first write it 20 : 4, and then, transposing the terms, as 4 : 20; or as $\frac{20}{4}$, and then inverting as $\frac{4}{20}$. Or, the reciprocals of the numbers may be taken, as explained above. To **invert** a ratio is to transpose its terms.

EXAMPLES FOR PRACTICE.

307. What is the value of the ratio of

(*a*)	98 to 49?	Ans.	(*a*)	2.
(*b*)	$45 to $9?		(*b*)	5.
(*c*)	$6\frac{1}{4}$ to $\frac{2}{4}$?		(*c*)	$12\frac{1}{2}$.
(*d*)	3.5 to 4.5?		(*d*)	$.77\frac{7}{9}$.
(*e*)	The inverse ratio of 76 to 19?		(*e*)	$\frac{1}{4}$.
(*f*)	The inverse ratio of 49 to 98?		(*f*)	2.
(*g*)	The inverse ratio of 18 to 24?		(*g*)	$1\frac{1}{3}$.
(*h*)	The inverse ratio of 9 to 15?		(*h*)	$1\frac{2}{3}$.
(*i*)	The ratio of 10 to 3, multiplied by 3?		(*i*)	10.
(*j*)	The ratio of 35 to 49, multiplied by 7?		(*j*)	5.
(*k*)	The ratio of 18 to 64, divided by 9?		(*k*)	$\frac{1}{32}$.
(*l*)	The ratio of 14 to 28, divided by 5?		(*l*)	$\frac{1}{10}$.

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308. Instead of expressing the value of a ratio by a single number, as above, it is customary to express it by

means of another ratio in which the consequent is 1. Thus, suppose that it is desired to find the ratio of the weights of two pieces of iron, one weighing 45 pounds and the other weighing 30 pounds. The ratio of the heavier to the lighter is then 45 : 30, an inconvenient expression. Using the fractional form, we have $\frac{45}{30}$. Dividing both terms by 30, the consequent, we obtain $\frac{1\frac{1}{2}}{1}$ or $1\frac{1}{2} : 1$. This is the same result as obtained above, for $1\frac{1}{2} \div 1 = 1\frac{1}{2}$, and $45 \div 30 = 1\frac{1}{2}$.

309. A ratio may be squared, cubed, or raised to any power, or any root of it may be taken. Thus, if the ratio of two numbers is 105 : 63, and it is desired to cube this ratio, the cube may be expressed as $105^3 : 63^3$. That this is correct is readily seen; for, expressing the ratio in the fractional form, it becomes $\frac{105}{63}$, and the cube is $\left(\frac{105}{63}\right)^3 = \frac{105^3}{63^3} = 105^3 : 63^3$. Also, if it is desired to extract the cube root of the ratio $105^3 : 63^3$, it may be done by simply dividing the exponents by 3, obtaining 105 : 63. This may be proved in the same way as in the case of cubing the ratio. Thus, $105^3 : 63^3 = \left(\frac{105}{63}\right)^3$, and $\sqrt[3]{\left(\frac{105}{63}\right)^3} = \frac{105}{63} = 105 : 63$.

310. Since $\left(\frac{105}{63}\right)^3 = \left(\frac{5}{3}\right)^3$, it follows that $105^3 : 63^3 = 5^3 : 3^3$ (this expression is read: the ratio of 105 cubed to 63 cubed equals the ratio of 5 cubed to 3 cubed), and, hence, that the antecedent and consequent may both be multiplied or both divided by the same number, irrespective of any indicated powers or roots, without altering the value of the ratio. Thus, $24^2 : 18^2 = 4^2 : 3^2$. For, performing the operations indicated by the exponents, $24^2 = 576$ and $18^2 = 324$. Hence, $576 : 324 = 1\frac{7}{9}$ or $1\frac{7}{9} : 1$. Also, $4^2 = 16$ and $3^2 = 9$; hence, $16 : 9 = 1\frac{7}{9}$ or $1\frac{7}{9} : 1$, the same result as before. Also, $24^2 : 18^2 = \frac{24^2}{18^2} = \left(\frac{24}{18}\right)^2 = \left(\frac{4}{3}\right)^2 = \frac{4^2}{3^2} = 4^2 : 3^2$.

The statement may be proved for roots in the same manner. Thus, $\sqrt[3]{24^3} : \sqrt[3]{18^3} = \sqrt[3]{4^3} : \sqrt[3]{3^3}$. For the $\sqrt[3]{24^3} = 24$ and $\sqrt[3]{18^3} = 18$; and, $24 : 18 = 1\frac{1}{3}$ or $1\frac{1}{3} : 1$. Also, $\sqrt[3]{4^3} = 4$ and $\sqrt[3]{3^3} = 3$; $4 : 3 = 1\frac{1}{3}$ or $1\frac{1}{3} : 1$.

NOTE.—If the numbers composing the antecedent and consequent have different exponents, or if different roots of those numbers are indicated, the operations described in Art. **310** cannot be performed. This is evident; for, consider the ratio $4^2 : 8^3$. When expressed in the fractional form, it becomes $\frac{4^2}{8^3}$, which cannot be expressed either as $\left(\frac{4}{8}\right)^2$ or as $\left(\frac{4}{8}\right)^3$, and, hence, cannot be reduced as described above.

PROPORTION.

311. **Proportion** is an equality of ratios, the equality being indicated by the double colon (: :) or by the sign of equality (=). Thus, to write in the form of a proportion the two equal ratios, $8 : 4$ and $6 : 3$, which both have the same value 2, we may employ one of the three following forms:

$$8 : 4 :: 6 : 3 \qquad (1)$$
$$8 : 4 = 6 : 3 \qquad (2)$$
$$\frac{8}{4} = \frac{6}{3} \qquad (3)$$

312. The first form is the one most extensively used, by reason of its having been exclusively employed in all the older works on mathematics. The second and third forms are being adopted by all modern writers on mathematical subjects, and, in time, will probably entirely supersede the first form. In this paper we shall adopt the second form, unless some statement can be made clearer by using the third form.

313. A proportion may be *read* in two ways. The old way to read the above proportion was—*8 is to 4 as 6 is to 3;* the new way is—*the ratio of 8 to 4 equals the ratio of 6 to 3.* The student may read it either way, but we recommend the latter.

314. Each ratio of a proportion is termed a **couplet.** In the above proportion, $8 : 4$ is a couplet, and so is $6 : 3$.

315. The numbers forming the proportion are called **terms;** and they are numbered consecutively from left to right, thus:

$$\overset{\textit{first}}{8} : \overset{\textit{second}}{4} = \overset{\textit{third}}{6} : \overset{\textit{fourth}}{3}$$

Hence, in any proportion, the ratio of the first term to the second term equals the ratio of the third term to the fourth term.

316. The first and fourth terms of a proportion are called the **extremes,** and the second and third terms, the **means.** Thus, in the foregoing proportion, 8 and 3 are the extremes and 4 and 6 are the means.

317. A **direct proportion** is one in which both couplets are direct ratios.

318. An **inverse proportion** is one which requires one of the couplets to be expressed as an inverse ratio. Thus, 8 is to 4 inversely as 3 is to 6 must be written $8 : 4 = 6 : 3$; i. e., the second ratio (couplet) must be inverted.

319. Proportion forms one of the most useful sections of Arithmetic. In our grandfathers' Arithmetics, it was called "The rule of three."

320. Rule I.—*In any proportion, the product of the extremes equals the product of the means.*

Thus, in the proportion,

$$17 : 51 = 14 : 42.$$

$17 \times 42 = 51 \times 14$, since both products equal 714.

321. Rule II.—*The product of the extremes divided by either mean gives the other mean.*

EXAMPLE.—What is the third term of the proportion $17 : 51 = \quad : 42$?

SOLUTION.—Applying rule II, $17 \times 42 = 714$, and $714 \div 51 = 14$. Ans.

322. Rule III.—*The product of the means divided by either extreme gives the other extreme.*

EXAMPLE.—What is the first term of the proportion $\quad : 51 = 14 : 42$?

SOLUTION.—Applying rule III, $51 \times 14 = 714$, and $714 \div 42 = 17$. Ans.

323. When stating a proportion in which one of the terms is unknown, represent the missing term by a letter, as x. Thus, the last example would be written,

$$x : 51 = 14 : 42$$

and for the value of x we have $x = \frac{51 \times 14}{42} = 17$.

324. If the same (addition and subtraction excepted) operations be performed upon *all* of the terms of a proportion, the proportion is not thereby destroyed. In other words, if all of the terms of a proportion be (1) multiplied or (2) divided by the same number; (3) if all the terms be raised to the same power; if (4) the same root of all the terms be taken, or (5) if both couplets be inverted, the proportion still holds. We will prove these statements by a numerical example, and the student can satisfy himself by other similar ones. The fractional form will be used, as it is better suited to the purpose. Consider the proportion $8 : 4 = 6 : 3$. Expressing it in the third form, it becomes $\frac{8}{4} = \frac{6}{3}$. What we are to prove is that, if any of the five operations enumerated above be performed upon all of the terms of this proportion, the first fraction will still equal the second fraction.

1. Multiplying all the terms by any number, say 7, $\frac{8 \times 7}{4 \times 7} = \frac{6 \times 7}{3 \times 7}$; or $\frac{56}{28} = \frac{42}{21}$. Now $\frac{56}{28}$ evidently equals $\frac{42}{21}$, since the value of either ratio is 2, and the same is true of the original proportion.

2. Dividing all the terms by any number, say 7, $\frac{8 \div 7}{4 \div 7} = \frac{6 \div 7}{3 \div 7}$; or $\frac{\frac{8}{7}}{\frac{4}{7}} = \frac{\frac{6}{7}}{\frac{3}{7}}$. But $\frac{8}{7} \div \frac{4}{7} = 2$, and $\frac{6}{7} \div \frac{3}{7} = 2$ also, the same as in the original proportion.

3. Raising all the terms to the same power, say the cube, $\frac{8^3}{4^3} = \frac{6^3}{3^3}$. This is evidently true, since $\frac{8^3}{4^3} = \left(\frac{8}{4}\right)^3 = 2^3 = 8$, and $\frac{6^3}{3^3} = \left(\frac{6}{3}\right)^3 = 2^3 = 8$ also.

4. Extracting the same root of all the terms, say the cube root, $\frac{\sqrt[3]{8}}{\sqrt[3]{4}} = \frac{\sqrt[3]{6}}{\sqrt[3]{3}}$. It is evident that this is likewise true, since $\frac{\sqrt[3]{8}}{\sqrt[3]{4}} = \sqrt[3]{\frac{8}{4}} = \sqrt[3]{2}$, and $\frac{\sqrt[3]{6}}{\sqrt[3]{3}} = \sqrt[3]{\frac{6}{3}} = \sqrt[3]{2}$ also.

5. Inverting both couplets, $\frac{4}{8} = \frac{3}{6}$, which is true, since both equal $\frac{1}{2}$.

325. If both terms of either couplet be multiplied or both divided by the same number, the proportion is not destroyed. This should be evident from the preceding article, and also from Art. **304.** Hence, in any proportion, equal factors may be canceled from the terms of a couplet, before applying rules II or III. Thus, the proportion $45 : 9 = x : 7.1$, we may divide both terms of the first couplet by 9 (that is, cancel 9 from both terms), obtaining $5 : 1 = x : 7.1$, whence $x = 7.1 \times 5 \div 1 = 35.5$. (See note in Art. **310.**)

326. The principle of all calculations in proportion is this: *Three of the terms are always given, and the remaining one is to be found.*

327. EXAMPLE.—If 4 men can earn \$25 in one week, how much can 12 men earn in the same time?

SOLUTION.—The required term must bear the same relation to the given term of the same kind as one of the remaining terms bears to the other remaining term. We can then form a proportion by which the required term may be found.

The first question the student must ask himself in every calculation by proportion is:

"What is it I want to find?"

In this case it is dollars. We have two sets of men, one set earning \$25, and we want to know how many dollars the other set earns. It is evident that the *amount* 12 men earn bears the same relation to the *amount* that 4 men earn as 12 men bears to 4 men. Hence, we have the proportion, the amount 12 men earn is to \$25 as 12 men is to 4 men; or, since either extreme equals the product of the means divided by the other extreme, we have

$$\text{The amount 12 men earn} : \$25 = 12 \text{ men} : 4 \text{ men},$$

or the amount 12 men earn $= \frac{\$25 \times 12}{4} = \$75$. Ans.

Since it matters not which place x or the required term occupies, the problem could be stated as any of the following forms, the value of x being the same in each :

(*a*) \$25 : the amount 12 men earn = 4 men : 12 men ; or the amount 12 men earn $= \frac{\$25 \times 12}{4}$, or \$75, since either mean equals the product of the extremes divided by the other mean.

(*b*) 4 men : 12 men = \$25 : the amount 12 men earn ; or the amount that 12 men earn $= \frac{\$25 \times 12}{4}$, or \$75, since either extreme equals the product of the means divided by the other extreme.

(*c*) 12 men : 4 men = the amount 12 men earn : \$25 ; or the amount that 12 men earn $= \frac{\$25 \times 12}{4}$, or \$75, since either mean equals the product of the extremes divided by the other mean.

328. If the proportion is an inverse one, first form it as though it were a direct proportion, and then invert one of the couplets.

EXAMPLES FOR PRACTICE.

329. Find the value of x in each of the following:

(*a*)	\$16 : \$64 :: x : \$4.	Ans. (*a*)	$x =$ \$1.
(*b*)	x : 85 :: 10 : 17.	(*b*)	$x = 50$.
(*c*)	24 : x :: 15 : 40.	(*c*)	$x = 64$.
(*d*)	18 : 94 :: 2 : x.	(*d*)	$x = 10\frac{4}{9}$.
(*e*)	\$75 : \$100 = x : 100.	(*e*)	$x = 75$.
(*f*)	15 pwt. : x = 21 : 10.	(*f*)	$x = 7\frac{1}{7}$ pwt.
(*g*)	x : 75 yd. = \$15 : \$5.	(*g*)	$x = 225$ yd.

1. If 75 pounds of lead cost \$2.10, what would 125 pounds cost at the same rate ? Ans. \$3.50.

2. If A does a piece of work in 4 days and B does it in 7 days, how long will it take A to do what B does in 63 days ? Ans. 36 days.

3. The circumferences of any two circles are to each other as their diameters. If the circumference of a circle 7 inches in diameter is 22 inches, what will be the circumference of a circle 31 inches in diameter ? Ans. $97\frac{3}{7}$ inches.

INVERSE PROPORTION.

330. In Art. **318,** an inverse proportion was defined as one which required one of the couplets to be expressed as an inverse ratio. Sometimes the word *inverse* occurs in the

statement of the example; in such cases the proportion can be written directly, merely inverting one of the couplets. But it frequently happens that only by carefully studying the conditions of the example can it be ascertained whether the proportion is direct or inverse. When in doubt, the student can always satisfy himself as to whether the proportion is direct or inverse by first ascertaining what is required, and stating the proportion as a direct proportion. Then, in order that the proportion may be true, if the first term is smaller than the second term, the third term must be smaller than the fourth; or if the first term is larger than the second term, the third term must be larger than the fourth term. Keeping this in mind, the student can always tell whether the required term will be larger or smaller than the other term of the couplet to which the required term belongs. Having determined this, the student then refers to the example, and ascertains from its conditions whether the required term is to be larger or smaller than the other term of the same kind. If the two determinations agree, the proportion is direct; otherwise, it is inverse, and one of the couplets must be inverted.

331. EXAMPLE.—If A's *rate* of doing work is to B's as 5 : 7, and A does a piece of work in 42 days, in what time will B do it?

SOLUTION.—The required term is the number of days it will take B to do the work. Hence, stating as a direct proportion,

$$5 : 7 = 42 : x.$$

Now, since 7 is greater than 5, x will be greater than 42. But, referring to the statement of the example, it is easy to see that B works faster than A; hence it will take B a less number of days to do the work than A. Therefore, the proportion is an inverse one, and should be stated

$$5 : 7 = x : 42,$$

from which $x = \frac{5 \times 42}{7} = 30$ days. Ans.

Had the example been stated thus: The time that A requires to do a piece of work is to the time that B requires, as 5 : 7; A can do it in 42 days, in what time can B do it? it is evident that it would take B a longer time to do the work than it would A; hence, x would be greater than 42, and the proportion would be direct, the value of x being $\frac{7 \times 42}{5}$ = 58.8 days.

EXAMPLES FOR PRACTICE.

332. Solve the following:

1. If a pump which discharges 4 gal. of water per min. can fill a tank in 20 hr., how long will it take a pump discharging 12 gal. per min. to fill it? Ans. $6\frac{2}{3}$ hr.

2. If a pump discharges 90 gal. of water in 20 hr., in what time will it discharge 144 gal.? Ans. 32 hr.

3. If 50 cu. ft. of air weigh 4.2 pounds when the absolute temperature is 562°, what will be the absolute temperature when the same volume weighs 5.8 pounds, the pressure being the same in both cases? Ans. 407°, very nearly.

CAUSES AND EFFECTS.

333. Many examples in proportion may be more easily solved by using the principle of *cause and effect*. That which may be regarded as producing a change or alteration in something, or as accomplishing something, may be called **a cause,** and the change or alteration, or thing accomplished, is the **effect.**

334. *Like causes produce like effects.* Hence, when two causes of the same kind produce two effects of the same kind, the ratio of the causes equals the ratio of the effects; in other words, the first cause is to the second cause as the first effect is to the second effect. Thus, in the question, if 3 men can lift 1,400 pounds, how many pounds can 7 men lift? we call 3 men and 7 men the *causes* (since they accomplish something, viz., the lifting of the weight), the number of pounds lifted, viz., 1,400 pounds and x pounds, are the effects. If we call 3 men the first cause, 1,400 pounds is the first effect; 7 men is the second cause, and x pounds is the second effect. Hence, we may write

$$\underset{\textit{1st cause}}{3} : \underset{\textit{2d cause}}{7} = \underset{\textit{1st effect}}{1{,}400} : \underset{\textit{2d effect}}{x},$$

whence $x = \frac{7 \times 1{,}400}{3} = 3{,}266\frac{2}{3}$ pounds.

The principle of cause and effect is extremely useful in the solution of examples in compound proportion, as we shall now show.

COMPOUND PROPORTION.

335. All the cases of proportion so far considered have been cases of **simple proportion**; i. e., each term has been composed of but one number. There are many cases, however, in which two or all of the terms have more than one number in them; all such cases belong to **compound proportion.** In all examples in compound proportion, both causes or both effects or all four consist of more than two numbers. We will illustrate this by an

EXAMPLE.—If 40 men earn \$1,280 in 16 days, how much will 36 men earn in 31 days?

SOLUTION.—Since 40 men earn something, 40 men is a cause, and since they take 16 days in which to earn something, 16 days is also a cause. For the same reason, 36 men and 31 days are also causes. The effects, that which is earned, are 1,280 dollars and x dollars. Then, 40 men and 16 days make up the first cause, and 36 men and 31 days make up the second cause. \$1,280 is the first effect and \$$x$ is the second effect. Hence, we write

$$\begin{array}{ccccccc} \textit{1st cause} & & \textit{2d cause} & & \textit{1st effect} & & \textit{2d effect} \\ \begin{matrix}40\\16\end{matrix} & : & \begin{matrix}36\\31\end{matrix} & = & 1{,}280 & : & x. \end{array}$$

Now, instead of using the colon to express the ratio, we shall use the vertical line (see Art. **299**), and the above becomes

$$\begin{matrix}40\\16\end{matrix} \left| \begin{matrix}36\\31\end{matrix} = 1{,}280 \right| x.$$

In the last expression, the product of all of the numbers included between the vertical lines must equal the product of all the numbers without them; i. e., $36 \times 31 \times 1{,}280 = 40 \times 16 \times x$.

$$\text{Or } x = \frac{36 \times 31 \times \overset{\overset{2}{\cancel{80}}}{\cancel{1280}}}{\cancel{40} \times \cancel{16}} = \$2{,}232. \text{ Ans.}$$

336. The above might have been solved by canceling factors of the numbers in the original proportion. For if any number within the lines has a factor common to any number without the lines, that factor may be canceled from both numbers. Thus,

$$\begin{matrix}\cancel{40}\\\cancel{16}\end{matrix} \left| \begin{matrix}36\\31\end{matrix} = \overset{\overset{2}{\cancel{80}}}{\cancel{1280}} \right| x,$$

16 is contained in 1,280, 80 times. Cancel 16 and 1,280, and

write 80 above 1,280. 40 is contained in 80, 2 times. Cancel 40 and 80 and write 2 above 80. Now, since there are no more numbers that can be canceled, $x = 36 \times 31 \times 2 =$ \$2,232, the same result as was obtained in the last article.

337. Rule.—*Write all the numbers forming the first cause in a vertical column, and draw a vertical line; on the other side of this line write in a vertical column all of the numbers forming the second cause. Write the sign of equality to the right of the second column, and on the right of this form a third column of the numbers composing the first effect, drawing a vertical line to the right; on the other side of this line, write, for a fourth column, the numbers composing the second effect. There must be as many numbers in the second cause as in the first cause, and in the second effect as in the first effect; hence, if any term is wanting, write x in its place. Multiply together all of the numbers within the vertical lines, and also all those without the lines (canceling previously, if possible), and divide the product of those numbers which do not contain x by the product of the others in which x occurs, and the result will be the value of x.*

EXAMPLE.—If 40 men can dig a ditch 720 feet long, 5 feet wide, and 4 feet deep in a certain time, how long a ditch 6 feet deep and 3 feet wide could 24 men dig in the same time?

SOLUTION.—Here 40 men and 24 men are the causes and the two ditches are the effects. Hence,

$$\begin{array}{c|c|l} \cancel{40} & 24 = \begin{matrix}\cancel{3}\\ \cancel{18}\\ \cancel{720}\\ 5\\ 4\end{matrix} & \begin{matrix} x\\ \cancel{3}\\ \cancel{6}\end{matrix} \end{array}$$ whence, $x = 24 \times 5 \times 4 = 480$ feet. Ans.

EXAMPLE.—The volume of a cylinder varies directly as its length and directly as the square of its diameter. If the volume of a cylinder 10 inches in diameter and 20 inches long is 1,570.8 cubic inches, what is the volume of another cylinder 16 inches in diameter and 24 inches long?

SOLUTION.—In this example, either the dimensions or the volumes may be considered the causes; say we take the dimensions for the causes. Then, squaring the diameters,

$$\begin{array}{c|c|c} \begin{matrix}10^2\\ 20\end{matrix} & \begin{matrix}16^2\\ 24\end{matrix} = 1{,}570.8 & x, \end{array} \text{ or } \begin{array}{c|c|c} \begin{matrix}100\\ \cancel{20}\\ 5\end{matrix} & \begin{matrix}256\\ \cancel{24}\\ 6\end{matrix} = 1{,}570.8 & x; \end{array}$$

whence, $x = \dfrac{256 \times 6 \times 1{,}570.8}{5 \times 100} = 4{,}825.4976$ cubic inches. Ans.

EXAMPLE.—If a block of granite 8 ft. long, 5 ft. wide, and 3 ft. thick weighs 7,200 lb., what will be the weight of a block of granite 12 ft. long, 8 ft. wide, and 5 ft. thick?

SOLUTION.—Taking the weights as the effects, we have

$$\begin{array}{c|l|l} & 4 & \\ \cancel{8} & \cancel{12} & \\ \cancel{5} & \cancel{8} = 7{,}200 & x, \text{ or } x = 4 \times 7{,}200 = 28{,}800 \text{ pounds. Ans.} \\ \cancel{3} & \cancel{5} & \end{array}$$

EXAMPLE.—If 12 compositors in 30 days of 10 hours each set up 25 sheets of 16 pages each, 32 lines to the page, in how many days 8 hours long can 18 compositors set up, in the same type, 64 sheets of 12 pages each, 40 lines to the page?

SOLUTION.—Here compositors, days, and hours compose the causes, and sheets, pages, and lines the effects. Hence,

$$\begin{array}{c|ccc|l} 8 & \cancel{3} & & \cancel{5} & 2 \\ \cancel{12} & \cancel{18} & & \cancel{25} & \cancel{64} \\ & & & \cancel{4} & \\ \cancel{30} & x & = & \cancel{16} & \cancel{12}, \text{ or } x = 3 \times 10 \times 2 = 60 \text{ days. Ans.} \\ \cancel{6} & & & & \cancel{4} \\ 10 & \cancel{8} & & \cancel{32} & \cancel{40} \\ & & & & \cancel{5} \end{array}$$

338. In examples stated like the second in Art. **337**, should an inverse proportion occur, write the various numbers as in the preceding examples, and then transpose those numbers which are said to vary inversely from one side of the vertical line to the other side.

EXAMPLE.—The centrifugal force of a revolving body varies directly as its weight, as the square of its velocity, and inversely as the radius of the circle described by the center of the body. If the centrifugal force of a body weighing 15 pounds is 187 pounds when the body revolves in a circle having a radius of 12 inches, with a velocity of 20 feet per second, what will be the centrifugal force of the same body when the radius is increased to 18 inches and the speed is increased to 24 feet per second?

SOLUTION.—Calling the centrifugal force the effect, we have

$$\begin{array}{c|l|l} 15 & 15 & \\ 20^2 & 24^2 = 187 & x. \\ 12 & 18 & \end{array}$$

Transposing 12 and 18 (since the radii are to vary inversely) and squaring 20 and 24,

$$\begin{array}{c|l|l} \cancel{15} & \cancel{15} & \\ & 2 & \\ 25 & \cancel{36} = 187 & x, \text{ or } x = \dfrac{12 \times 2 \times 187}{25} = 179.52 \text{ pounds. Ans.} \\ \cancel{400} & \cancel{576} & \\ \cancel{18} & 12 & \end{array}$$

EXAMPLES FOR PRACTICE.

339. Solve the following by compound proportion:

1. If 12 men dig a trench 40 rods long in 24 days of 10 hours each, how many rods can 16 men dig in 18 days of 9 hours each?

Ans. 36 rods.

2. If a piece of iron 7 ft. long, 4 in. wide, and 6 in. thick weighs 600 lb., how much will a piece of iron weigh that is 16 ft. long, 8 in. wide, and 4 in. thick? Ans. $1,828\frac{4}{7}$ lb.

3. If 24 men can build a wall 72 rods long, 6 feet wide, and 5 feet high in 60 days of 10 hours each, how many days will it take 32 men to build a wall 96 rods long, 4 feet wide, and 8 feet high, working 8 hours a day? Ans. 80 days.

4. The horsepower of an engine varies as the mean effective pressure, as the piston speed, and as the square of the diameter of the cylinder. If an engine having a cylinder 14 inches in diameter develops 112 horsepower when the mean effective pressure is 48 pounds per square inch and the piston speed is 500 feet per minute, what horsepower will another engine develop if the cylinder is 16 inches in diameter, piston speed is 600 feet per minute, and mean effective pressure is 56 pounds per square inch? Ans. 204.8 horsepower.

5. Referring to the second example in Art. **337**, what will be the volume of a cylinder 20 inches in diameter and 24 inches long?

Ans. 7,539.84 cubic inches.

6. Knowing that the product of $3 \times 5 \times 7 \times 9$ is 945, what is the product of $6 \times 15 \times 14 \times 36$? Ans. 45,360.

7. An engine of 15 horsepower can pump out $\frac{3}{5}$ of the water contained in a sump by working 8 hours per day for 9 days. In how many days of 12 hours could an engine of 16 horsepower perform the remainder of the work? Ans. $3\frac{3}{4}$ days.

8. If 8 men can make a mining (bearing in) 112 feet long, 1 foot high at the front, and 4 feet deep in a longwall face in 7 hours, how many men will be required to make a mining 80 feet long in the same face in 5 hours, if the mining must be 1.3 feet at the front and 5 feet deep? Ans. 13 men.

MENSURATION AND USE OF LETTERS IN FORMULAS.

FORMULAS.

1. The term **formula,** as used in mathematics and in technical books, may be defined as *a rule in which symbols are used instead of words;* in fact, a formula may be regarded as a shorthand method of expressing a rule. Any formula can be expressed in words, and when so expressed it becomes a rule.

Formulas are much more convenient than rules; they show at a glance all the operations that are to be performed; they do not require to be read three or four times, as is the case with most rules, to enable one to understand their meaning; they take up much less space, both in the printed book and in one's note book, than rules; in short, whenever a rule can be expressed as a formula, the formula is to be preferred.

As the term "quantity" is a very convenient one to use, we will define it. In mathematics, the word **quantity** is applied to anything that it is desired to subject to the ordinary operations of addition, subtraction, multiplication, etc., when we do not wish to be more specific and state exactly what the thing is. Thus, we can say "two or more numbers," or "two or more quantities"; the word quantity is more general in its meaning than the word number.

2. The signs used in formulas are the ordinary signs indicative of operations and the signs of aggregation. All

these signs are explained in arithmetic, but some of them will here be explained in order to refresh the student's memory.

3. The signs indicative of operations are six in number, viz.: $+$, $-$, $\times$, $\div$, $|$, $\sqrt{}$.

Division is indicated by the sign $\div$, or by placing a straight line between the two quantities. Thus, 25 | 17, 25 / 17, and $\frac{25}{17}$ all indicate that 25 is to be divided by 17. When both quantities are placed on the same horizontal line, the straight line indicates that the quantity on the left is to be divided by that on the right. When one quantity is below the other, the straight line between indicates that the quantity above the line is to be divided by the one below it.

The sign ($\sqrt{}$) indicates that some root of the quantity on the right is to be taken; it is called the **radical sign.** To indicate what root is to be taken, a small figure, called the **index,** is placed within the sign, this being always omitted when the square root is to be indicated. Thus, $\sqrt{25}$ indicates that the square root of 25 is to be taken; $\sqrt[3]{25}$ indicates that the cube root of 25 is to be taken; etc.

4. The signs of aggregation are four in number; viz., ——, ( ), [ ], and { }, respectively called the **vinculum,** the **parenthesis,** the **brackets,** and the **brace;** they are used when it is desired to indicate that all the quantities included by them are to be subjected to the same operation. Thus, if we desire to indicate that the sum of 5 and 8 is to be multiplied by 7, and we do not wish to actually add 5 and 8 before indicating the multiplication, we may employ any one of the four signs of aggregation as here shown: $\overline{5+8} \times 7$, $(5+8) \times 7$, $[5+8] \times 7$, $\{5+8\} \times 7$. The vinculum is placed above those quantities which are to be treated as one quantity and subjected to the same operations.

5. While any one of the four signs may be used as shown above, custom has restricted their use somewhat. The vinculum is rarely used except in connection with the radical sign. Thus, instead of writing $\sqrt[3]{}(5+8)$, $\sqrt[3]{}[5+8]$, or $\sqrt[3]{}\{5+8\}$ for the cube root of 5 plus 8, all of which would be correct, the vinculum is nearly always used, $\sqrt[3]{5+8}$.

In cases where but one sign of aggregation is needed (except, of course, when a root is to be indicated), the parenthesis is always used. Hence, $(5 + 8) \times 7$ would be the usual way of expressing the product of 5 plus 8, and 7.

If two signs of aggregation are needed, the brackets and parenthesis are used, so as to avoid having a parenthesis within a parenthesis, the brackets being placed outside. For example, $[(20 - 5) \div 3] \times 9$ means that the difference between 20 and 5 is to be divided by 3, and this result multiplied by 9.

If three signs of aggregation are required, the brace, brackets, and parenthesis are used, the brace being placed outside, the brackets next, and the parenthesis inside. For example, $\{[(20 - 5) \div 3] \times 9 - 21\} \div 8$ means that the quotient obtained by dividing the difference between 20 and 5 by 3 is to be multiplied by 9, and that after 21 has been subtracted from the product thus obtained, the result is to be divided by 8.

Should it be necessary to use all four of the signs of aggregation, the brace would be put outside, the brackets next, the parenthesis next, and the vinculum inside. For example, $\{[(\overline{20 - 5} \div 3) \times 9 - 21] \div 8\} \times 12$.

6. As stated in *Arithmetic*, when several quantities are connected by the various signs indicating addition, subtraction, multiplication, and division, the operation indicated by the sign of multiplication must always be performed first. Thus, $2 + 3 \times 4$ equals 14, 3 being multiplied by 4 before adding to 2. Similarly, $10 \div 2 \times 5$ equals 1, since 2×5 equals 10, and $10 \div 10$ equals 1. Hence, in the above case, if the brace were omitted, the result would be $\frac{1}{4}$, whereas, by inserting the brace, the result is 36.

Following the sign of multiplication comes the sign of division in order of importance. For example, $5 - 9 \div 3$ equals 2, 9 being divided by 3 before subtracting from 5. The signs of addition and subtraction are of equal value; that is, if several quantities are connected by plus and minus signs, the indicated operations may be performed in the order in which the quantities are placed.

7. There is one other sign used, which is neither a sign of aggregation nor a sign indicative of an operation to be performed; it is (=), and is called the sign of **equality;** it means that all on one side of it is exactly equal to all on the other side. For example, $2 = 2$, $5 - 3 = 2$, $5 \times (14 - 9) = 25$.

8. Having called particular attention to certain signs used in formulas, the formulas themselves will now be explained. First, consider the well-known rule for finding the horsepower of a steam engine, which may be stated as follows:

Divide the continued product of the mean effective pressure in pounds per square inch, the length of the stroke in feet, the area of the piston in square inches, and the number of strokes per minute, by 33,000; the result will be the horsepower.

This is a very simple rule, and very little, if anything, will be saved by expressing it as a formula, so far as clearness is concerned. The formula, however, will occupy a great deal less space, as we shall show.

An examination of the rule will show that four quantities (viz., the mean effective pressure, the length of the stroke, the area of the piston, and the number of strokes) are multiplied together, and the result is divided by 33,000. Hence, the rule might be expressed as follows:

$$\text{Horsepower} = \begin{matrix}\text{mean effective pressure}\\ \text{(in pounds per square inch)}\end{matrix} \times \begin{matrix}\text{stroke}\\ \text{(in feet)}\end{matrix}$$

$$\times \begin{matrix}\text{area of piston}\\ \text{(in square inches)}\end{matrix} \times \begin{matrix}\text{number of strokes}\\ \text{(per minute)}\end{matrix} \div 33{,}000.$$

This expression could be shortened by representing each quantity by a single letter; thus, representing horsepower by the letter "H," the mean effective pressure in pounds per square inch by "P," the length of stroke in feet by "L," the area of the piston in square inches by "A," the number of strokes per minute by "N," and substituting these letters for the quantities that they represent, the above expression would reduce to

$$H = \frac{P \times L \times A \times N}{33{,}000},$$

a much simpler and shorter expression. This last expression is called a *formula.*

9. The formula just given shows, as we stated in the beginning, that a formula is really a shorthand method of expressing a rule. It is customary, however, to omit the sign of multiplication between two or more quantities when they are to be multiplied together, or between a number and a letter representing a quantity, it being always understood that, when two letters are adjacent with no sign between them, the quantities represented by these letters are to be multiplied. Bearing this fact in mind, the formula just given can be further simplified to

$$H = \frac{P L A N}{33,000}.$$

10. The sign of multiplication, evidently, cannot be omitted between two or more numbers, as it would then be impossible to distinguish the numbers. A near approach to this, however, may be attained by placing a dot between the numbers which are to be multiplied together, and this is frequently done in works on mathematics when it is desired to economize space. In such cases it is usual to put the dot higher than the position occupied by the decimal point. Thus, $2 \cdot 3$ means the same as 2×3; $542 \cdot 749 \cdot 1{,}006$ indicates that the numbers 542, 749, and 1,006 are to be multiplied together.

It is also customary to omit the sign of multiplication in expressions similar to the following: $a \times \sqrt{b + c}$, $3 \times (b + c)$, $(b + c) \times a$, etc., writing them $a \sqrt{b + c}$, $3\,(b + c)$, $(b + c)\,a$, etc. The sign is not omitted when several quantities are included by a vinculum and it is desired to indicate that the quantities so included are to be multiplied by another quantity. For example, $3 \times \overline{b + c}$, $\overline{b + c} \times a$, $\sqrt{b + c} \times a$, etc. are always written as here printed.

11. Before proceeding further, we will explain one other device that is used by formula makers and which is apt to puzzle one who encounters it for the first time—it is the use of what mathematicians call *primes* and *subs.*, and

what printers call *superior* and *inferior* characters. As a rule, formula makers designate quantities by the initial letters of the names of the quantities. For example, they represent volume by v, pressure by p, height by h, etc. This practice is to be commended, as the letter itself serves in many cases to identify the quantity which it represents. Some authors carry the practice a little further and represent all quantities of the same nature by the same letter throughout the book, always having the same letter represent the same thing. Now, this practice necessitates the use of the primes and subs. above mentioned when two quantities have the same name but represent different things. Thus, consider the word *pressure* as applied to steam at different stages between the boiler and the condenser. First, there is *absolute* pressure, which is equal to the gauge pressure in pounds per square inch plus the pressure indicated by the barometer reading (usually assumed in practice to be 14.7 pounds per square inch, when a barometer is not at hand). If this be represented by p, how shall we represent the gauge pressure? Since the absolute pressure is always greater than the gauge pressure, suppose we decide to represent it by a capital letter and the gauge pressure by a small (lower-case) letter. Doing so, P represents absolute pressure and p, gauge pressure. Further, there is usually a "drop" in pressure between the boiler and the engine, so that the initial pressure, or pressure at the beginning of the stroke, is less than the pressure at the boiler. How shall we represent the initial pressure? We may do this in one of three ways and still retain the letter p or P to represent the word pressure: First, by the use of the prime mark; thus, p' or P' (read *p prime* and *P major prime*) may be considered to represent the initial gauge pressure or the initial absolute pressure. Second, by the use of sub. figures; thus, p_1 or P_1 (read *p sub. one* and *P major sub. one*). Third, by the use of sub. letters; thus, p_i or P_i (read *p sub. i* and *P major sub. i*). In the same manner p'' (read *p second*), p_2, or p_r might be used to represent the gauge pressure at release, etc. The sub. letters have the advantage of still further identifying the quantity represented; in many instances,

however, it is not convenient to use them, in which case primes and subs. are used instead. The prime notation may be continued as follows: p''', p^{iv}, p^{v}, etc.; it is inadvisable to use superior figures, for example, p^1, p^2, p^3, p^n, etc., as they are liable to be mistaken for exponents.

12. The main thing to be remembered by the student is that *when a formula is given in which the same letters occur several times, all like letters having the same primes or subs. represent the same quantities, while those which differ in any respect represent different quantities.* Thus, in the formula

$$t = \frac{w_1 s_1 t_1 + w_2 s_2 t_2 + w_3 s_3 t_3}{w_1 s_1 + w_2 s_2 + w_3 s_3},$$

w_1, w_2, and w_3 represent the weights of three different bodies; s_1, s_2, and s_3, their specific heats; and t_1, t_2, and t_3, their temperatures; while t represents the final temperature after the bodies have been mixed together. It should be noted that those letters having the *same* subs. refer to the same bodies. Thus, w_1, s_1, and t_1 all refer to one of the three bodies; w_2, s_2, t_2, to another body, etc.

It is very easy to apply the above formula when the values of the quantities represented by the different letters are known. All that is required is to substitute the numerical values of the letters and then perform the indicated operations. Thus, suppose that the values of w_1, s_1, and t_1 are, respectively, 2 pounds, .0951, and 80°; of w_2, s_2, and t_2, 7.8 pounds, 1, and 80°; and of w_3, s_3, and t_3, $3\frac{1}{4}$ pounds, .1138, and 780°; then, the final temperature t is, substituting these values for their respective letters in the formula,

$$t = \frac{2 \times .0951 \times 80 + 7.8 \times 1 \times 80 + 3\frac{1}{4} \times .1138 \times 780}{2 \times .0951 + 7.8 \times 1 + 3\frac{1}{4} \times .1138}$$

$$= \frac{15.216 + 624 + 288.483}{.1902 + 7.8 + .36985} = \frac{927.699}{8.36005} = 110.97°.$$

In substituting the numerical values, the signs of multiplication are, of course, written in their proper places; all the multiplications are performed before adding, according to the rule previously given.

13. The student should now be able to apply any formula involving only algebraic expressions that he may meet with, and which does not require the use of logarithms for its solution. We will, however, call his attention to one or two other facts that he may have forgotten.

Expressions similar to $\dfrac{160}{\dfrac{660}{25}}$ sometimes occur, the heavy line indicating that 160 is to be divided by the quotient obtained by dividing 660 by 25. If both lines were light, it would be impossible to tell whether 160 was to be divided by $\frac{660}{25}$, or whether $\frac{160}{660}$ was to be divided by 25. If this latter result were desired, the expression would be written $\dfrac{\dfrac{160}{660}}{25}$. In every case the heavy line indicates that all above it is to be divided by all below it.

In an expression like the following, $\dfrac{160}{7+\dfrac{660}{25}}$, the heavy line is not necessary, since it is impossible to mistake the operation that is required to be performed. But, since $7+\frac{660}{25}=\frac{175+660}{25}$, if we substitute $\frac{175+660}{25}$ for $7+\frac{660}{25}$, the heavy line becomes necessary in order to make the resulting expression clear. Thus,

$$\frac{160}{7+\dfrac{660}{25}}=\frac{160}{\dfrac{175+660}{25}}=\frac{160}{\dfrac{835}{25}}.$$

14. Fractional exponents are sometimes used instead of the radical sign. That is, instead of indicating the square, cube, fourth root, etc. of some quantity, as 37, by $\sqrt{37}$, $\sqrt[3]{37}$, $\sqrt[4]{37}$, etc., these roots are indicated by $37^{\frac{1}{2}}$, $37^{\frac{1}{3}}$, $37^{\frac{1}{4}}$, etc. Should the numerator of the fractional exponent be some quantity other than 1, this quantity, whatever it may be, indicates that the quantity affected by the exponent is to be raised to the power indicated by the numerator; the

denominator is *always* the index of the root. Hence, instead of writing $\sqrt[3]{37^2}$ for the cube root of the square of 37, it may be written $37^{\frac{2}{3}}$, the denominator being the index of the root; in other words, $\sqrt[3]{37^2} = 37^{\frac{2}{3}}$. Likewise, $\sqrt[5]{(1 + a^2 b)^3}$ may also be written $(1 + a^2 b)^{\frac{3}{5}}$, a much simpler expression.

15. We will now give several examples showing how to apply some of the more difficult formulas that the student may encounter.

1. The area of any segment of a circle that is less than (or equal to) a semicircle is expressed by the formula

$$A = \frac{\pi r^2 E}{360} - \frac{c}{2}(r - h),$$

in which A = area of segment;
π = 3.1416;
r = radius;
E = angle obtained by drawing lines from the center to the extremities of arc of segment;
c = chord of segment;
h = height of segment.

Example.—What is the area of a segment whose chord is 10 inches long, angle subtended by chord is 83.46°, radius is 7.5 inches, and height of segment is 1.91 inches?

Solution.—Applying the formula just given,

$$A = \frac{\pi r^2 E}{360} - \frac{c}{2}(r - h) = \frac{3.1416 \times 7.5^2 \times 83.46}{360} - \frac{10}{2}(7.5 - 1.91)$$
$$= 40.968 - 27.95 = 13.018 \text{ sq. in., nearly. Ans.}$$

2. The area of any triangle may be found by means of the following formula, in which A = the area, and a, b, and c represent the lengths of the sides:

$$A = \frac{b}{2}\sqrt{a^2 - \left(\frac{a^2 + b^2 - c^2}{2b}\right)^2}.$$

Example.—What is the area of a triangle whose sides are 21 feet, 46 feet, and 50 feet long?

Solution.—In order to apply the formula, suppose we let a represent the side that is 21 feet long; b, the side that is 50 feet long; and c, the side that is 46 feet long. Then, substituting in the formula

$$A = \frac{b}{2}\sqrt{a^2 - \left(\frac{a^2 + b^2 - c^2}{2\,b}\right)^2} = \frac{50}{2}\sqrt{21^2 - \left(\frac{21^2 + 50^2 - 46^2}{2 \times 50}\right)^2}$$

$$= \frac{50}{2}\sqrt{441 - \left(\frac{441 + 2{,}500 - 2{,}116}{100}\right)^2} = 25\sqrt{441 - \left(\frac{825}{100}\right)^2}$$

$$= 25\sqrt{441 - 8.25^2} = 25\sqrt{441 - 68.0625} = 25\sqrt{372.9375}$$

$$= 25 \times 19.312 = 482.8 \text{ sq. ft., nearly. Ans.}$$

The operations in the above examples have been extended much farther than was necessary; it was done in order to show the student every step of the process. The last formula is perfectly general, and the same answer would have been obtained had the 50-foot side been represented by a, the 46-foot side by b, and the 21-foot side by c.

3. The Rankine-Gordon formula for determining the least load in pounds that will cause a long column to break is

$$P = \frac{S\,A}{1 + q\frac{l^2}{G^2}},$$

in which P = load (pressure) in pounds;
S = ultimate strength (in pounds per square inch) of the material composing the column;
A = area of cross-section of column in square inches;
q = a factor (multiplier) whose value depends upon the shape of the ends of the column and on the material composing the column;
l = length of column in inches;
G = least radius of gyration of cross-section of column.

The values of S, q, and G^2 are given in printed tables in books in which this formula occurs.

EXAMPLE.—What is the least load that will break a hollow steel column whose outside diameter is 14 inches; inside diameter, 11 inches; length, 20 feet, and whose ends are flat?

SOLUTION.—For steel, $S = 150{,}000$, and $q = \frac{1}{25{,}000}$ for flat-ended steel columns; A, the area of the cross-section, $= .7854\,(d_1^2 - d_2^2) = .7854\,(14^2 - 11^2)$, d_1 and d_2 being the outside and inside diameters,

respectively; $l = 20 \times 12 = 240$ inches; and $G^2 = \frac{d_1^2 + d_2^2}{16} = \frac{14^2 + 11^2}{16}$. Substituting these values in the formula

$$P = \frac{S\,A}{1 + q\,\frac{l^2}{G^2}} = \frac{150{,}000 \times .7854\,(14^2 - 11^2)}{1 + \frac{1}{25{,}000} \times \frac{240^2}{\frac{14^2 + 11^2}{16}}}$$

$$= \frac{150{,}000 \times 58.905}{1 + .1163} = \frac{8{,}835{,}750}{1.1163} = 7{,}915{,}211 \text{ lb. Ans.}$$

4. Example.—When $A = 10$, $B = 8$, $C = 5$, and $D = 4$, what is the value of E in the following?

(a) $E = \sqrt[3]{\frac{B\,C\,D}{A\left(2 + \frac{D^2}{C^2}\right)}}$; (b) $E = \frac{A - \frac{3}{4}D + \frac{4\,B^2}{A + C}}{A - \sqrt{\frac{2\,B^2}{A + 22}}}$.

Solution.—(a) Substituting.

$$E = \sqrt[3]{\frac{8 \times 5 \times 4}{10\left(2 + \frac{4^2}{5^2}\right)}}.$$

To simplify the denominator, square the 4 and 5, add the resulting fraction to 2, and multiply by 10. Simplifying, we have

$$E = \sqrt[3]{\frac{160}{10\left(2 + \frac{16}{25}\right)}} = \sqrt[3]{\frac{160}{10 \times \frac{66}{25}}} = \sqrt[3]{\frac{160}{\frac{660}{25}}} = \sqrt[3]{\frac{200}{33}}.$$

Reducing the fraction to a decimal before extracting the cube root,

$$E = \sqrt[3]{6.0606} = 1.823. \text{ Ans.}$$

(b) Substituting,

$$E = \frac{10 - \frac{3}{4} \times 4 + \frac{4 \times 8^2}{10 + 5}}{10 - \sqrt{\frac{2 \times 8^2}{10 + 22}}} = \frac{10 - 3 + \frac{4 \times 64}{15}}{10 - \sqrt{\frac{2 \times 64}{32}}}$$

$$= \frac{7 + 17.066+}{10 - \sqrt{4}} = \frac{24.066+}{8} = 3.008+. \text{ Ans.}$$

16. In the preceding pages, the unknown quantity has always been represented by the single letter at the left of the sign of equality, while the letters at the right have represented known values from which the required values could be found. It is possible, however, to find the value of the quantity represented by any letter in a formula, if the values represented by all the others are known. For example, let it be required to find how many strokes per minute an

engine having a piston area of 78.54 square inches must make in order to develop 60 horsepower, if the mean effective pressure is 40 pounds per square inch and the length of stroke is $1\frac{1}{4}$ ft. By substituting the given values in the formula $H = \frac{P\,L\,A\,N}{33{,}000}$, we have

$$60 = \frac{40 \times 1\frac{1}{4} \times 78.54 \times N}{33{,}000},$$

in which N, the number of strokes, is to be found.

But it is evident that the expression on the right of the sign of equality is equal to $\frac{40 \times 1\frac{1}{4} \times 78.54}{33{,}000} \times N$, a fraction whose numerator is composed of three factors. Reducing the numerator to a single number by performing the indicated multiplications, we obtain, after canceling,

$$60 = \frac{119}{1{,}000} \times N = .119\,N.$$

If 60 equals .119 N, then N equals 60 divided by .119; hence,

$$N = \frac{60}{.119} = 504.2 \text{ strokes per minute.}$$

The method of procedure is essentially the same when the unknown quantity occurs in the denominator of a formula. Thus, in the formula $f = \frac{m\,v^2}{r}$, suppose that $f = 375$, $m = 1.25$, and $v = 60$. Then, substituting,

$$375 = \frac{1.25 \times 60^2}{r} = \frac{4{,}500}{r}.$$

But, if 375 equals 4,500 divided by r, then $375 \times r = 4{,}500$; hence, r must equal 4,500 divided by 375, or $r = \frac{4{,}500}{375} = 12$.

EXAMPLES FOR PRACTICE.

Find the numerical values of x in the following formulas, when $A = 9$, $B = 8$, $d = 10$, $e = 3$, and $c = 2$:

1. $x = \frac{d + c^2}{d^2 - 40}$. Ans. $x = \frac{7}{30}$.

2. $x = \frac{\frac{2}{3}(A + c)}{c\,e}$. Ans. $x = 1\frac{2}{9}$.

3. $x = \sqrt{\frac{d^2}{2c}} + \sqrt{A B^2}$. Ans. $x = 29$.

4. $x = \frac{A e}{\sqrt{16 B c}} + \frac{5}{16}$. Ans. $x = 2$.

5. $x = (c + 2e)\left(\sqrt{B} - \frac{1}{c}\right) + \frac{e^2 - c^2}{e^2 + c^2}$. Ans. $x = 12\frac{5}{13}$.

6. $x = \sqrt{\frac{B c d}{A\left(2 + \frac{d^3}{e^2}\right)}}$ Ans. $x = .396+$.

MENSURATION.

17. **Mensuration** treats of the measurement of lines, angles, surfaces, and solids.

LINES AND ANGLES.

18. A **straight line** is one that does not change its direction throughout its whole length. To distinguish one straight line from another, two of its points are designated by letters. The line shown in Fig. 1 would be called the line *A B*.

FIG. 1.

19. A **curved line** changes its direction at every point. Curved lines are designated by three or more letters, as the curved line *A B C*, Fig. 2.

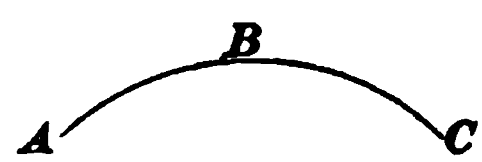


FIG. 2.

20. **Parallel lines** (Fig. 3) are those which are equally distant from each other at all points.


FIG. 3.

21. A line is **perpendicular** to another (see Fig. 4) when it meets that line so as not to incline towards it on either side.

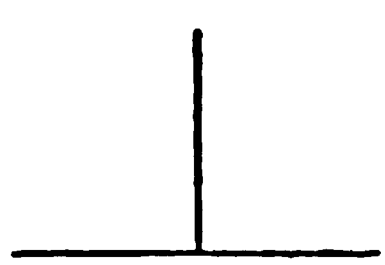


FIG. 4.

22. A **vertical line** is one that points towards the center of the earth, and is also known as a *plumb*-line.

23. A **horizontal line** (see Fig. 5) is one that makes a right angle with any vertical line.

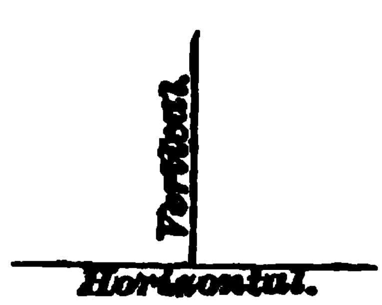


FIG. 5.

24. An **angle** is the opening between two lines which intersect or meet; the point of meeting is called the **vertex** of the angle. Angles are distinguished by naming the vertex and a point on each line. Thus, in Fig. 6, the angle formed by the lines $A\,B$ and $C\,B$ is called the angle $A\,B\,C$, or the angle $C\,B\,A$; the letter at the vertex is always placed in the middle. When an angle stands alone so that it cannot be mistaken for any other angle, only the vertex letter need be used. Thus, the angle referred to might be designated simply as the angle B.

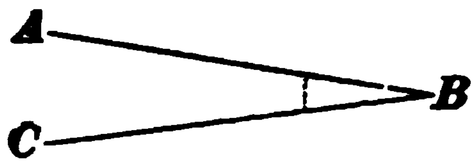


FIG. 6.

25. If one straight line meets another straight line at a point between its ends, as in Fig. 7, two angles, $A\ B\ C$ and $A\ B\ D$, are formed, which are called **adjacent angles.**

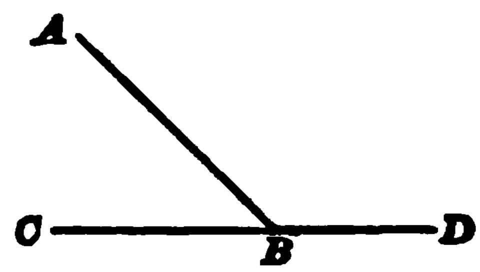


FIG. 7.

26. When these adjacent angles, $A\ B\ C$ and $A\ B\ D$, are equal, as in Fig. 8, they are called **right angles.**

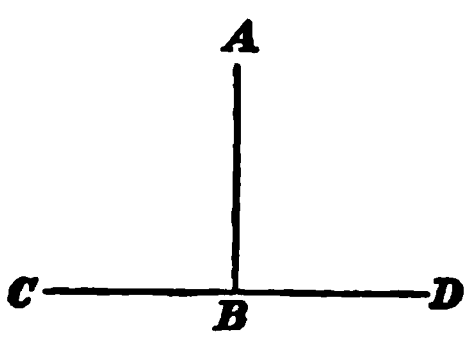


FIG. 8.

27. An **acute angle** is less than a right angle. $A\ B\ C$, Fig. 9, is an acute angle.

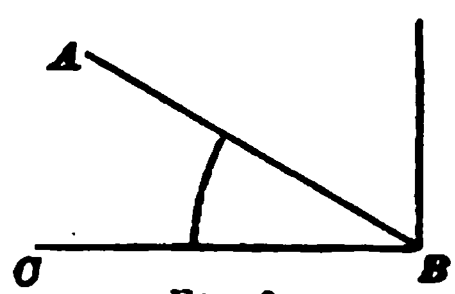


FIG. 9.

28. An **obtuse angle** is greater than a right angle. $A\,B\,D$ (Fig. 10) is an obtuse angle.

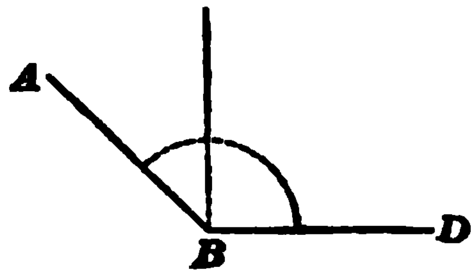


FIG. 10.

29. A **circle** (see Fig. 11) is a figure bounded by a curved line, called the **circumference,** every point of which is equally distant from a point within, called the **center.**

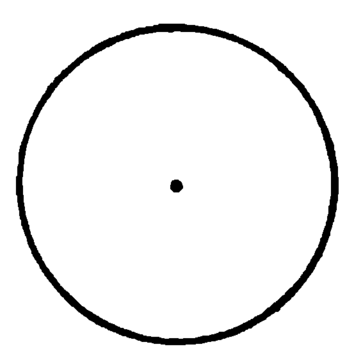


FIG. 11.

30. An **arc** of a circle is any part of its circumference; thus $a\ e\ b$, Fig. 12, is an arc of the circle.

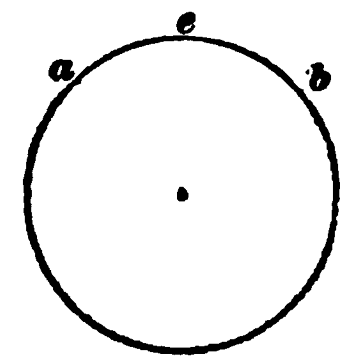


FIG. 12.

31. The circumference of every circle is considered to be divided into 360 equal parts, or arcs, called **degrees**; every degree is subdivided into 60 equal parts, called **minutes**, and every minute is again divided into 60 equal parts, called **seconds**.

Since 1 degree is $\frac{1}{360}$ of any circumference, it follows that the length of a degree will be different in circles of different sizes, but the proportion of the length of an arc of 1 degree to the whole circumference will always be the same, viz., $\frac{1}{360}$ of the circumference.

Degrees, minutes, and seconds are denoted by the symbols $^\circ$, $'$, $''$. Thus, the expression $37^\circ\ 14'\ 44''$ is read 37 degrees 14 minutes 44 seconds.

32. The arcs of circles are used to measure angles. An angle having its vertex at the center of a circle is measured by the arc included between its sides; thus, in Fig. 13, the arc $F B$ measures the angle $F O B$. If the arc $F B$ contains 20°, or $\frac{20}{360}$ of the circumference, the angle $F O B$ would be an angle of 20°; if it contained $20^\circ\ 14'\ 18''$, it would be an angle of $20^\circ\ 14'\ 18''$, etc.

FIG. 13.

In the figure, if the line $C D$ be drawn perpendicular to $A B$, the adjacent angles will be equal, and the circle will be divided into four equal angles, each of which will be a right angle. A right angle, therefore, is an angle of $\frac{360^\circ}{4}$, or 90°; two right angles are measured by 180°, or half the circumference, and four right angles by the whole circumference, or 360°. One-half of a right angle, as $E O B$, is an angle of 45°. An *acute* angle may now be defined as an angle of *less* than 90°, and an *obtuse* angle as one of *more*

than 90°. These values are important and should be remembered.

33. From the foregoing it will be evident that if a number of straight lines on the same side of a given straight line meet at the same point, the sum of all the angles formed is equal to two right angles, or 180°. Thus, in Fig. 14, angles $COB + DOC + EOD + FOE + AOF$ = 2 right angles, or 180°.

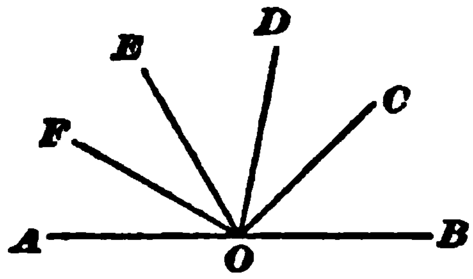


FIG. 14.

34. Also, if through a given point any number of straight lines be drawn, the sum of all the angles formed about the points of intersection equals four right angles, or 360°. Thus, in Fig. 15, angles $HOF + FOC + COA + AOG + GOE + EOD + DOB + BOH$ = 4 right angles, or 360°.

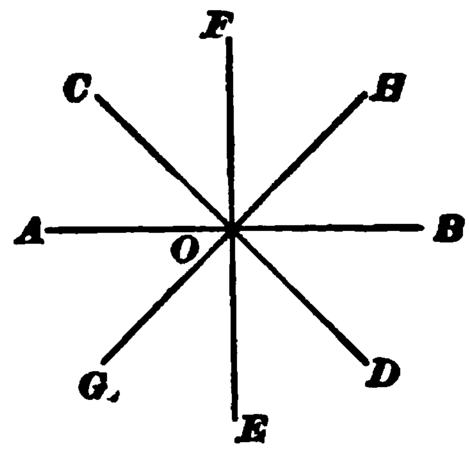


FIG. 15.

EXAMPLE.—In a flywheel with 12 arms, how many degrees are there in the angle included between the center lines of any two arms, the arms being spaced equally?

SOLUTION.—Since there are 12 arms, there are 12 angles, which together equal 360°. Hence, one angle equals $\frac{1}{12}$ of 360°, or $\frac{360°}{12} = 30°$. Ans.

EXAMPLES FOR PRACTICE.

1. How many seconds are in 32° 14′ 6″? Ans. 116,046 sec.

2. How many degrees, minutes, and seconds do 38,582 seconds amount to? Ans. 10° 43′ 2″.

3. How many right angles are there in an angle of 170°? Ans. $1\frac{8}{9}$ right angles.

4. In a pulley with five arms, what part of a right angle is included between the center lines of any two arms? Ans. $\frac{4}{5}$ of a right angle.

5. If one straight line meets another so as to form an angle of 20° 10′, what does its adjacent angle equal? Ans. 159° 50′.

6. If a number of straight lines meet a given straight line at a given point, all being on the same side of the given line, so as to form six equal angles, how many degrees are there in each angle? Ans. 30°.

QUADRILATERALS.

35. A **plane figure** is any part of a plane or flat surface bounded by straight or curved lines.

36. A **quadrilateral** is a plane figure bounded by four straight lines.

37. A **parallelogram** is a quadrilateral whose opposite sides are parallel.

There are four kinds of parallelograms: the **square,** the **rectangle,** the **rhombus,** and the **rhomboid.**

38. A **rectangle** (Fig. 16) is a parallelogram whose angles are all right angles.

FIG. 16.

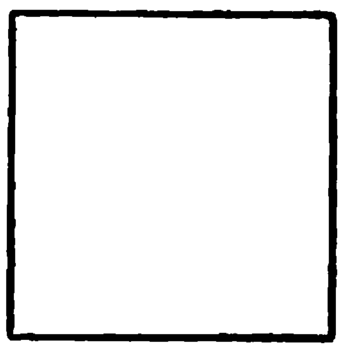


FIG. 17.

39. A **square** (Fig. 17) is a rectangle whose sides are all of the same length.

40. A **rhomboid** (Fig. 18) is a parallelogram whose opposite sides are equal and parallel, and whose angles are not right angles.

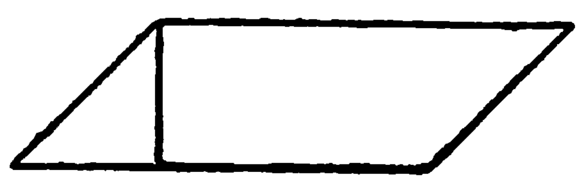


FIG. 18.

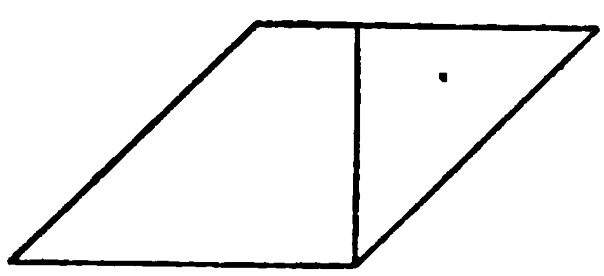


FIG. 19.

41. A **rhombus** (Fig. 19) is a parallelogram having equal sides, and whose angles are not right angles.

42. A **trapezoid** (Fig. 20) is a quadrilateral which has only two of its sides parallel.

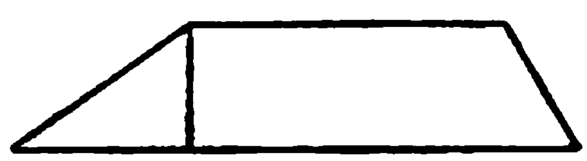


FIG. 20.

43. The **altitude** of a parallelogram or a trapezoid is the perpendicular distance between the parallel lines, as shown by the vertical lines in Figs. 18, 19, and 20.

44. The **base** of *any* plane figure is the side on which it is supposed to stand.

45. The **area** of a surface is expressed by the number of unit squares it will contain.

46. A **unit square** is the square having a unit for its side. For example, if the unit is 1 inch, the unit square is the square each of whose sides measures 1 inch in length, and the area of a surface would be expressed by the number of square inches it would contain. If the unit were 1 foot, the unit square would measure 1 foot on each side, and the area of the given surface would be the number of square feet it would contain, etc.

The square that measures 1 inch on a side is called a **square inch,** and the one that measures 1 foot on a side is called a **square foot.** Square inch and square foot are abbreviated to sq. in. and sq. ft.

47. To find the area of any parallelogram:

Rule 1.—*Multiply the base by the altitude.*

NOTE.—Before multiplying, the base and altitude must be reduced to the same kind of units; that is, if the base should be given in feet and the altitude in inches, they could not be multiplied together until either the altitude had been reduced to feet or the base to inches. This principle holds throughout the subject of mensuration.

EXAMPLE 1.—The sides of a square piece of sheet iron are each $10\frac{1}{4}$ inches long. How many square inches does it contain?

SOLUTION.— $10\frac{1}{4}$ inches $= 10.25$ inches when reduced to a decimal. The base and altitude are each 10.25 inches. Multiplying them together, $10.25 \times 10.25 = 105.06+$ sq. in. Ans.

EXAMPLE 2.—What is the area in square rods of a piece of land in the shape of a rhomboid, one side of which is 8 rods long and whose length, measured on a line perpendicular to this side, is 200 feet?

SOLUTION.—The base is 8 rods and the altitude 200 feet. As the answer is to be in rods, the 200 feet should be reduced to rods. Reducing $200 \div 16\frac{1}{2} = 200 \div \frac{33}{2} = 12.12$ rods. Hence, area $= 8 \times 12.12 = 96.96$ sq. rd. Ans.

48. To find the area of a trapezoid:

Rule 2.—*Multiply one-half the sum of the parallel sides by the altitude.*

EXAMPLE.—A board 14 feet long is 20 inches wide at one end and 16 inches wide at the other. If the ends are parallel, how many square feet does the board contain?

SOLUTION.—One-half the sum of the parallel sides $= \frac{20 + 16}{2}$ $= 18$ inches $= 1\frac{1}{2}$ feet. The length of the board corresponds to the altitude of a trapezoid. Hence, $14 \times 1\frac{1}{2} = 21$ sq. ft. Ans.

49. Having given the area of a parallelogram and one dimension, to find the other dimension:

Rule 3.—*Divide the area by the given dimension.*

EXAMPLE.—What is the width of a parallelogram whose area is 212 square feet and whose length is $26\frac{1}{2}$ feet?

SOLUTION.— $212 \div 26\frac{1}{2} = 212 \div \frac{53}{2} = 8$ ft. Ans.

The following examples illustrate a few special cases:

EXAMPLE 1.—An engine room is 22 feet by 32 feet. The engine bed occupies a space of 3 feet by 12 feet; the flywheel pit, a space of 2 feet by 6 feet, and the outer bearing a space of 2 feet by 4 feet. How many square feet of flooring will be required for the room?

SOLUTION.—Area of engine bed $= 3 \times 12 = 36$ sq. ft.
Area of flywheel pit $= 2 \times 6 = 12$ sq. ft.
Area of outer bearing $= 2 \times 4 = 8$ sq. ft.
Total, 56 sq. ft.
Area of engine room $= 22 \times 32 = 704$ sq. ft.

$704 - 56 = 648$ sq. ft. of flooring required. Ans.

EXAMPLE 2.—How many square yards of plaster will it take to cover the sides and ceiling of a room 16×20 feet and 11 feet high, having four windows, each 7×4 feet, and three doors, each 9×4 feet over all, the baseboard coming 6 inches above the floor?

SOLUTION.—

Area of ceiling $= 16 \times 20 = 320$ sq. ft.
Area of end walls $= 2(16 \times 11) = 352$ sq. ft.
Area of side walls $= 2(20 \times 11) = 440$ sq. ft.
Total area $= 1{,}112$ sq. ft.

From the above must be deducted:

Windows $= 4(7 \times 4) = 112$ sq. ft.
Doors $= 3(9 \times 4) = 108$ sq. ft.

Baseboard less the width of three doors $= (72 - 12) \times \frac{6}{12} = 30$ sq. ft.

Total number of feet to be deducted $= 112 + 108 + 30 = 250$ sq. ft.

Hence, number of square feet to be plastered $= 1{,}112 - 250$ $= 862$ sq. ft., or $95\frac{7}{9}$ sq. yd. Ans.

EXAMPLE 3.—How many acres are contained in a rectangular tract of land 800 rods long and 520 rods wide?

SOLUTION.— $800 \times 520 = 416{,}000$ sq. rd. Since there are 160 square rods in 1 acre, the number of acres $= 416{,}000 \div 160 = 2{,}600$ acres. Ans.

EXAMPLES FOR PRACTICE.

1. What is the area in square feet of a rhombus whose base is 84 inches and whose altitude is 3 feet? Ans. 21 sq. ft.

2. A flat roof, 46 feet by 80 feet in size, is covered by tin roofing weighing one-half pound per square foot; what is the total weight of the roofing? Ans. 1,840 lb.

3. One side of a room measures 16 feet. If the floor contains 240 square feet, what is the length of the other side? Ans. 15 ft.

4. How many square feet in a board 12 feet long, 18 inches wide at one end, and 12 inches wide at the other end? Ans. 15 sq. ft.

5. How much would it cost to lay a sidewalk a mile long and 8 feet 6 inches wide, at the rate of 20 cents per square foot? How much at the rate of $1.80 per square yard? Ans. $8,976 in each case.

6. How many square yards of plastering will be required for the ceiling and walls of a room 10 ft. × 15 ft. and 9 feet high; the room contains one door 3½ ft. × 7 ft., three windows 3½ ft. × 6 ft., and a baseboard 8 inches high? Ans. 53.5 sq. yd.

THE TRIANGLE.

50. A **triangle** is a plane figure having three sides.

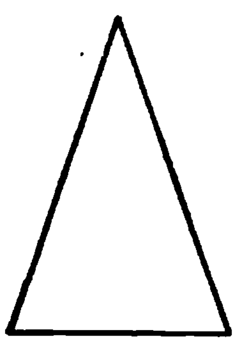


FIG. 21.

51. An **isosceles** triangle is one having two of its sides equal; see Fig. 21.

52. An **equilateral** triangle (Fig. 22) is one having all of its sides of the same length.

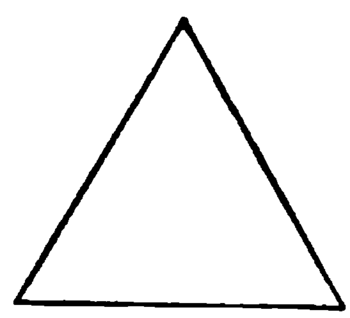


FIG. 22.

FIG. 23.

53. A **scalene** triangle (Fig. 23) is one having no two of its sides equal.

54. A **right-angled** triangle (Fig. 24) is any triangle having one right angle. The side opposite the right angle is called the **hypotenuse.** A right-angled triangle is now usually called a **right triangle.**

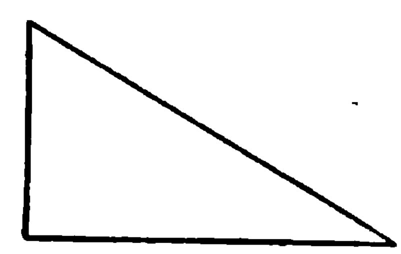


FIG. 24.

55. In any triangle the sum of the three angles equals two right angles, or 180°. Thus, in Fig. 25, the sum of the angles A, B, and C equals two right angles, or 180°. Hence, if any two angles of a triangle are given and it is required to find the third angle:

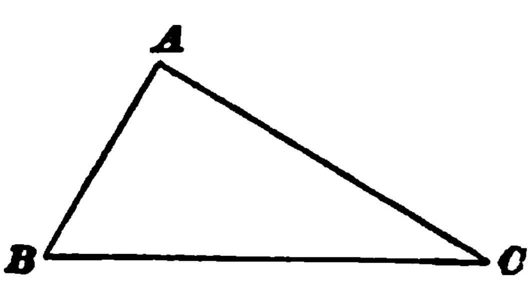


FIG. 25.

Rule 4.—*Add the two given angles and subtract their sum from 180°; the remainder will be the third angle.*

EXAMPLE.—If two angles of a triangle are 48° 16′ and 47° 50′, what does the third angle equal ?

SOLUTION.—First reduce 48° 16′ and 47° 50′ to minutes, for convenience in adding and subtracting the angles. $48° = 48 \times 60' = 2{,}880'$; $2{,}880' + 16' = 2{,}896'$; hence, $48°\ 16' = 2{,}896'$. In like manner, $47°\ 50' = 47 \times 60' + 50' = 2{,}820' + 50' = 2{,}870'$. Adding the two angles and subtracting the sum from 180° reduced to minutes, $2{,}896' + 2{,}870' = 5{,}766'$; $180° = 180 \times 60' = 10{,}800'$; $10{,}800 - 5{,}766 = 5{,}034'$. Reducing this last number to degrees and minutes, $\frac{5{,}034}{60} = 83\frac{54}{60}° = 83°\ 54'$. Hence, the third angle in the triangle $= 83°\ 54'$. Ans.

56. In any right triangle there can be but one right angle, and since the sum of all the angles is two right angles, it is evident that the sum of the two acute angles must equal one right angle, or 90°. Therefore, if in any right triangle one acute angle is known, to find the other acute angle:

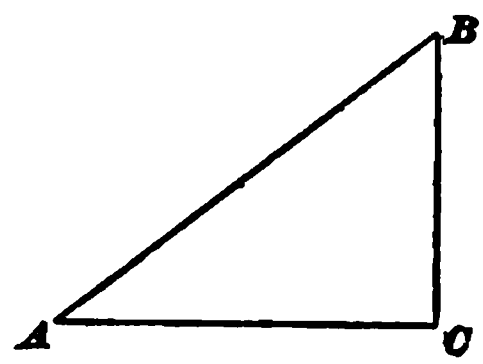


FIG. 26.

Rule 5.—*Subtract the known acute angle from 90°; the result will be the other acute angle.*

EXAMPLE.—If one acute angle, as A, of the right triangle $A\ B\ C$, Fig. 26, equals 30°, what does the angle B equal ?

SOLUTION.— $90° - 30° = 60°$. Ans.

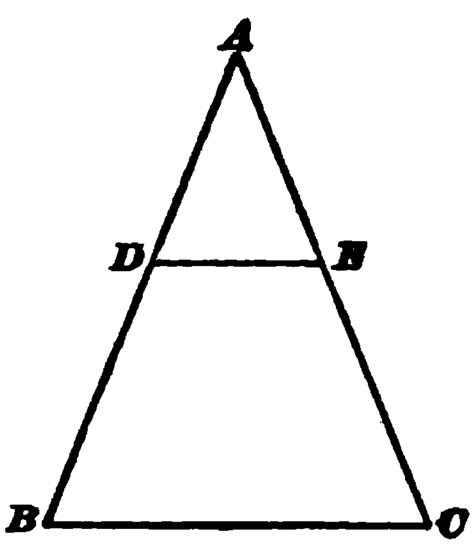


FIG. 27.

57. If a straight line be drawn through two sides of a triangle, parallel to the third side, a second triangle will be formed whose sides will be proportional to the corresponding sides of the first triangle. Thus, in the triangle $A\ B\ C$, Fig. 27, if the line $D\ E$ be drawn parallel to the side $B\ C$, the triangle $A\ D\ E$ will be formed and we shall have

(1) Side $A\ D$: side $D\ E$ = side $A\ B$: side $B\ C$; and,

(2) Side $A\ E$: side $D\ E$ = side $A\ C$: side $B\ C$; also,

(3) Side $A\ D$: side $A\ E$ = side $A\ B$: side $A\ C$.

EXAMPLE.—In Fig. 27, if $A\,B = 24$, $B\,C = 18$, and $D\,E = 8$, what does $A\,D$ equal?

SOLUTION.—Writing these values for the sides in (1),

$$A\,D : 8 = 24 : 18; \text{ whence, } A\,D = \frac{24 \times 8}{18} = 10\tfrac{2}{3}. \quad \text{Ans.}$$

58. In any right triangle, the square described on the hypotenuse is equal to the sum of the squares described upon the other two sides. If $A\,B\,C$, Fig. 28, is a right triangle right-angled at B, then the square described upon the hypotenuse $A\,C$ is equal to the sum of the squares described upon the sides $A\,B$ and $B\,C$. Hence, having given the two sides forming the right angle in a right triangle, to find the hypotenuse:

FIG. 28.

Rule 6.—*Square each of the sides forming the right angle; add the squares together and take the square root of the sum.*

EXAMPLE.—If $A\,B = 3$ inches and $B\,C = 4$ inches, what is the length of the hypotenuse $A\,C$?

SOLUTION.—Squaring each of the given sides, $3^2 = 9$ and $4^2 = 16$. Taking the square root of the sum of 9 and 16, the hypotenuse $= \sqrt{9 + 16} = \sqrt{25} = 5$ in. Ans.

59. If the hypotenuse and one side are given, the other side can be found as follows:

Rule 7.—*Subtract the square of the given side from the square of the hypotenuse, and extract the square root of the remainder.*

EXAMPLE 1.—The side given is 3 inches, the hypotenuse is 5 inches; what is the length of the other side?

SOLUTION.— $3^2 = 9$; $5^2 = 25$. $25 - 9 = 16$, and $\sqrt{16} = 4$ in. Ans.

EXAMPLE 2.—If from a church steeple which is 150 feet high a rope is to be attached to the top and to a stake in the ground, which is 85 feet from the center of the base (the ground being supposed to be level), what must be the length of the rope?

SOLUTION.—In Fig. 29, $A\ B$ represents the steeple, 150 feet high; C a stake 85 feet from the foot of the steeple, and $A\ C$ the rope. Here we have a right triangle right-angled at B, and $A\ C$ is the hypotenuse. The square of $A\ B = 150^2 = 22{,}500$; of $C\ B$, $85^2 = 7{,}225$. $22{,}500 + 7{,}225 = 29{,}725$; $\sqrt{29{,}725} = 172.4$ ft., nearly. Ans.

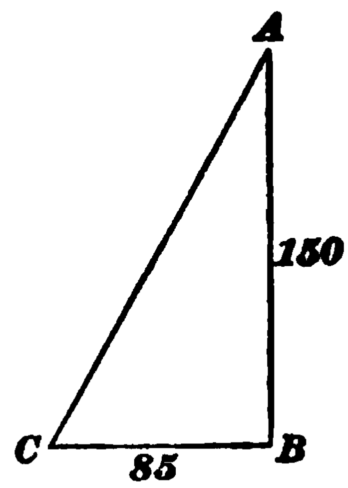


FIG. 29.

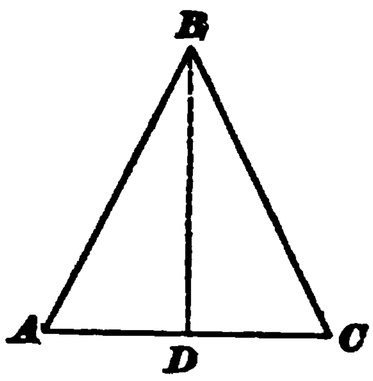


FIG. 30.

60. The **altitude** of any triangle is a line, as $B\ D$, drawn from the vertex B of the angle opposite the base $A\ C$, perpendicular to the base, as in Fig. 30, or to the base extended, as in Fig. 31.

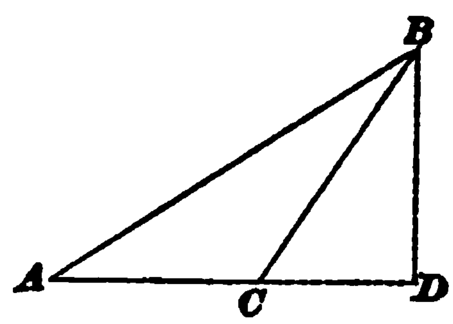


FIG. 31.

61. If in any parallelogram a straight line, called the **diagonal**, be drawn, connecting two opposite corners, it will divide the parallelogram into two equal triangles, as $A\ D\ B$ and $D\ B\ C$ in Fig. 32. The area of each triangle will equal one-half the area of the parallelogram, i. e., one-half the product of the base and the altitude. Hence, to find the area of any triangle:

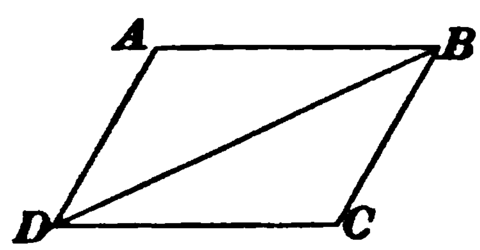


FIG. 32.

Rule 8.—*Multiply the base by the altitude and divide the product by 2.*

EXAMPLE.—What is the area in square feet of a triangle whose base is 18 feet and whose altitude is 7 feet 9 inches?

SOLUTION.— 7 ft. 9 in. $= 7\frac{3}{4}$ ft. $= \frac{31}{4}$ ft. $18 \times \frac{31}{4} = 139\frac{1}{2}$, and one-half of $139\frac{1}{2} = 69\frac{3}{4}$ sq. ft. Ans.

To find the altitude or base of a triangle, having given the area and the base or altitude:

Rule 9.—*Multiply the area by 2 and divide by the given dimension.*

EXAMPLE.—What must be the height of a triangular piece of sheet metal to contain 100 square inches, if the base is 10 inches long?

SOLUTION.— $100 \times 2 = 200$; $200 \div 10 = 20$ in. Ans.

EXAMPLES FOR PRACTICE.

1. What is the area of a triangle whose base is 18 feet long and whose altitude is 10 feet 6 inches? Ans. 94.5 sq. ft.

2. Two angles of a scalene triangle together equal 100° 4′. What is the size of the third angle? Ans. 79° 56′.

3. One angle of a right triangle equals 20° 10′ 5″. What is the size of the other acute angle? Ans. 69° 49′ 55″.

4. A ladder 65 feet long reaches to the top of a wall when its foot is 25 feet from the wall. How high is the wall? Ans. 60 ft.

5. Draw a triangle, and through two of its sides draw a line parallel to the base. Letter the different lines, and then, without referring to the text, write out the proportions existing between the sides of the two triangles.

6. A triangular piece of sheet metal weighs 24 pounds. If the base of the triangle is 4 feet and its height 6 feet, how much does the metal weigh per square foot? Ans. 2 lb.

7. The area of a triangle is 16 square inches. If the altitude is 4 inches, what does the base measure? Ans. 8 in.

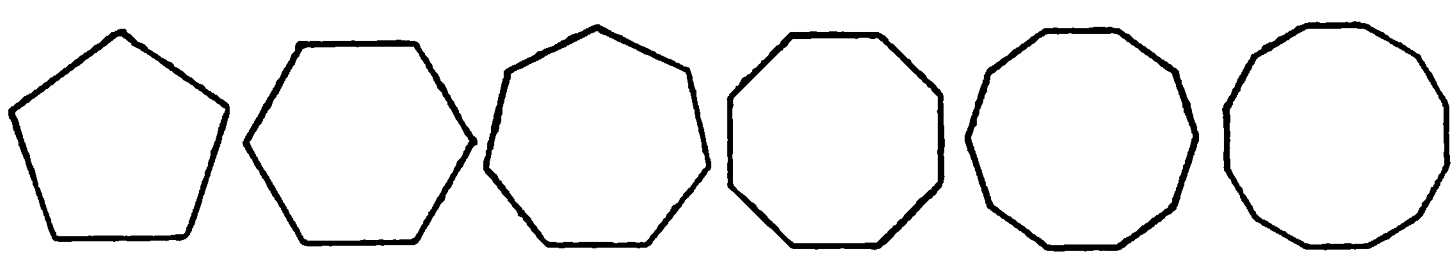
8. Two sides of a right triangle are 92 feet and 69 feet long. How long is the hypotenuse? Ans. 115 ft.

POLYGONS.

62. A **polygon** is a plane figure bounded by straight lines. The term is usually applied to a figure having more than four sides. The bounding lines are called the **sides**, and the sum of the lengths of all the sides is called the **perimeter** of the polygon.

63. A **regular polygon** is one in which all the sides and all the angles are equal.

64. A polygon of five sides is called a **pentagon**; one of six sides, a **hexagon**; one of seven sides, a **heptagon**,



Pentagon. Hexagon. Heptagon. Octagon. Decagon. Dodecagon.

FIG. 33.

etc. Regular polygons having from five to twelve sides are

shown in Fig. 33. In any polygon, the sum of all the interior angles, as $A+B+C+D+E$, Fig. 34, equals 180° multiplied by a number which is two less than the number of sides in the polygon. Hence, to find the size of any one of the interior angles of a *regular* polygon:

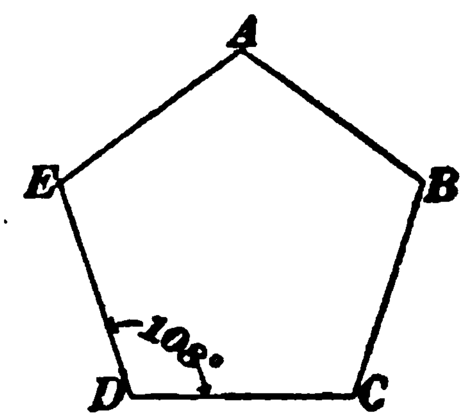


FIG. 34.

Rule 10.—*Multiply 180° by the number of sides less two and divide the result by the number of sides; the quotient will be the number of degrees in each interior angle.*

EXAMPLE 1.—If Fig. 34 is a regular pentagon, how many degrees are there in each interior angle?

SOLUTION.—In a pentagon there are five sides; hence, $5-2=3$ and $180 \times 3 = 540$; $540 \div 5 = 108°$ in each angle. Ans.

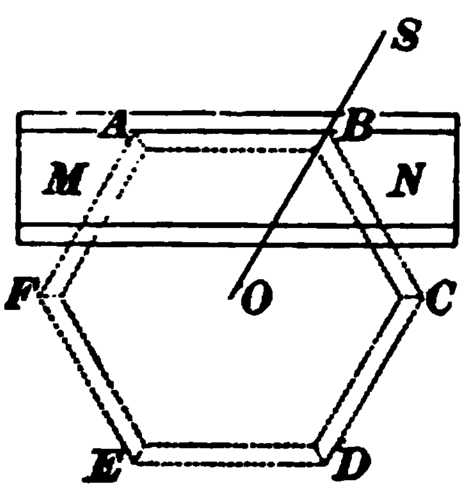


FIG. 35.

EXAMPLE 2.—It is desired to make a miter-box in which to cut a strip of molding to fit around a column having the shape of a regular hexagon. At what angle should the saw run across the miter-box?

SOLUTION.—In Fig. 35, let $A\,B$, $B\,C$, $C\,D$, etc. represent the pieces of molding as they will fit around the column. First find the size of one of the equal angles of the polygon by the above rule. Number of sides $=6$; $6-2=4$; hence, $180 \times 4 = 720$, and $720 \div 6 = 120°$ in each angle. Now, let $M\,N$ represent the miter-box and $O\,S$ the direction in which the saw should run; then, $A\,B\,O$ is the angle made by the saw with the side of the miter-box; but as the polygon is a regular one, this angle is one-half the interior angle $A\,B\,C$, which we have found to be 120°. Hence, the saw should run at an angle of $\frac{120}{2} = 60°$ with the side of the miter-box. Ans.

65. The area of any regular polygon may be found by drawing lines from the center to each angle and computing the area of each triangle thus formed. Hence, to find the area of any regular polygon:

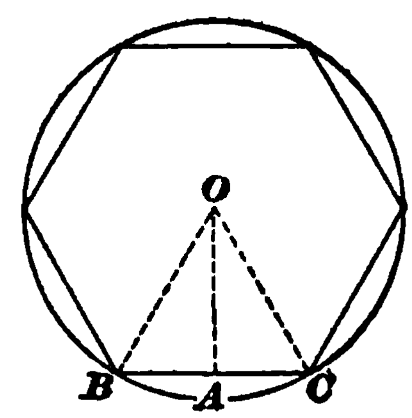


FIG. 36.

Rule 11.—*Multiply the length of a side by half the distance from the side to the center, and that product by the number of sides. The last product will be the area of the figure.*

EXAMPLE.—In Fig. 36 the side $B C$ of the regular hexagon is 12 inches and the distance $A O$ is 10.4 inches; required the area of the polygon.

SOLUTION.— $10.4 \div 2 = 5.2$; $12 \times 5.2 \times 6 = 374.4$ sq. in. Ans.

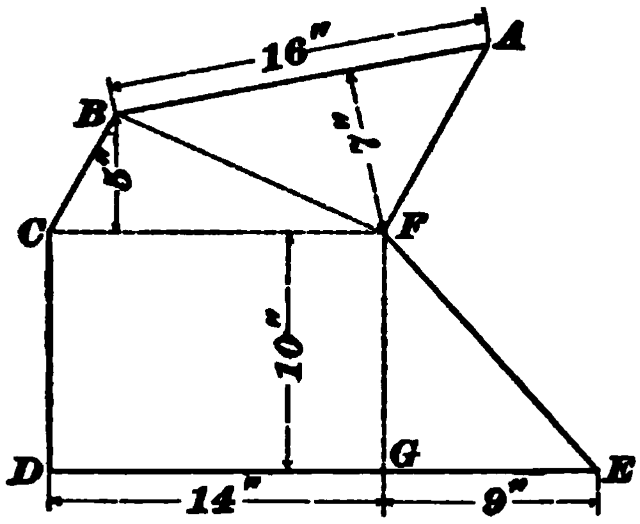


FIG. 37.

66. To obtain the area of any irregular polygon, draw diagonals dividing the polygon into triangles and quadrilaterals, and compute the areas of these separately; their sum will be the area of the figure.

EXAMPLE.—It is required to find the area of the polygon $A B C D E F$, Fig. 37.

SOLUTION.—Draw the diagonals $B F$ and $C F$ and the line $F G$ perpendicular to $D E$, dividing the figure into the triangles $A B F$, $B C F$, and $F G E$ and the rectangle $F C D G$. Let it be supposed that the altitudes of the figures and the lengths of the sides $A B$, $D G$, and $G E$ are as indicated in the polygon above. Then,

$$\text{Area } A B F = \frac{16 \times 7}{2} = 56 \text{ sq. in.}$$

$$\text{Area } B C F = \frac{14 \times 5}{2} = 35 \text{ sq. in.}$$

$$\text{Area } F C D G = 14 \times 10 = 140 \text{ sq. in.}$$

$$\text{Area } F G E = \frac{9 \times 10}{2} = 45 \text{ sq. in.}$$

Total area $= 56 + 35 + 140 + 45 = 276$ sq. in. Ans.

EXAMPLES FOR PRACTICE.

1. How many degrees are there in one of the angles of a regular octagon? Ans. 135°.

2. Find the area of the polygon $A B C D E F$ (see Fig. 37), supposing each of the given dimensions to be increased to $1\frac{1}{2}$ times the length given in the figure. Ans. 621 sq. in.

3. What is the area of a regular heptagon whose sides are 4 inches long, the distance from one side to the center being 4.15 inches? Ans. 58.1 sq. in.

4. At what angle should the saw run in a miter-box to cut strips to fit around the edge of a table top made in the shape of a regular pentagon? Ans. 54°.

THE CIRCLE.

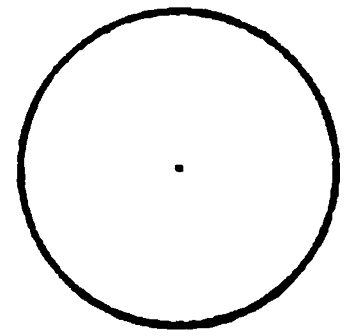


FIG. 38.

67. A **circle** (Fig. 38) is a figure bounded by a curved line, called the **circumference,** every point of which is equally distant from a point within, called the **center.**

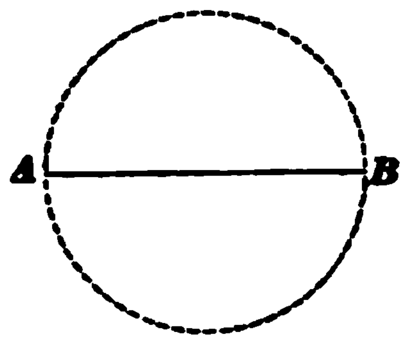


FIG. 39.

68. The **diameter** of a circle is a straight line passing through the center and terminated at both ends by the circumference; thus, *A B* (Fig. 39) is a diameter of the circle.

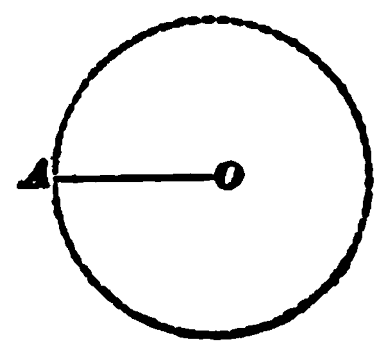


FIG. 40.

69. The **radius** of a circle, *A O* (Fig. 40), is a straight line drawn from the center *O* to the circumference. It is equal in length to one-half the diameter. The plural of radius is **radii,** and all radii of a circle are equal.

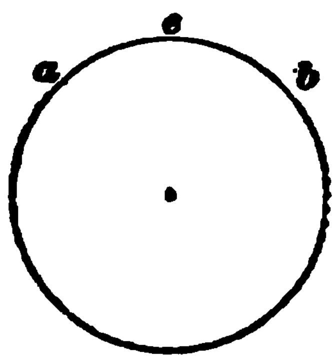


FIG. 41.

70. An **arc** of a circle (see *a e b*, Fig. 41) is any part of its circumference.

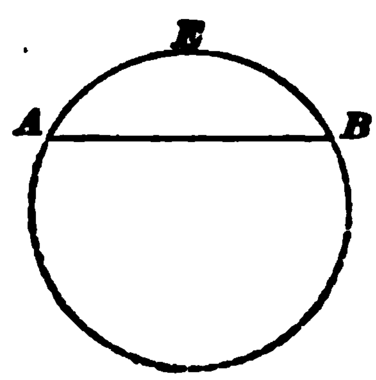


FIG. 42.

71. A **chord** is a straight line joining any two points in a circumference; or it is a straight line joining the extremities of an arc; thus, the straight line *A B*, Fig. 42, is a chord of the circle whose corresponding arc is *A E B*.

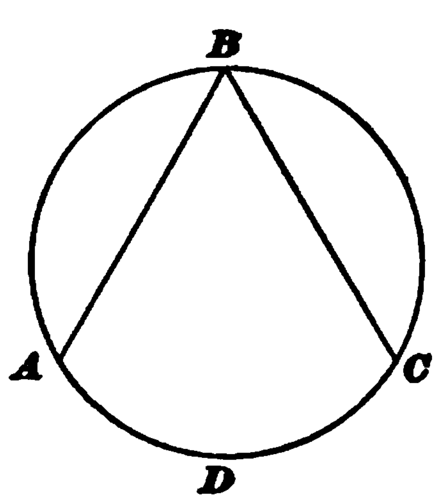


FIG. 43.

72. An **inscribed angle** is one whose vertex lies on the circumference of a circle and whose sides are chords. It is measured by one-half the intercepted arc. Thus, in Fig. 43, *A B C* is an inscribed angle, and it is measured by one-half the arc *A D C*.

EXAMPLE.—If in Fig. 43, the arc $A\ D\ C = \frac{2}{5}$ of the circumference, what is the measurement of the inscribed angle $A\ B\ C$?

SOLUTION.—Since the angle is an inscribed angle, it is measured by one-half the intercepted arc, or $\frac{2}{5} \times \frac{1}{2} = \frac{1}{5}$ of the circumference. The whole circumference $= 360°$; hence, $360° \times \frac{1}{5} = 72°$; therefore, angle $A\ B\ C$ is an angle of 72°.

73. If a circle is divided into halves, each half is called a **semicircle,** and each half circumference is called a **semi-circumference.**

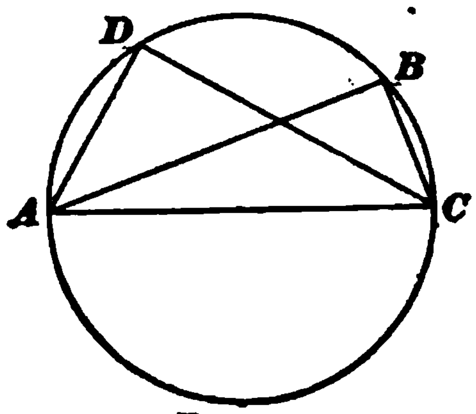


FIG. 44.

Any angle inscribed in a semicircle is a right angle, since it is measured by one-half a semi-circumference, or $180° \div 2 = 90°$. Thus, the angles $A\ D\ C$ and $A\ B\ C$, Fig. 44, are right angles, since they are inscribed in a semicircle.

74. An **inscribed** polygon is one whose vertexes lie on the circumference of a circle and whose sides are chords, as $A\ B\ C\ D\ E$, Fig. 45.

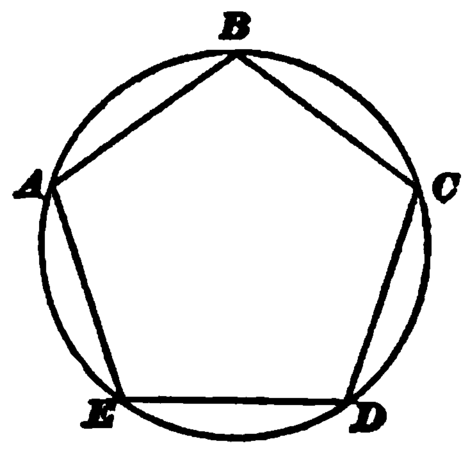


FIG. 45.

The sides of an inscribed regular hexagon have the same length as the radius of the circle.

If, in any circle, a radius be drawn perpendicular to any chord, it bisects (cuts in halves) the chord. Thus, if the radius $O\ C$, Fig. 46, is perpendicular to the chord $A\ B$, $A\ D = D\ B$.

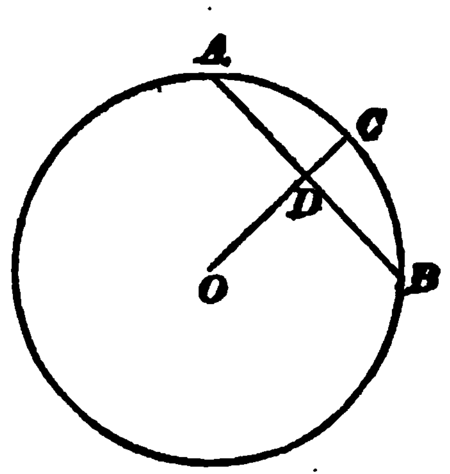


FIG. 46.

EXAMPLE.—If a regular pentagon is inscribed in a circle and a radius is drawn perpendicular to one of the sides, what are the lengths of the two parts of the side, the perimeter of the pentagon being 27 inches?

SOLUTION.—A pentagon has five sides, and since it is a regular pentagon, all the sides are of equal lengths; the perimeter of the pentagon, which equals the distance around it, or equals the sum of all the sides, is 27 inches. Therefore, the length of one side $= 27 \div 5 = 5\frac{2}{5}$ inches. Since the pentagon is an inscribed pentagon, its sides are chords, and as a radius perpendicular to a chord bisects it, we have $5\frac{2}{5} \div 2 = 2\frac{7}{10}$ inches, which equals the length of each of the parts of the side cut by a radius perpendicular to it. Ans.

75. If, from any point on the circumference of a circle, a perpendicular is let fall upon a diameter, it will divide the diameter into two parts, one of which will be in the same ratio to the perpendicular as the perpendicular is to the other part. That is, the perpendicular will be a *mean proportional* between the two parts of the diameter.

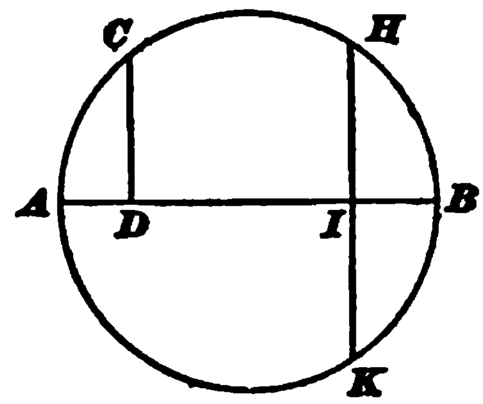


FIG. 47.

If $A\,B$, Fig. 47, is the given diameter and C any point on the circumference, then $A\,D : C\,D = C\,D : D\,B$, $C\,D$ being a mean proportional between $A\,D$ and $D\,B$.

EXAMPLE.—If $H\,K = 30$ feet and $I\,B = 8$ feet, what is the diameter of the circle, $H\,K$ being perpendicular to $A\,B$?

SOLUTION.— 30 feet ÷ 2 feet = 15 feet = $I\,H$. And $B\,I : I\,H = I\,H : I\,A$, or $8 : 15 = 15 : I\,A$.

Therefore, $I\,A = \frac{15^2}{8} = \frac{225}{8} = 28\frac{1}{8}$ feet and $I\,A + I\,B = 28\frac{1}{8} + 8 = 36\frac{1}{8}$ feet $= A\,B$, the diameter of the circle. Ans.

76. When the diameter of a circle and the lengths of the two parts into which it is divided are given, the length of the perpendicular may be found by multiplying the lengths of the two parts together and extracting the square root of the product.

EXAMPLE.—In Fig. 47, the diameter of the circle $A\,B$ is $36\frac{1}{8}$ feet and the distance $B\,I$ is 8 feet; what is the length of the line $H\,K$?

SOLUTION.—As the diameter of the circle is $36\frac{1}{8}$ feet and as $B\,I$ is 8 feet, $I\,A$ is equal to $36\frac{1}{8} - 8 = 28\frac{1}{8}$ feet. The two parts, therefore, are 8 and $28\frac{1}{8}$ feet, and their product $= 8 \times 28\frac{1}{8} = 8 \times \frac{225}{8} = 225$; the square root of their product $= \sqrt{225} = 15$ feet, and as $H\,K = I\,H + I\,K$, or $2\,I\,H$, $H\,K = 15 \times 2 = 30$ ft. Ans.

77. To find the circumference of a circle, the diameter being given:

Rule 12.—*Multiply the diameter by 3.1416.*

EXAMPLE.—What is the circumference of a circle whose diameter is 15 inches?

SOLUTION.— $15 \times 3.1416 = 47.124$ in. Ans.

78. To find the diameter of a circle, the circumference being given:

Rule 13.—*Divide the circumference by 3.1416.*

EXAMPLE.—What is the diameter of a circle whose circumference is 65.973 inches?

SOLUTION.— $65.973 \div 3.1416 = 21$ in. Ans.

79. To find the length of an arc of a circle:

Rule 14.—*Multiply the length of the circumference of the circle of which the arc is a part by the number of degrees in the arc and divide by 360.*

EXAMPLE.—What is the length of an arc of 24°, the radius of the arc being 18 inches?

SOLUTION.— $18 \times 2 = 36$ in. = the diameter of the circle. $36 \times 3.1416 = 113.1$ in., the circumference of the circle of which the arc is a part.

$113.1 \times \frac{24}{360} = 7.54$ in., or the length of the arc. Ans.

80. To find the area of a circle:

Rule 15.—*Square the diameter and multiply by .7854.*

EXAMPLE.—What is the area of a circle whose diameter is 15 inches?

SOLUTION.— $15^2 = 225$; and $225 \times .7854 = 176.72$ sq. in. Ans.

81. Given the area of a circle, to find its diameter:

Rule 16.—*Divide the area by .7854 and extract the square root of the quotient.*

EXAMPLE 1.—The area of a circle = 17,671.5 square inches. What is its diameter in feet?

SOLUTION.— $\sqrt{\frac{17,671.5}{.7854}} = 150$ inches.

$\frac{150}{12} = 12\frac{1}{2}$ feet, or the diameter. Ans.

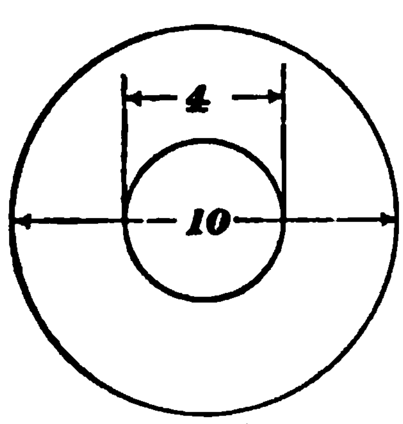


FIG. 48.

EXAMPLE 2.—What is the area of a flat circular ring, Fig. 48, whose outside diameter is 10 inches and inside diameter is 4 inches?

SOLUTION.—The area of the large circle $= 10^2 \times .7854 = 78.54$ sq. in.; the area of the small circle $= 4^2 \times .7854 = 12.57$ sq. in. The area of the ring is the difference between these areas, or $78.54 - 12.57 = 65.97$ sq. in. Ans.

82. To find the area of a sector (a **sector** of a circle is the area included between two radii and the circumference, as, for example, the area $BACO$, Fig. 36):

Rule 17.—*Divide the number of degrees in the arc of the sector by 360. Multiply the result by the area of the circle of which the sector is a part.*

EXAMPLE.—The number of degrees in the angle formed by drawing radii from the center of a circle to the extremities of the arc of the circle is 75°. The diameter of the circle is 12 inches; what is the area of the sector?

SOLUTION.— $\frac{75}{360} = \frac{5}{24}$; and $12^2 \times .7854 = 113.1$ sq. in.

$$113.1 \times \frac{5}{24} = 23.56 \text{ sq. in., the area. Ans.}$$

83. To find the area of a segment of a circle (a **segment** of a circle is the area included between a chord and its arc; for example, the area *A B C*, Fig. 49) when its chord and height are given. There is no exact method, except by applying principles of trigonometry. The following rule gives results that are exact enough for practical purposes.

Rule 18.—*Divide the diameter by the height of the segment; subtract .608 from the quotient and extract the square root of the remainder. This result multiplied by 4 times the square of the height of the segment and then divided by 3 will give the area, very nearly.*

The rule, expressed as a formula, is as follows, where D = the diameter of the circle and h = the height of the segment (see Fig. 49):

$$\text{Area of } A\,B\,C\,A = \frac{4\,h^2}{3}\sqrt{\frac{D}{h} - .608}.$$

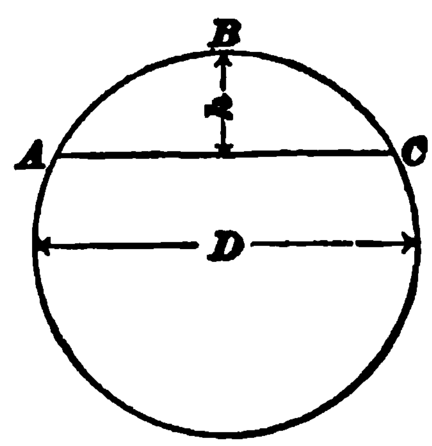


FIG. 49.

EXAMPLE.—What is the area of the segment of a circle whose diameter is 54 inches, the height of the segment being 20 inches?

SOLUTION.—Substituting in the formula,

$$\text{Area} = \frac{4 \times 20^2}{3}\sqrt{\frac{54}{20} - .608}. \quad \frac{4 \times 20^2}{3} = \frac{4 \times 400}{3}$$

$$= \frac{1{,}600}{3};\ \sqrt{\frac{54}{20} - .608} = \sqrt{2.092} = 1.447;\ \frac{1{,}600}{3} \times 1.447 = 771.7 \text{ sq. in.}$$

Ans.

NOTE.—Had the chord *A C*, Fig. 49, been given instead of the diameter, the diameter would have been found as explained in Art. **75.**

EXAMPLES FOR PRACTICE.

1. An angle inscribed in a circle intercepts one-third of the circumference. How many degrees are there in the angle? Ans. 60°.

2. Suppose that in Fig. 47, the diameter $AB = 15$ feet and the distance $BI = 3$ feet. What is the length of the line HK? Ans. 12 ft.

3. The diameter of a flywheel is 18 feet. What is the distance around it to the nearest 16th of an inch? Ans. 56 ft. $6\frac{9}{16}$ in.

4. A carriage wheel was observed to make $71\frac{5}{8}$ turns while going 300 yards. What was its diameter? Ans. 4 ft., nearly.

5. What is the length of an arc of 64°, the radius of the arc being 30 inches? Ans. 33.51 in.

6. Find the area of a circle 2 feet 3 inches in diameter. Ans. 3.976 sq. ft.

7. What must be the diameter of a circle to contain 100 square inches? Ans. 11.28 in.

8. Compute the area of a segment whose height is 11 inches and the radius of whose arc is 21 inches. Ans. 289.04 sq. in.

9. Find the area of a flat circular ring whose outside diameter is 12 inches and whose inside diameter is 6 inches. Ans. 84.82 sq. in.

THE PRISM AND CYLINDER.

84. A **solid,** or body, has three dimensions: length, breadth, and thickness. The sides which enclose it are called the **faces,** and their lines of intersection are called the **edges.**

85. A **prism** is a solid whose ends are equal and parallel polygons and whose sides are parallelograms. Prisms take their names from the form of their bases. Thus, a triangular prism is one having a triangle for its base; a hexagonal prism is one having a hexagon for its base, etc.

86. A **cylinder** is a body of uniform diameter whose ends are equal parallel circles.

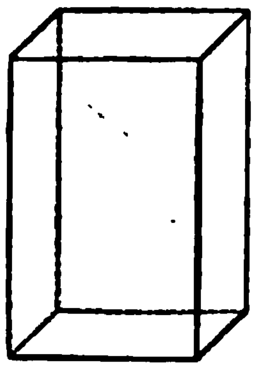


Fig. 50.

87. A **parallelopipedon** (Fig. 50) is a prism whose bases (ends) are parallelograms.

88. A **cube** (Fig. 51) is a prism whose faces and ends are squares. All the faces of a cube are equal.

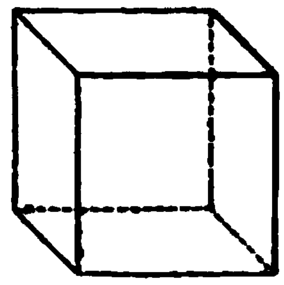


Fig. 51.

In the case of plane figures, we are concerned with perimeters and areas. In the case of solids, we are

concerned with the areas of their outside surfaces and with their contents or volumes.

89. The **entire surface** of any solid is the area of the whole outside of the solid, including the ends.

The **convex surface** of a solid is the same as the entire surface, except that the areas of the ends are not included.

90. A **unit of volume** is a cube each of whose edges is equal in length to the unit. The **volume** is expressed by the number of times it will contain a *unit of volume.*

Thus, if the unit of length is 1 inch, the unit of volume will be the cube whose edges each measure 1 inch, this cube being 1 *cubic inch;* and the number of cubic inches the solid contains will be its volume. If the unit of length is 1 foot, the unit of volume will be 1 *cubic foot*, etc. Cubic inch, cubic foot, and cubic yard are abbreviated to cu. in., cu. ft., and cu. yd., respectively.

Instead of the word *volume*, the expression **cubical contents** is sometimes used.

91. To find the area of the convex surface of a prism or cylinder:

Rule 19.—*Multiply the perimeter of the base by the altitude.*

EXAMPLE 1.—A block of marble is 24 inches long and its ends are 9 inches square. What is the area of its convex surface?

SOLUTION.— $9 \times 4 = 36 =$ the perimeter of the base; $36 \times 24 = 864$ sq. in., the convex area. Ans.

To find the entire area of the outside surface, add the areas of the two ends to the convex area. Thus, the area of the two ends $= 9 \times 9 \times 2 = 162$ sq. in.; $864 + 162 = 1{,}026$ sq. in. Ans.

EXAMPLE 2.—How many square feet of sheet iron will be required for a pipe $1\frac{1}{2}$ feet in diameter and 10 feet long, neglecting the amount necessary for lapping?

SOLUTION.—The problem is to find the convex surface of a cylinder $1\frac{1}{2}$ feet in diameter and 10 feet long. The perimeter, or circumference, of the base $= 1\frac{1}{2} \times 3.1416 = 1.5 \times 3.1416 = 4.712$ ft. The convex surface $= 4.712 \times 10 = 47.12$ sq. ft. of metal. Ans.

92. To find the volume of a prism or a cylinder:

Rule 20.—*Multiply the area of the base by the altitude.*

EXAMPLE 1.—What is the weight of a length of wrought-iron shafting 16 feet long and 2 inches in diameter? Wrought iron weighs .28 pound per cubic inch.

SOLUTION.—The shaft is a cylinder 16 ft. long. The area of one end, or the base, $= 2^2 \times .7854 = 3.1416$ sq. in. Since the weight of the iron is given per cubic inch, the contents of the shaft must be found in cubic inches. The length, 16 ft., reduced to inches $= 16 \times 12 = 192$ in.; $3.1416 \times 192 = 603.19$ cu. in. = the volume. The weight $= 603.19 \times .28 = 168.89$ lb. Ans.

EXAMPLE 2.—Find the cubical contents of a hexagonal prism, Fig. 52, 12 inches long, each edge of the base being 1 inch long.

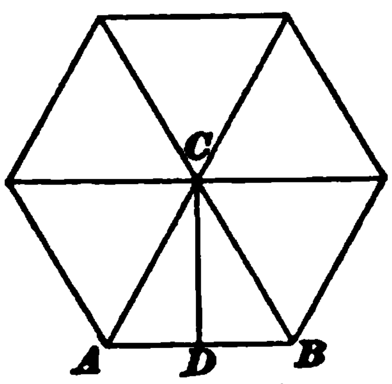


FIG. 52.

SOLUTION.—In order to obtain the area of one end, the distance $C D$ from the center C to one side must be found.

In the right triangle $C D A$, side $A D = \frac{1}{2} A B$, or $\frac{1}{2}$ inch, and since the polygon is a hexagon, side $C A$ = distance $A B$, or 1 inch (Art. **74**). Hence, $C A$ being the hypotenuse, the length of side $C D = \sqrt{1^2 - (\frac{1}{2})^2} = \sqrt{1^2 - .5^2} = \sqrt{.75}$, or .866 inch. Area of triangle $A C B = \frac{1 \times .866}{2} = .433$ sq. in.; area of the whole polygon $= .433 \times 6 = 2.598$ sq. in. Hence, the contents of the prism $= 2.598 \times 12 = 31.176$ cu. in. Ans.

EXAMPLE 3.—It is required to find the number of cubic feet of steam space in the boiler shown in Fig. 53. The boiler is 16 feet long between

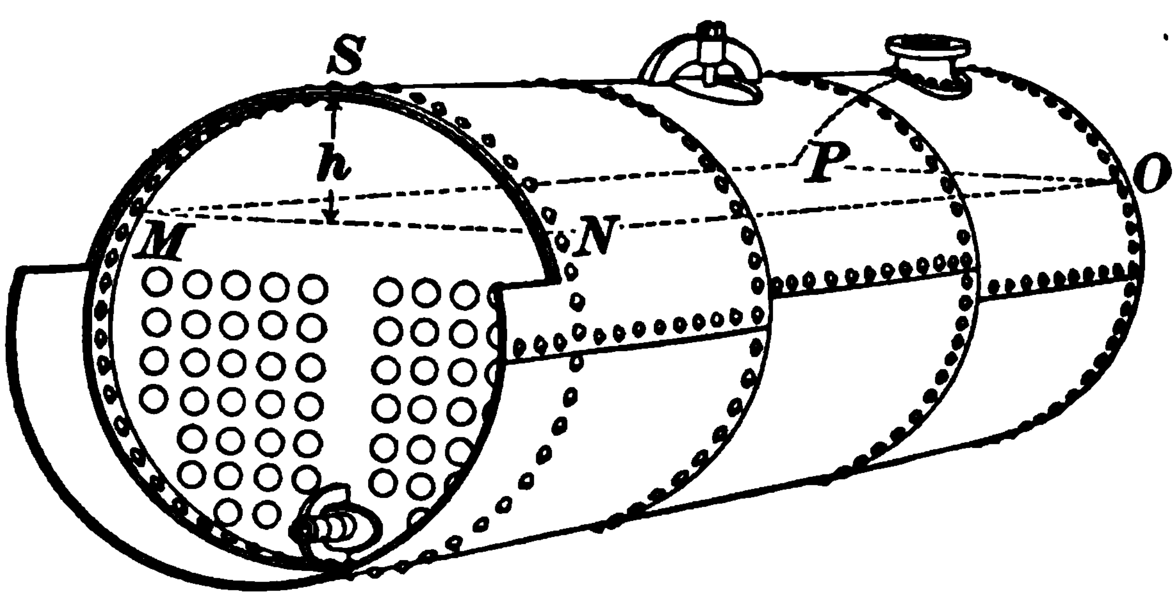


FIG. 53.

heads, 54 inches in diameter, and the mean water-line $M N$ is at a distance of 16 inches from the top of the boiler. The volume of the steam outlet casting may be neglected.

SOLUTION.—The volume of the steam space, which is that space within the boiler above the surface $M N O P$ of the water, is found by the rule for finding the volume of a prism or cylinder, the area $M N S$ being the base and the length $N O$ the altitude. First obtain the area of the segment $M N S$, whose height h is 16 inches, in square feet; then multiply the result by 16, the length of the boiler.

By the formula given in Art. **83**, the area of the segment =

$$\frac{4\,h^2}{3}\sqrt{\frac{D}{h} - .608} = \frac{4 \times 16^2}{3}\sqrt{\frac{54}{16} - .608}.$$

$$\frac{4 \times 16^2}{3} = 341.33;\ \sqrt{\frac{54}{16} - .608} = \sqrt{2.767} = 1.663.$$

Hence, the area $= 341.33 \times 1.663 = 567.63$ sq. in. This reduced to sq. ft. $= 567.63 \div 144 = 3.942$ sq. ft., and the volume therefore $= 3.942 \times 16 = 63.07$ cu. ft. Ans.

In the above solution, the space occupied by the stays is not considered, for sake of simplicity. They are not shown in the figure.

EXAMPLE 4.—In the above boiler there are 60 tubes, $3\frac{1}{4}$ inches outside diameter. How many gallons of water will it take to fill the boiler up to the mean water level, there being 231 cubic inches in a gallon?

SOLUTION.—Find the volume in cubic inches of that part of the boiler below the surface of the water $M\,N\,O\,P$, since the contents of a gallon is given in cubic inches, and from it subtract the volume of the tubes in cubic inches.

This may be done by first finding the *total* area of one end of the boiler in square inches, from it subtracting the area of the segment $M\,N\,S$, and the areas of the ends of the tubes in square inches, and then by multiplying the result by the length of the boiler *in inches*.

Total area of one end $= 54^2 \times .7854 = 2{,}290.23$ sq. in.

Area of segment $M\,N\,S$, as found in last example, $= 567.63$ sq. in.

Area of the end of one tube $= 3.25^2 \times .7854 = 8.2958$ sq. in.

Area of the ends of the 60 tubes $= 8.2958 \times 60 = 497.75$ sq. in.

Hence, the area to be subtracted $= 567.63 + 497.75 = 1{,}065.38$ sq. in. Subtracting, $2{,}290.23 - 1{,}065.38 = 1{,}224.85$ sq. in. = net area.

The cubical contents $= 1{,}224.85 \times 16 \times 12 = 235{,}171.2$ cu. in. This divided by 231 will give the number of gallons; whence $235{,}171.2 \div 231 = 1{,}018.06$ gal. of water. Ans.

EXAMPLES FOR PRACTICE.

1. Find the area in square inches of the convex surface of a bar of iron $4\frac{1}{4}$ inches in diameter and 8 feet 5 inches long. Ans. 1,348.53 sq. in.

2. Find the area of the entire surface of the above bar.
Ans. 1,376.9 sq. in.

3. What is the area of the entire surface of the hexagonal prism whose base is shown in Fig. 52? Ans. 77.196 sq. in.

4. A multitubular boiler has the following dimensions: diameter, 50 inches; length between heads, 15 feet; number of tubes, 56; outside diameter of tubes, 3 inches; distance of mean water-line from top of boiler, 16 inches. (*a*) Compute the steam space in cubic feet. (*b*) Find the number of gallons of water required to fill the boiler up to the mean water-line.

Ans. $\begin{cases} (a) & 56.4 \text{ cu. ft.} \\ (b) & 800 \text{ gal.} \end{cases}$

THE PYRAMID AND CONE.

93. A **pyramid** (Fig. 54) is a solid whose base is a polygon and whose sides are triangles uniting at a common point, called the **vertex.**

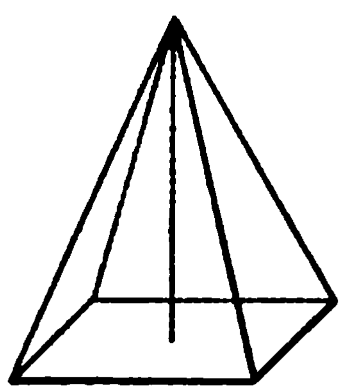


FIG. 54.

94. A **cone** (Fig. 55) is a solid whose base is a circle and whose convex surface tapers uniformly to a point called the **vertex.**

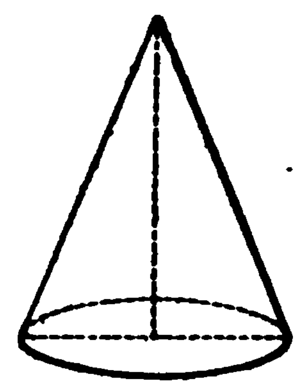


FIG. 55.

95. The **altitude** of a pyramid or cone is the perpendicular distance from the vertex to the base.

96. The **slant height** of a *pyramid* is a line drawn from the vertex perpendicular to one of the sides of the base. The slant height of a *cone* is any straight line drawn from the vertex to the circumference of the base.

97. To find the convex area of a pyramid or cone:

Rule 21.—*Multiply the perimeter of the base by one-half the slant height.*

EXAMPLE 1.—What is the convex area of a pentagonal pyramid if one side of the base measures 6 inches and the slant height is 14 inches?

SOLUTION.—The base of a pentagonal pyramid is a pentagon, and, consequently, has fives sides.

$6 \times 5 = 30$ inches, or the perimeter of the base. $30 \times \frac{14}{2} = 210$ sq. in., or the convex area. Ans.

EXAMPLE 2.—What is the entire area of a right cone whose slant height is 17 inches and whose base is 8 inches in diameter?

SOLUTION.—The perimeter of the base $= 8 \times 3.1416 = 25.1328$ in.

$$\text{Convex area} = 25.1328 \times \frac{17}{2} = 213.63 \text{ sq. in.}$$
$$\text{Area of base} = 8^2 \times .7854 = \underline{50.27} \text{ sq. in.}$$
$$\text{Entire area} = 263.90 \text{ sq. in. Ans.}$$

98. To find the volume of a pyramid or cone:

Rule 22.—*Multiply the area of the base by one-third of the altitude.*

EXAMPLE 1.—What is the volume of a triangular pyramid, each edge of whose base measures 6 inches and whose altitude is 8 inches?

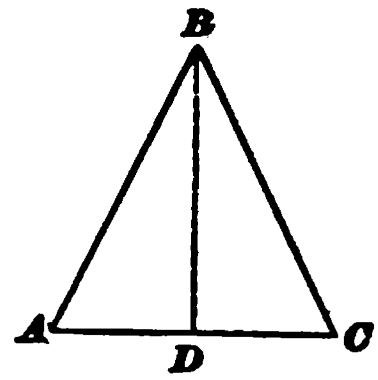


FIG. 56.

SOLUTION.—Draw the base as shown in Fig. 56; it will be an equilateral triangle, all of whose sides are 6 inches long.

Draw a perpendicular BD from the vertex to the base; it will divide the base into two equal parts, since an equilateral triangle is also isosceles, and will be the altitude of the triangle. In order to obtain the area of the base, this altitude must be determined.

In the right triangle BDA, the hypotenuse $BA = 6$ inches and side $AD = 3$ inches, to find the other side,

$$BD = \sqrt{6^2 - 3^2} = 5.2 \text{ in., nearly.}$$

Area of the base, or BAC, $= \frac{6 \times 5.2}{2} = 15.6$ sq. in. Hence, the volume $= 15.6 \times \frac{8}{3} = 41.6$ cu. in. Ans.

EXAMPLE 2.—What is the volume of a cone whose altitude is 18 inches and whose base is 14 inches in diameter?

SOLUTION.—Area of the base $= 14^2 \times .7854 = 153.94$ sq. in. Hence, the volume $= 153.94 \times \frac{18}{3} = 923.64$ cu. in. Ans.

EXAMPLES FOR PRACTICE.

1. Find the convex surface of a square pyramid whose slant height is 28 inches and one edge of whose base is $7\frac{1}{2}$ inches long.
Ans. 420 sq. in.

2. What is the volume of a triangular pyramid, one edge of whose base measures 3 inches and whose altitude is 4 inches? Ans. 5.2 cu. in.

3. Find the volume of a cone whose altitude is 12 inches and the circumference of whose base is 31.416 inches. Ans. 314.16 cu. in.

NOTE.—Find the diameter of the base and then its area.

THE FRUSTUM OF A PYRAMID OR CONE.

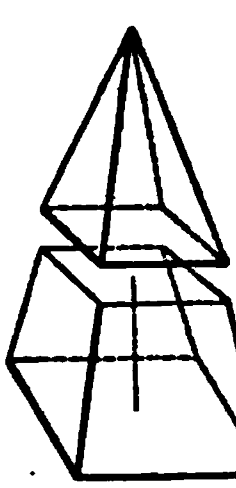


FIG. 57.

99. If a pyramid be cut by a plane, parallel to the base, so as to form two parts, as in Fig. 57, the lower part is called the **frustum** of the pyramid.

If a cone be cut in a similar manner, as in Fig. 58, the lower part is called the **frustum** of the cone.

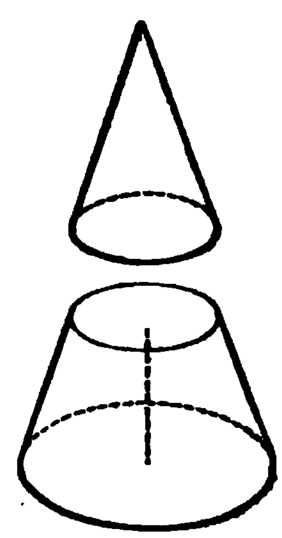


FIG. 58.

100. The upper end of the frustum of a pyramid or cone is called the **upper base,** and the lower end the **lower base.** The **altitude** of a frustum is the perpendicular distance between the bases.

101. To find the convex surface of a frustum of a pyramid or cone:

Rule 23.—*Multiply one-half the sum of the perimeters of the two bases by the slant height of the frustum.*

EXAMPLE 1.—Given, the frustum of a triangular pyramid, in which one side of the lower base measures 10 inches, one side of the upper base measures 6 inches, and whose slant height is 9 inches; find the area of the convex surface.

SOLUTION.— 10 in. $\times$ 3 = 30 in., the perimeter of the lower base. 6 in. $\times$ 3 = 18 in., the perimeter of the upper base. $\frac{30 + 18}{2} = 24$ in., or one-half the sum of the perimeters of the two bases. $24 \times 9 = 216$ sq. in., the convex area. Ans.

EXAMPLE 2.—If the diameters of the two bases of a frustum of a cone are 12 inches and 8 inches, respectively, and the slant height is 12 inches, what is the entire area of the frustum?

SOLUTION.— $\frac{(12 \times 3.1416) + (8 \times 3.1416)}{2} \times 12 = 376.99$ sq. in., the area of the convex surface.

Area of the upper base = $8^2 \times .7854 = 50.27$ sq. in.

Area of the lower base = $12^2 \times .7854 = 113.1$ sq. in.

The entire area of the frustum $= 376.99 + 50.27 + 113.1 = 540.36$ sq. in. Ans.

102. To find the volume of the frustum of a pyramid or cone:

Rule 24.—*Add together the areas of the upper and lower bases and the square root of the product of the two areas; multiply the sum by one-third of the altitude.*

EXAMPLE 1.—Given, a frustum of a square pyramid (one whose base is a square); each edge of the lower base is 12 inches, each edge of the upper base is 5 inches, and its altitude is 16 inches; what is its volume?

SOLUTION.—Area of upper base = $5 \times 5 = 25$ sq. in.; area of lower base = $12 \times 12 = 144$ sq. in.; the square root of the product of the two areas = $\sqrt{25 \times 144} = 60$. Adding these three results, and multiplying by one-third the altitude, $25 + 144 + 60 = 229$; $229 \times \frac{16}{3} = 1{,}221\frac{1}{3}$ cu. in. = the volume. Ans.

EXAMPLE 2.—How many gallons of water will a round tank hold which is 4 feet in diameter at the top, 5 feet in diameter at the bottom, and 8 feet deep?

SOLUTION.—There are 231 cubic inches in a gallon, and the volume of the tank should be found in cubic inches. The tank is in the shape of the frustum of a cone. The upper diameter $= 4 \times 12 = 48$ inches; the lower diameter $= 5 \times 12 = 60$ inches, and the depth $= 8 \times 12 = 96$ inches. Area of upper base $= 48^2 \times .7854 = 1{,}809.56$ sq. in., area of lower base $= 60^2 \times .7854 = 2{,}827.44$ sq. in., $\sqrt{1{,}809.56 \times 2{,}827.44} = 2{,}261.95$.

Whence, $1{,}809.56 + 2{,}827.44 + 2{,}261.95 = 6{,}898.95$; $6{,}898.95 \times \frac{96}{3} = 220{,}766.4$ cu. in. = contents. Now, since there are 231 cu. in. in 1 gallon, the tank will hold $220{,}766.4 \div 231 = 955.7$ gal., nearly. Ans.

EXAMPLES FOR PRACTICE.

1. Find the convex surface of the frustum of a square pyramid, one edge of whose lower base is 15 inches long, one edge of whose upper base is 14 inches long, and whose slant height is 1 inch. Ans. 58 sq. in.

2. Find the volume of the above frustum, supposing its altitude to be 3 inches. Ans. 631 cu. in.

3. Find the volume of the frustum of a cone whose altitude is 12 feet and the diameters of whose upper and lower bases are 8 and 10 feet, respectively. Ans. 766.55 cu. ft.

4. If a tank had the dimensions of example 3, how many gallons would it hold? Ans. 5,734.2 gal., nearly.

THE SPHERE AND CYLINDRICAL RING.

103. A **sphere** (Fig. 59) is a solid bounded by a uniformly curved surface, every point of which is equally distant from a point within, called the center.

The word **ball,** or **globe,** is generally used instead of sphere.

FIG. 59.

104. To find the area of the surface of a sphere:

Rule 25.—*Square the diameter and multiply the result by 3.1416.*

EXAMPLE.—What is the area of the surface of a sphere whose diameter is 14 inches?

SOLUTION.—Diameter squared $\times 3.1416 = 14^2 \times 3.1416 = 14 \times 14 \times 3.1416 = 615.75$ sq. in. Ans.

From this it will be seen that the surface of a sphere equals the circumference of a great circle multiplied by the diameter, a rule often used; a *great circle* of a sphere is the intersection of its surface with a plane passing through its center; for instance, the *great circle* of a sphere 6 inches diameter is a circle of 6 inches diameter. Any number of *great circles* could be described on a given sphere.

105. To find the volume of a sphere:

Rule 26.—*Cube the diameter and multiply the result by .5236.*

EXAMPLE.—What is the weight of a lead ball 12 inches in diameter, a cubic inch of lead weighing .41 pound?

SOLUTION.—Diameter cubed $\times$ .5236 $= 12 \times 12 \times 12 \times .5236 = 904.78$ cu. in., or the volume of the ball. The weight, therefore, $= 904.78 \times .41 = 370.96$ lb. Ans.

106. To find the convex area of a cylindrical ring:

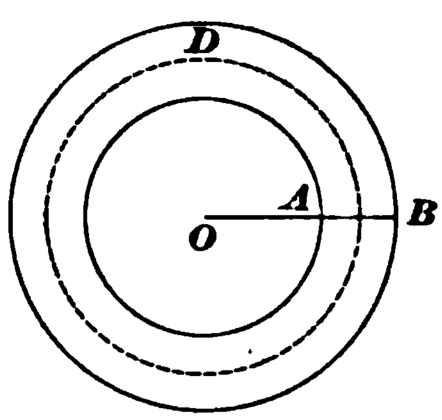


FIG. 60.

A **cylindrical ring** (Fig. 60) is a cylinder bent to a circle. The **altitude** of the cylinder before bending is the same as the length of the dotted center line *D*. The **base** will correspond to a cross-section on the line *A B* drawn from the center *O*. Hence, to find the convex area:

Rule 27.—*Multiply the circumference of an imaginary cross-section on the line A B by the length of the center line D.*

EXAMPLE.—If the outside diameter of the ring is 12 inches and the inside diameter is 8 inches, what is its convex area?

SOLUTION.—The diameter of the center circle equals one-half the sum of the inside and outside diameters $= \frac{12 + 8}{2} = 10$, and $10 \times 3.1416 = 31.416$ in., the length of the center line.

The radius of the inner circle is 4 inches; of the outside circle, 6 inches; therefore, the diameter of the cross-section on the line *A B* is 2 inches. Then, $2 \times 3.1416 = 6.2832$ in., and $6.2832 \times 31.416 = 197.4$ sq. in., the convex area. Ans.

107. To find the volume of a cylindrical ring:

Rule 28.—*The volume will be the same as that of a cylinder whose altitude equals the length of the dotted center line D* (Fig. 61) *and whose base is the same as a cross-section of the ring on the line A B drawn from the center O. Hence, to find the volume of a cylindrical ring, multiply the area of an imaginary cross-section on the line A B by the length of the center line D.*

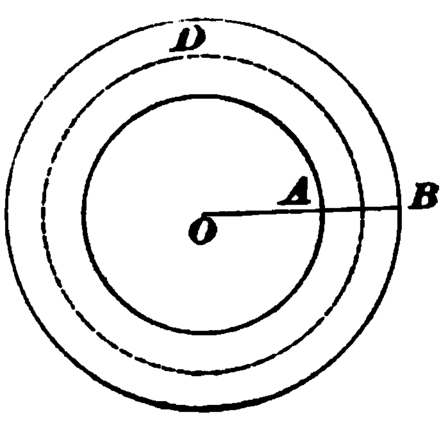


FIG. 61.

EXAMPLE.—What is the volume of a cylindrical ring whose outside diameter is 12 inches and whose inside diameter is 8 inches?

SOLUTION.—The diameter of the center circle equals one-half the sum of the inside and outside diameters $= \frac{12 + 8}{2} = 10$.

$10 \times 3.1416 = 31.416$ inches, the length of the center line.

The radius of the outside circle = 6 inches; of the inside circle = 4 inches; therefore, the diameter of the cross-section on the line $A\ B$ = 2 inches.

Then, $2^2 \times .7854 = 3.1416$ sq. in., the area of the imaginary cross-section.

And $3.1416 \times 31.416 = 98.7$ cu. in., the volume. Ans.

EXAMPLES FOR PRACTICE.

1. What is the volume of a sphere 30 inches in diameter?
Ans. 14,137.2 cu. in.

2. How many square inches in the surface of the above sphere?
Ans. 2,827.44 sq. in.

3. Required the area of the convex surface of a circular ring, the outside diameter of the ring being 10 inches and the inside diameter $7\frac{1}{2}$ inches. Ans. 107.95 sq. in.

4. Find the cubical contents of the ring in the last example.
Ans. 33.73 cu. in.

5. The surface of a sphere contains 314.16 square inches. What is the volume of the sphere? Ans. 523.6 cu. in.

ELEMENTARY ALGEBRA
AND
TRIGONOMETRIC FUNCTIONS.

ELEMENTS OF ALGEBRA.

USE OF LETTERS.

433. In arithmetic, numbers are represented by the figures 1, 2, 3, 4, etc. There is no reason, however, why numbers may not be represented by other symbols, such as letters, if rules are provided for their use.

434. In algebra, numbers are represented by both figures and letters. It will be seen later that the use of letters often simplifies the solution of examples and shortens calculations.

435. The principal advantage of letters is that they are general in their meaning. Thus, unlike figures, the letter a does not stand for the number 1, the letter b for 2, c for 3, etc., but *any* letter may be taken to represent *any* number, it being only necessary that a letter shall always stand for the same number *in the same example.*

436. To illustrate this difference between letters and figures, we may take an example, as follows: If a farmer exchanges 20 bushels of oats, worth 40 cents per bushel, for 8 bushels of wheat, what is the price of the wheat per bushel? A rule for solving this problem, and others like it, would be as follows: *Multiply the number of bushels of oats by the price per bushel, and divide the result by the number of bushels of wheat.*

This rule is *general*, because it tells us what to do with the number of bushels and with the prices of the oats and the wheat, *whatever they may be.*

A more concise way of stating this rule is to use letters in the same manner as in formulas. Thus :

Let $a =$ number of bushels of oats ;
$b =$ price per bushel of oats ;
$c =$ number of bushels of wheat ;
$d =$ price per bushel of wheat.

Then, according to the rule,

$$\frac{\text{bushels of oats} \times \text{price per bushel}}{\text{bushels of wheat}} = \text{price of wheat},$$

or
$$\frac{a \times b}{c} = d.$$

In the example in question, $a = 20$, $b = 40$, and $c = 8$. Hence, writing for a, b, and c their values, 20, 40, and 8, d, the price per bushel of wheat $= \frac{20 \times 40}{8} = 100$. Here the expression $\frac{20 \times 40}{8}$ corresponds to $\frac{a \times b}{c}$, but this difference is to be noticed : $\frac{20 \times 40}{8}$ applies only to *this* example, and by performing the operations indicated only one answer can be obtained, while $\frac{a \times b}{c}$ is *general* in its application, in the same way that the rule previously given is general. That is, while a, b, and c stand for the numbers 20, 40, and 8 in *this* example, they may stand for other numbers in *another* example ; hence, by writing their values in place of the letters, and by performing the operations indicated, the answer to any example of the same kind may be obtained. Consequently, while figures or combinations of figures always represent the same numbers, letters are more general, and may represent any numbers, according to the conditions of the example.

437. An **equation** is a statement of equality between two expressions. Thus, $x + y = 8$ is an equation, and

means that the sum of the numbers represented by x and y is equal to 8. Examples are solved in algebra by the aid of equations, in which numbers are represented both by letters and by figures. The following simple example will give an idea of the method of solution :

EXAMPLE.—If an iron rail 30 feet long is cut in two so that one part is four times as long as the other, how long is the shorter part?

SOLUTION.—Since any letter may represent any number,
Let $x =$ the length of the shorter part.
Then, $4 \times x$ (written $4x$) = the length of the longer part.
But the sum of the two parts must equal the total length, 30 feet.
Hence, $x + 4x = 30$.
Adding x and $4x$, $5x = 30$.
Whence, dividing by 5, $x = 6$ feet. Ans.

438. The student has probably noticed the similarity between an equation and a **formula.** All formulas are equations, and the same rules apply to both. An equation is not called a formula, however, unless it is a statement of a general rule.

439. **Algebra** treats of the equation and its use. Since the use of equations involves the use of letters, it will be necessary before considering equations to take up addition, subtraction, multiplication, etc., of expressions in which letters are used.

NOTATION.

440. The term **quantity** is used to designate any number that is to be subjected to mathematical processes. A quantity is strictly a concrete number ; as, 6 books, 5 pounds, 10 yards. *Symbols* used to *represent* numbers, and expressions containing two or more such symbols, as a, x, bd, 10, $(c + 12)$, etc., are often called *quantities*, the term being a convenient one to use.

441. The **signs** $+$, $-$, $\times$, $\div$ are the same in algebra as in arithmetic. The sign of multiplication $\times$ is usually omitted, however, multiplication being indicated by simply writing the quantities together. Thus, abc means $a \times b \times c$;

$2xy$ means $2 \times x \times y$. Evidently the sign can not be omitted between two *figures*, as addition instead of multiplication would then be indicated. Thus, 24 means $20 + 4$ instead of 2×4.

442. A **coefficient** is a figure or letter prefixed to a quantity ; it shows how many times the latter is to be taken. Thus, in the expression $4a$, 4 is the coefficient of a, and indicates that a is to be taken four times ; that is, $4a$ is equal to $a + a + a + a$. When several quantities are multiplied together, any of them may be regarded as the coefficient of the others. Thus, in $6axy$, 6 is the coefficient of axy; $6a$, of xy; $6ax$, of y, etc. In general, however, when a coefficient is spoken of, the numerical coefficient only is meant, as the 6 above. When no numerical coefficient is written it is understood to be 1. Thus, cd is the same as $1cd$.

443. The **factors** of a quantity are the quantities which, when multiplied together, will produce it. Thus, 2, 3, and 3 are the factors of 18, since $2 \times 3 \times 3 = 18$; 2, a, and b are the factors of $2ab$, since $2 \times a \times b = 2ab$.

444. An **exponent** is a small figure placed at the right and a little above a quantity ; it shows how many times the latter is to be taken *as a factor*. Thus, $4^3 = 4 \times 4 \times 4 = 64$, the exponent 3 showing that the number 4 is to be used three times as a factor ; likewise $a^5 = aaaaa$. Any quantity written without an exponent is understood to have the exponent 1 ; thus, $b^1 = b$.

445. The difference between a coefficient and an exponent should be clearly understood. A coefficient *multiplies* the quantity which it precedes ; it shows that the quantity is to be *added to itself*. Thus, $3a = 3 \times a$, or $a + a + a$. An exponent indicates that a quantity is to be *multiplied by itself*. Thus, $a^3 = a \times a \times a$. A more complete definition of an exponent will be given later.

446. A **power** is the result obtained by taking a quantity two or more times *as a factor*. For example, 16 is

the fourth power of 2, because 2 multiplied by itself until it has been taken four times as a factor produces 16 ; a^3 is the third power of a, because $a \times a \times a = a^3$.

447. A **root** of a quantity is one of its equal factors. Thus, 2 is the root of 4, 8, and 16, since $2 \times 2 = 4$, $2 \times 2 \times 2 = 8$, and $2 \times 2 \times 2 \times 2 = 16$, 2 being one of the equal factors in each case. In like manner, a is a root of a^2, a^3, a^6, etc. The symbol which denotes that the second, or square, root is to be extracted is $\sqrt{\ }$; it is called the **radical sign,** and the quantity under the sign is called the **radical.** For other roots the same symbol is used, but with a figure, called the *index* of the root, written above it to indicate the root. Thus, $\sqrt{a}$, $\sqrt[3]{a}$, $\sqrt[4]{a}$, etc., signify the square root, cube root, fourth root, etc., of a.

448. The use of the parenthesis, bracket, brace, and vinculum is explained in Art. **341.** These symbols are called **symbols of aggregation,** meaning that the quantities enclosed within them are aggregated, or collected, into one quantity.

449. The **terms** of an algebraic expression are those parts which are connected by the signs + and −. Thus, x^2, $-2xy$, and y^2 are terms of the expression $x^2 - 2xy + y^2$. When a term contains both figures and letters, the part consisting of letters is called the **literal** part of the term ; thus, xy is the literal part of the term $2xy$.

450. Like terms are those which differ only in their numerical coefficients ; all others are **unlike terms.** Thus, $2ab^2$ and $5ab^2$ are like terms ; $5ab$ and $5ab^2$ are unlike terms, because one contains b and the other b^2.

451. A **monomial** is an expression consisting of only one term ; as, $4abc$, $3x^2$, $2ax^3$, etc.

452. A **binomial** is an expression consisting of two terms ; as, $a + b$, $2a + 5b$, etc.

453. A **trinomial** is an expression consisting of three terms ; as, $a^2 + 2ab + b^2$, $(a + x)^2 - 2(a + x)y + y^2$, etc., the expression $(a+x)$ being treated as one quantity. (Art. **341.**)

454. A **polynomial** is an expression consisting of more than one term. The name is usually applied only to an expression consisting of four or more terms.

455. The polynomial $a + a^2b + 2a^3 - 3a^4b - a^5$ is said to be arranged according to the *increasing powers* of a, because the exponents of a increase in each term from left to right, the exponent of the first a being 1 understood. (Art. **444.**) The polynomial $a^2b^3 + ab^2 + 4a^4b + 1$ is arranged according to the *decreasing powers* of b, the exponents of b decreasing in order from left to right.

456. The arrangement of the terms of a polynomial does not affect its value. Thus, $x^2 + 2xy + y^2$ has the same value as $2xy + y^2 + x^2$, just as $2 + 6 + 4$ has the same value as $6 + 4 + 2$.

READING ALGEBRAIC EXPRESSIONS.

457. Quantities like a, x, b^2, etc., are read "*a*," "*x*," "*b* square," etc. In reading monomials in which multiplication is indicated, the word "times" is not used. Thus, abc is read "*abc*"; $7ad^2b^3$ is read "7*ad* square *b* cube."

458. The polynomial $a + a^2b + 2x^3 - 3a^4b - a^5$ is read "*a*, plus *a* square *b*, plus 2*a* cube, minus 3*a* fourth *b*, minus *a* fifth." Considerable care is required when reading expressions containing polynomials. Thus, if $4(a - b)$ were read "4*a* minus *b*," the binomial $4a - b$ would be understood. It *should* be read "4 times $a - b$," or "4 times the parenthesis *a* minus *b*," in which case it will be understood that 4 multiplies the whole quantity $a - b$, since the word "times" is not used with monomials. Again, $m(m^2 + 2mn + n^2)$ and $m(m^2 + 2mn) + n^2$ should each be so read that there can be no doubt as to whether the n^2 is to be multiplied by m or not.

Let the distinction to be made in reading the following be observed:

$$\sqrt{\frac{m+n}{x-y^2}} \text{ and } \sqrt{m + \frac{n}{x-y^2}}.$$

In the first case, the whole quantity $m + n$ is divided by $x - y^2$, and it would be clear to say, "the square root of the fraction $m + n$ over $x - y^2$." In the second case, where the n only is divided by $x - y^2$, it may be read, "the square root of the quantity, m plus the fraction n over $x - y^2$." The word "quantity" shows that the square root of the whole expression is taken, and the word "fraction" after "plus" shows that only the n is divided by $x - y^2$.

459. When a polynomial is affected by an exponent, it should be indicated clearly. Thus,

$$(3a - d^2)\,(3a - d)^2\,(3a - d^2)^2$$

should be read, "$3a - d$ square, times the square of $3a - d$, times the square of $3a - d$ square."

460. Sometimes expressions like A', B'', c', d'', C_1, a_2, etc., appear in formulas or elsewhere in algebraic problems when it is desirable to have the same letter represent different quantities that are similar, or correspond to one another. The marks $'$, $''$, $'''$, $_1$, $_2$, etc., serve to distinguish the letters. (See Art. **343.**) The expressions are also used to designate similar or corresponding lines in geometrical figures, as will appear in Mechanical Drawing. A', B'', C''', etc., are read "*a major prime*, *b major second*, *c major third*," etc.; a', b'', c''', etc., are read "*a minor prime*, *b minor second*, *c minor third*," etc.; a_1, B_2, C_3, d_4, etc., are read "*a minor sub-one*, *b major sub-two*, *c major sub-three*, *d minor sub-four*," etc.

The words *major* and *minor* are used only when capitals and small letters are employed in the same problem. Otherwise they are dropped, and a', b_2, for example, are read "*a prime*, *b sub-two.*"

POSITIVE AND NEGATIVE QUANTITIES.

461. Positive and **negative** are terms applied to quantities of opposite character; as, money earned and money owed, water running into a tank and water running out, a distance up-stream and a distance down-stream, the height of a tower and the depth of a well, the pull on a lifting-rope and the weight of the load, etc.

462. Positive quantities are preceded by the sign plus, as $+2xy$, $+ab$, etc., and negative quantities by the sign minus, as $-2xy$, $-ab$, etc. Thus, if money earned is + \$50, a like amount owed is − \$50. If the quantity of water running into a tank is denoted by $+a$, the same quantity running out should be denoted by $-a$.

463. It really does not matter which quantity be taken as positive and which as negative, so long as the characters of the positive and the negative quantity are opposite; but it is customary to call something gained positive and something lost negative. Thus, money earned is usually regarded as positive, money owed as negative; distance up, positive, distance down, negative.

464. The signs + and − may be used in two entirely different senses; heretofore we have used them exclusively as symbols of operation; thus + placed between two quantities indicates that they are to be added, etc. In the distinction between positive and negative quantities, however, we denote the positive quantity by the sign + and the negative quantity by the sign −. Hence, under different circumstances, these signs may denote addition and subtraction, or they may denote positive and negative quantities. Suppose we write the expression \$500 − \$200 = \$300; the sign − in this case indicates that the \$200 is subtracted from the \$500. Suppose, however, that a man has in his possession \$500 and owes \$200. The former amount we may denote by + \$500, and the latter, since it is owed, by − \$200; in this case the sign − before the \$200 indicates the negative character of the quantity. To find how much the man is worth we add the two, thus:

$$+\$500+(-\$200)=+\$300.$$

In this addition the second + sign denotes the operation of addition, while the three signs immediately before the quantities denote the positive or negative character of the quantities.

465. It is usual to consider a quantity as increasing in a *positive* direction; *any positive quantity, no matter how small, is always considered greater than any negative quantity, no matter how large.*

The value of a negative quantity is conceived to *increase* as its numerical value *decreases*. A man who is $10 in debt is better off than one who is $50 in debt; and the man who has $5 in the bank is better off than either. Thus, 5 is greater than -10, and -10 is greater than -50.

466. When writing algebraic expressions, if a positive term stands alone, or if the first term of an expression is positive, the plus sign is omitted, it being understood that the term is positive. Thus, $3a$ means the same as $+3a$, and $a-b$ the same as $+a-b$. The minus sign must never be omitted. Polynomials are usually written with a positive term first, and monomials with the letters arranged alphabetically.

EXAMPLES FOR PRACTICE.

467. Express the following algebraically:

1. Three x square y square, minus two cd into a plus b.
Ans. $3x^2y^2 - 2cd(a+b)$.

2. The quantity m square plus two mn plus n square in parenthesis, times a square b cube c fourth. Ans. $(m^2 + 2mn + n^2)a^2b^3c^4$.

3. A, plus the square root of D, times the parenthesis X plus Y.
Ans. $A + \sqrt{D}(X+Y)$.

4. A, plus the radical D times the parenthesis X plus Y.
Ans. $A + \sqrt{D(X+Y)}$.

5. Ten x plus y, minus seven times the quantity x minus the fraction y over 4 in parenthesis, plus the fraction x square minus y square over two cd.
Ans. $10x + y - 7\left(x - \frac{y}{4}\right) + \frac{x^2 - y^2}{2cd}$.

When $a = 6$, $b = 5$, and $c = 4$, find the numerical values of:

6. $a^2 + 2ab + b^2$. Ans. $6^2 + 2 \times 6 \times 5 + 5^2 = 121$.
7. $2a^2 + 3bc - 5$. Ans. $72 + 60 - 5 = 127$.
8. $2ac^5 - a^2(a+b)$. Ans. 11,892.
9. $abc^2 + ab^2c - a^2bc$. Ans. 360

When $x = 8$ and $y = 6$, what do the following equal:

10. $(x+y)(x-y) - \sqrt{\frac{x+y^2}{11}}$?

Ans. $(8+6)(8-6) - \sqrt{\frac{8+6^2}{11}} = 26.$

11. $\sqrt{(x+y^2)(x^2+y)} - (x-y)(\sqrt[3]{x}+y)$? Ans. 39.5.

12. $\frac{x^2y^2}{x+y} + \frac{x^2y(x+y^2)}{\sqrt{3xy}}$? Ans. 1,572.57.

ADDITION AND SUBTRACTION.

PRELIMINARY IDEAS.

468. Suppose we take a point, as A on the line shown in Fig. 62, and lay off equal distances in opposite directions.

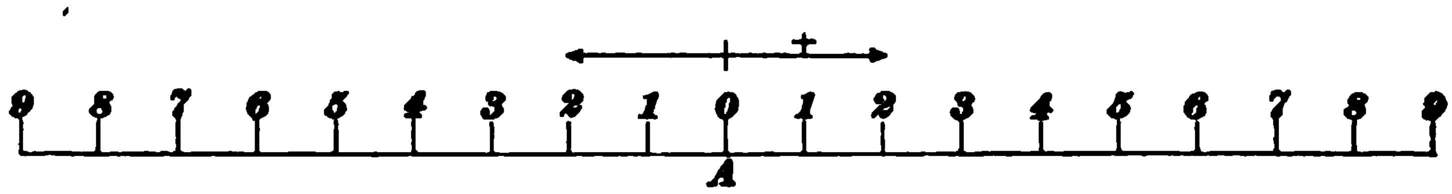


FIG. 62.

Now, let us call distances to the right $+$, or *positive*, and distances to the left $-$, or *negative*. Let us also call a movement to the right positive, and one to the left negative. Suppose the positive direction east, the negative west, and the distances to represent miles.

Suppose a man starts from A and walks east 6 miles, and after a pause walks 3 miles farther east. His distance from A is $+6 + (+3) = +9$ miles, the plus sign being taken because the motion is in the positive direction. Suppose, however, the man starts from A and walks west 3 miles, and after a pause walks 5 miles farther west. His distance from A is 8 miles west of A, or 8 miles in a negative direction; that is $-3 + (-5) = -8$. As a third case, imagine the man to walk 6 miles east, and then turn around and walk 4 miles west. Counting 4 west from 6, we see that he would still be 2 miles east of A, or 2 miles in a positive direction; that is, $+6 + (-4) = 2$. If, instead of walking back 4 miles, he had walked back 10 miles, we find by counting 10 miles west from 6 that he would have been 4 miles west of

A, or 4 miles in a negative direction, $+6+(-10)=-4$. For reference, the above results are collected:

$$\begin{aligned}+6+(+3)&=+9\\ -3+(-5)&=-8\\ +6+(-4)&=+2\\ +6+(-10)&=-4\end{aligned}$$

The student should observe carefully that in each of these additions the signs immediately before the numbers denote their positive or negative character, while the $+$ sign in front of the parenthesis denotes the operation of addition.

469. From these illustrations we have the following important principle: *If all the terms to be added are positive, the sum is positive; if all are negative, the sum is negative. If one term is positive and the other is negative, the sum has the sign of the numerically greater.* If there are several terms to be added, part of which are positive and part negative, the sum is positive if the sum of the positive terms is numerically greater than the sum of the negative terms.

When the terms have the same sign, the numerical sum is that which would be obtained if the signs were disregarded. Thus, in the first case above, the signs of 6 and 3 are the same, therefore the sum is numerically $6+3=9$; likewise in the second, the signs of the 3 and the 5 are alike and the sum is numerically $3+5=8$. When, however, one term is positive and the other is negative, the sum is the numerical difference between the terms. Thus, in the third case above, the signs of 6 and 4 are different, and the sum, 2, is the numerical difference; likewise in the fourth case, the 6 and the 10 have different signs, and the sum, 4, is the numerical difference.

470. To add like terms containing letters, we simply add the coefficients, having regard for the proper signs, and annex the literal part. Thus, the sum of $6ax^2y$ and $3ax^2y$ is $9ax^2y$; the sum of $3ab$ and $-11ab$ is $-8ab$; and the sum of $9m^2n$, $3m^2n$, $-8m^2n$, and $2m^2n$ is $6m^2n$.

ADDITION OF MONOMIALS.

471. Like Quantities.—To add like quantities having the same sign:

Rule I.—*Add the coefficients, give the sum the common sign, and annex the common literal part.*

To add like quantities having different signs:

Rule II.—*Add the positive and the negative coefficients separately, and from the greater sum subtract the less. Give the remainder the sign of the greater sum, and annex the common literal part.*

EXAMPLE.—Find the sum of $-2abxy$, $-abxy$, $-3abxy$, and $-6abxy$.

SOLUTION.—The sum of the coefficients is 12 (remember that the coefficient of $-abxy$ is 1), and the common sign is $-$. The common literal part, $abxy$, annexed to these gives as the result $-12abxy$. (Rule I.)

EXAMPLE.—Combine xy^2, $-2xy^2$, $8xy^2$, and $-4xy^2$.

SOLUTION.—The sum of the coefficients of the positive terms is 9, and of the negative terms, 6. Their difference is 3, and the sign of the greater sum is $+$. The common literal part, xy^2, annexed to these gives as the result $3xy^2$. (Rule II.)

SUBTRACTION OF MONOMIALS.

472. Referring again to Fig. 62, suppose two men, C and D, to start from point A and travel eastward. At the end of a certain time, C has walked 8 miles and D has walked 5 miles. Now, the distance between C and D is 3 miles; to pass from $+5$ to $+8$, we must walk $+3$ miles east, or in a *positive* direction; or $+8-(+5)=+3$. Observe that the minus sign here denotes subtraction, while the three plus signs denote that the three quantities which they precede are positive. The difference between 5 and 8 means how far, and in what direction we must go to pass from 5 to 8 or from 8 to 5. If we pass from $+5$ to $+8$, we move in a positive direction, and we say $+5$ from $+8$ is equal to $+3$. Suppose, however, we pass from $+8$ to $+5$; we move westward, or in a *negative* direction, a distance of 3 miles. Hence, we say that $+8$ from $+5$ is -3, or $+5$

$-(+8) = -3$. We shall always consider the point we pass *from* as the subtrahend, or quantity to be subtracted, and the point we *approach* as the minuend, or quantity we subtract from.

Suppose C has walked 7 miles east and D has walked 4 miles west, how far apart are they? To pass from D to C we must travel 11 miles east, or in a positive direction. Therefore, $7-(-4) = +11$. To pass from C to D, we travel 11 miles west, or in a negative direction; $-4 -(+7) = -11$.

The following exercises may be studied in connection with Fig. 62:

From $+3$ to $+8$ is $+5$, or $+8-(+3) = +5$.
From $+10$ to $+6$ is -4, or $+6-(+10) = -4$.
From -5 to $+4$ is $+9$, or $+4-(-5) = +9$.
From -9 to $-2 = +7$, or $-2-(-9) = +7$.
From -3 to $-7 = -4$, or $-7-(-3) = -4$.
From $+2$ to $-4 = -6$, or $-4-(+2) = -6$.

473. In every case, the difference is what must be added to the subtrahend to obtain the minuend. Thus, if I am 3 miles east of A, Fig. 62, how far must I go to be 8 miles east of A? Evidently 5 miles, since 5 miles added to 3 miles gives 8 miles. Similarly, the difference between -5 and $+4$ is $+9$, since $+9$ must be added to -5 to obtain $+4$; that is, we must travel 9 miles east in passing from -5 to $+4$.

474. We have seen that if we add $+6$ and -4 we obtain $+2$ as the sum. (Art. **468.**) If we subtract $+4$ from $+6$, the difference is $+6-(+4) = +2$, since $+2$ must be added to $+4$ to make $+6$. If we add $+6$ and -10, the sum is -4; if we subtract $+10$ from $+6$, the difference is -4. Therefore, it appears that if we wish to subtract one quantity from another, we obtain the same result if we change the sign of the quantity to be subtracted and *add* it to the other quantity. Thus, $+7$ subtracted from $+12$ is the same as -7 added to $+12$, the result being 5 in either case. $-3mn$ subtracted from $4mn$ gives the

same result as $+3mn$ added to $4mn$. The following exercises are given as illustrations:

$+5-(-2)=+5+(+2)=+7.$
$-6-(+3)=-6+(-3)=-9.$
$+2a-(+a)=+2a+(-a)=+a.$
$+5xy-(+15xy)=+5xy+(-15xy)=-10xy.$

475. To subtract like quantities:

Rule.—*Change the sign of the subtrahend, and proceed as in addition.*

EXAMPLE.—From $-3ab^2x$ take $7ab^2x$.

SOLUTION.—Changing the sign of the subtrahend, $7ab^2x$, and *adding*, we have $-3ab^2x+(-7ab^2x)=-10ab^2x$. Ans.

476. Unlike Quantities.—In arithmetic, unlike numbers, as 5 books and 3 dollars, can not be added or subtracted. So, in algebra, unlike terms, as $3ab^2$, $4xy$, $2m$, etc., can not be combined or subtracted, except by indicating the operations by signs.

Expressions in algebra are composed of quantities between which operations of addition, multiplication, etc., are indicated. The trinomial $m^2-2mn+n^2$, for example, is the indicated sum of m^2, $-2mn$, and n^2, *and it is to be considered as one quantity, in the same way that an arithmetical sum, obtained by actually performing the addition, is considered.*

EXAMPLE.—What does $7cd^3-8cx-cd^3+6adx+2cd^3$ equal?

SOLUTION.—In this case, part of the terms are like and part unlike. Combining like terms, $7cd^3+2cd^3-cd^3=8cd^3$. Connecting the unlike terms with this result by their respective signs, we have as the final result $8cd^3-8cx+6adx$. Ans.

EXAMPLE.—Subtract $2m-3$ from $7m+2xy$.

SOLUTION.—As in the example above, we have like and unlike terms. Subtracting like terms, Art. **475**, we have $7m-2m=5m$. We must now connect the unlike terms by their respective signs. Since -3 is in the subtrahend, its sign will be changed, giving us $5m+2xy+3$. Ans.

EXAMPLES FOR PRACTICE.

477. Find the sum of the following:

1. $-6a^2$, $2a^2$, $-5a^2$, $4a^2$, $-3a^2$, and a^2. Ans. $-7a^2$.
2. $2a^2b$, $-a^2b$, $11a^2b$, $-5a^2b$, $4a^2b$, and $-9a^2b$. Ans. $2a^2b$.

3. $2x^2$, $3xy$, $-x^2$, $8y^2$, $-5xy$, and $-7y^2$. Ans. $x^2-2xy+y^2$.

NOTE.—Combine like terms and connect with respective signs.

4. a^2bc, $-2ab^2c$, $3abc^2$, $-4a^2bc$, and $5ab^2c$.

Ans. $3ab^2c-3a^2bc+3abc^2$.

Solve the following:

5. From $17a$ take $-11a$. Ans. $28a$.

6. From $-11a$ take $17a$. Ans. $-28a$.

7. Subtract $5cd$ from $-4cd$. Ans. $-9cd$.

8. Subtract $-10b^2$ from $-10b^2$. Ans. 0.

9. What quantity added to $10xy$ will produce $-12xy$? Ans. $-22xy$.

10. What, then, does $10xy$ subtracted from $-12xy$ equal?

Ans. $-22xy$.

ADDITION AND SUBTRACTION OF POLYNOMIALS.

478. To add polynomials:

Rule.—*Write the expressions underneath one another, with like terms in the same vertical column. Add each column separately, and connect the sums by their proper signs.*

EXAMPLE.—Find the sum of $5a^2+6ac-3b^2-2xy$, $7ac-3a^2+4b^2+3xy$, and $4xy-5b^2+8ac-a^2$.

SOLUTION.—Writing like terms in the same vertical column, we have

$$\begin{array}{rrrrr} & 5a^2 & +\ 6ac & -\ 3b^2 & -\ 2xy \\ & -\ 3a^2 & +\ 7ac & +\ 4b^2 & +\ 3xy \\ & -\ a^2 & +\ 8ac & -\ 5b^2 & +\ 4xy \\ \hline \textit{sum} & a^2 & +21ac & -\ 4b^2 & +\ 5xy. \end{array} \quad \text{Ans.}$$

EXAMPLE.—Find the sum of $a^2x-ax^2-x^2$, $ax-x^2-a^2$, $-2a^2-2a^2x-2ax^2$, and $3a^2-3a^2x+3ax^2$.

SOLUTION.—

$$\begin{array}{rrrrrr} & a^2x & -\ ax^2 & -\ x^2 & & \\ & & & -\ x^2 & -\ a^2 & +\ ax \\ & -\ 2a^2x & -\ 2ax^2 & & -\ 2a^2 & \\ & -\ 3a^2x & +\ 3ax^2 & & +\ 3a^2 & \\ \hline \textit{sum} & -\ 4a^2x & +\ 0 & -\ 2x^2 & +\ 0 & +\ ax = \end{array}$$

$ax-4a^2x-2x^2$. Ans. (Arts. **455** and **456**.)

479. To subtract one polynomial from another:

Rule.—*Write the subtrahend underneath the minuend, with like terms in the same vertical column. Change the sign of each term of the subtrahend, and proceed as in addition.*

EXAMPLE.—From $3ac - 2b$ subtract $ac - b - d$.

SOLUTION.— $3ac - 2b$

$- ac + b + d$, subtrahend with signs changed.

difference $2ac - b + d$. Ans.

EXAMPLE.—From $2x^3 - 3x^2y + 2xy^2$ subtract $x^3 - xy^2 + y^3$.

SOLUTION.— $2x^3 - 3x^2y + 2xy^2$

$- x^3 \qquad + xy^2 - y^3$, subtrahend with signs changed.

difference $x^3 - 3x^2y + 3xy^2 - y^3$. Ans.

EXAMPLES FOR PRACTICE.

480. Find the sum of the following:

1. $ax + 2bx + 4by - 3ay$, $2ax + bx + 2ay - by$, and $4ax + 3by$.
   Ans. $7ax + 3bx + 6by - ay$.
2. $a - x + 4y - 3z + w$, $z + 3a - 2x - y - w$, and $x + y + z$.
   Ans. $4a - 2x + 4y - z$.
3. $2a - 3b + 4d$, $2b - 3d + 4c$, $2d - 3c + 4a$, and $2c - 3a + 4b$.
   Ans. $3a + 3b + 3c + 3d$.
4. $6x - 3y + 7m$, $2n - x + y$, $2y - 4x - 5m$, and $m + n - y$.
   Ans. $x - y + 3m + 3n$.

Solve the following:

5. From $7a + 5b - 3c$ take $a - 7b + 5c - 4$.
   Ans. $6a + 12b - 8c + 4$.
6. From $3m - 5n + r - 2s$ take $2r + 3n - m - 5s$.
   Ans. $4m - 8n - r + 3s$.
7. Subtract $2x - 2y + 2$ from $y - x$. Ans. $3y - 3x - 2$.
8. Subtract $3x^3 + 4x^2y - 7xy^2 + y^3 - xy^3$ from $5x^3 + x^2y - 6xy^2 + y^3$.
   Ans. $2x^3 - 3x^2y + xy^2 + xy^3$.

SYMBOLS OF AGGREGATION.

481. Parentheses, brackets, etc., being used to enclose expressions that are to be treated as one quantity, the sign before the symbol affects the *entire expression*, not the first term only. Thus, $-(a^2 - 2ab + b^2)$ signifies that all the terms are to be subtracted from what precedes, not a^2 only.

482. When combining the terms of any expression without parentheses, we proceed as in addition of monomials. When we have a parenthesis preceded by a minus sign, we must consider the expression within the parenthesis as a

subtrahend, and change all signs before removing the parenthesis.

If, on the contrary, the sign of the parenthesis is plus, we may remove the symbols, but we *must not* change the signs of the expression, because we do not change the signs of an expression to be added.

483. When a quantity is enclosed by a parenthesis, the first term is understood to have the plus sign, unless the minus sign is given; thus, in the expression $-(8x + 5 - 2b)$, the minus sign refers to the whole quantity. The sign of $8x$ is $+$, and the expression if written in full would be $-(+8x + 5 - 2b)$.

EXAMPLE.—Remove the parenthesis from $4c - (3a + 4ab - d)$.

SOLUTION.—Changing the sign of each enclosed term, and remembering that the sign of $3a$ is $+$, understood, we have as the result $4c - 3a - 4ab + d$. Ans.

EXAMPLE.—Remove the parentheses from $4a - 5x - (a - 4x) + (x - 8a)$.

SOLUTION.— $4a - 5x - (a - 4x) + (x - 8a) = 4a - 5x - a + 4x + x - 8a$. Adding the like terms, we have

$$\begin{array}{r} 4a - 5x \\ -\ \ a + 4x \\ -8a + \ \ x \\ \hline -5a + 0 \end{array} = -5a. \quad \text{Ans.}$$

484. Symbols of aggregation will often be found enclosing others. In such cases they may be removed in succession, ***always beginning with the innermost pair.***

EXAMPLE.—Remove all the symbols of aggregation from $6a - \{b - [7cd - 4a + (2cd - \overline{a - b})]\}$.

SOLUTION.—We first remove the vinculum. This being in effect the same as the parenthesis, the minus sign before the a indicates that $+a$ and $-b$ are to be subtracted.

Hence, we have

$$6a - \{b - [7cd - 4a + (2cd - a + b)]\}.$$

Removing the parenthesis we have

$$6a - \{b - [7cd - 4a + 2cd - a + b]\}.$$

This, with the brackets removed, is equal to

$$6a - \{b - 7cd + 4a - 2cd + a - b\},$$

which, in turn, is equal to

$$6a - b + 7cd - 4a + 2cd - a + b.$$

Combining like terms,

$$6a - 4a - a - b + b + 7cd + 2cd = a + 9cd. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

485. Remove the parentheses from the following:

1. $-(2mn - m^2 - n^2)$. Ans. $m^2 - 2mn + n^2$.
2. $1 - (-b + c + 3)$. Ans. $b - c - 2$.
3. $5a - 4b + 3c - (-3a + 2b - c)$. Ans. $8a - 6b + 4c$.
4. $3x - (2x - 5) + (7 - x)$. Ans. 12.

Remove the symbols of aggregation from the following:

5. $m - [4n - k - (m + n - 2k)]$. Ans. $2m - 3n - k$.
6. $5x - (2x - 3y) - (x + 5y)$. Ans. $2x - 2y$.
7. $3a - [7a - (5a - b - a)] - (-a - 4b)$. Ans. $a + 3b$.
8. $3x + \{2y - [5x - (3y + \overline{x - 4y})]\}$. Ans. $y - x$.
9. $100x - \{200x - [500x - (-100x) - 300x] - 400x\}$. Ans. $600x$.
10. $7cx - \{4cy - [(4cx + 3cy) + cy - cx]\}$. Ans. $10cx$.

NOTE.—Observe that the sign before the parenthesis is + understood.

MULTIPLICATION.

PRELIMINARY IDEAS.

486. In algebra, **multiplication** is often indicated only, and the final answer is frequently nothing more than the symbols so written that it is shown they should be multiplied when their numerical values are substituted. Thus the product of m and n is mn, the absence of a sign between the two quantities denoting the operation of multiplication. If we multiply $2a$ by $3b$, the four factors which form the product are 2, a, 3, and b; hence, $2a \times 3b = 2 \times a \times 3 \times b$. Since we are at liberty to arrange the factors in any order, we have

$$2a \times 3b = 2 \times 3 \times a \times b = 6 \times a \times b = 6ab.$$

Hence, in finding the product of two quantities, *the coefficients are multiplied together and prefixed to the literal factors.*

487. When two quantities are multiplied together, *the sign of the product is positive if the two quantities have the*

same sign; and it is negative if the quantities have opposite signs.

Thus,
$$(+5n) \times (+3m) = +15mn.$$
$$(-5n) \times (+3m) = -15mn.$$
$$(-5n) \times (-3m) = +15mn.$$
$$(+5n) \times (-3m) = -15mn.$$

488. Assume that the $3m$ above is the multiplier, and that $5n$ is the multiplicand. The multiplier must be considered as an abstract number, and simply shows how many times the multiplicand is to be taken; thus, $3m$ may represent 15, 21, 30, etc., depending upon the value of m. The multiplicand, $5n$, may represent any concrete number; as, \$20, 60 feet, 100 miles. The sign of the multiplicand shows the character of the quantity; thus, if $+5n$ represents \$20, $-5n$ may be taken to represent a debt of \$20. The sign of the multiplier, on the other hand, is a symbol of operation and shows how the result is to be treated after it is obtained. If the multiplier is positive, the result obtained is to be added when taken in connection with other quantities; the multiplier is given the negative sign to show that the result is to be subtracted.

Multiplying $+5n$ by $3m$, considering the multiplier simply as an abstract number and disregarding its sign, we obtain $+15mn$. Similarly, $-5n$ multiplied by $3m$ gives $-15mn$. The product in each case has the same character as the multiplicand. Now, if the multiplier has the positive sign, the product is to be added when taken in connection with other quantities, and, therefore, the signs remain as above. That is, $(+5n) \times (+3m) = +15mn$, and $(-5n) \times (+3m) = -15mn$.

If, on the other hand, the multiplier has the negative sign, the product is to be subtracted, and, therefore, its sign must be changed when combined with other quantities. Thus, when the multiplier, $3m$, has the negative sign, we have $(+5n) \times (-3m) = -15mn$ and $(-5n) \times (-3m) = +15mn$.

489. Exponents.—It has been shown that $a^2 = a \times a$ and $a^3 = a \times a \times a$. The product $a^2 \times a^3$, or $a^2a^3 =$

$a \times a \times a \times a \times a = a^5$. The exponent 2 shows that a is used twice as a factor, and the exponent 3 shows that a is used three times as a factor. In the product, a must be used five times as a factor, or the exponent of a is 5. *The exponent of the product is the sum of the exponents of the factors.*

Thus,
$$a^4 \times a^3 = a^{4+3} = a^7.$$
$$c^2 \times c^4 = c^{2+4} = c^6.$$
$$m \times m^2 \times m^3 = m^{1+2+3} = m^6.$$
$$n^2 \times n^4 \times n^5 \times n^3 = n^{14}.$$

MULTIPLICATION OF MONOMIALS.

490. Rule.—*To the product of the coefficients annex the letters of both factors; give each letter an exponent equal to the sum of the exponents of that letter.*

Make the sign of the product plus, when the signs of the factors are alike; and minus, when they are unlike.

EXAMPLE.—Multiply $4a^2b$ by $-5a^3bc$.

SOLUTION.—The product of the coefficients is 20, and the letters to be annexed are a, b, and c. The new exponent of a is 5, and of b, 2, since $a^{2+3} = a^5$, and $b^{1+1} = b^2$. The sign of the product is minus, since the two factors have different signs. Hence, $4a^2b \times -5a^3bc = -20a^5b^2c$. Ans.

491. When there are more than two factors, we have simply three or more examples in multiplication to solve in succession, each to be performed by the foregoing rule.

EXAMPLE.—Find the continued product of $6x^2yz^3$, $-9x^2y^2z^2$, and $-3x^4yz$.

SOLUTION.— $6x^2yz^3 \times -9x^2y^2z^2 = -54x^{2+2}y^{1+2}z^{3+2}$, or $-54x^4y^3z^5$. Now, multiplying this product by $-3x^4yz$, we have $-54x^4y^3z^5 \times -3x^4yz = 162x^8y^4z^6$. Ans.

EXAMPLES FOR PRACTICE.

492. Find the product of:

1. a^3b^2 and $-5abd$. Ans. $-5a^4b^3d$.
2. $-7xy$ and $-7x^2y^2$. Ans. $49x^3y^3$.
3. $-15m^5n^6$ and $3mn$. Ans. $-45m^6n^7$.
4. $3a(x-y)^2$ and $2a^2(x-y)$. Ans. $6a^3(x-y)^3$.

SUGGESTION.—Treat the $(x-y)$ as though it were a single letter.

5. Find the continued product of $2a^3m^2x$, $-3a^2mx^3$, and $4am^3x^2$.
Ans. $-24a^6m^6x^6$.

6. What does $-a^2bn \times -2cdn \times -3bdc^2 \times -2acn^2$ equal?
Ans. $12a^3b^2c^4d^2n^4$.

MULTIPLICATION OF POLYNOMIALS.

493. When one of the factors is a monomial:

Rule.—*Multiply each term of the polynomial by the monomial, and connect the separate products by their proper signs.*

EXAMPLE.—Find the product of $-9a^5+3a^3b^2-4a^2b^3-b^5$ and $-3ab^4$.

SOLUTION.—

$$\begin{array}{l}
-9a^5+3a^3b^2-4a^2b^3-b^5 \\
\underline{-3ab^4 \qquad\qquad\qquad\qquad\quad} \\
27a^6b^4-9a^4b^6+12a^3b^7+3ab^9. \quad \text{Ans.}
\end{array}$$

494. When both factors are polynomials:

Rule.—*Multiply each term of one polynomial by each term of the other, and add the partial products.*

EXAMPLE.—Multiply $6a-4b$ by $4a-2b$.

SOLUTION.—Write the multiplier under the multiplicand, and begin to multiply *at the left* instead of at the right, as in arithmetic, since polynomials are always written and read from the left, and there are no numbers to carry.

$$\begin{array}{lrrrl}
 & 6a & -4b & & (1) \\
 & 4a & -2b & & \\
\hline
\text{Multiplying (1) by } 4a \text{ gives} & 24a^2 & -16ab & & (2) \\
\text{Multiplying (1) by } -2b \text{ gives} & & -12ab & +8b^2 & (3) \\
\hline
\text{Adding (2) and (3) gives} & 24a^2 & -28ab & +8b^2. & \text{Ans.}
\end{array}$$

It will be noticed that the like terms, $-16ab$ and $-12ab$, are written under each other, so that it will be easier to add them.

EXAMPLE.—Multiply $x^3-x+1+x^2$ by $1-x^2+x$.

SOLUTION.—With a view to bringing like terms in the same columns, arrange both multiplicand and multiplier either according to the increasing or the decreasing powers of the same letter. (Art. **455.**) Arranging in this case according to the increasing powers of x, we have

$$\begin{array}{lrrrrrrl}
 & 1 & -x & +x^2 & +x^3 & & & (1) \\
 & 1 & +x & -x^2 & & & & \\
\hline
\text{Multiplying (1) by 1 gives} & 1 & -x & +x^2 & +x^3 & & & (2) \\
\text{Multiplying (1) by } +x \text{ gives} & & x & -x^2 & +x^3 & +x^4 & & (3) \\
\text{Multiplying (1) by } -x^2 \text{ gives} & & & -x^2 & +x^3 & -x^4 & -x^5 & (4) \\
\hline
\text{Adding (2), (3), and (4) gives} & 1 & & -x^2 & +3x^3 & & -x^5. & \text{Ans}
\end{array}$$

495. Multiplication is frequently indicated by enclosing each of the quantities to be multiplied in a parenthesis. The sign of multiplication is not placed between the parentheses, multiplication being understood. When the quantities are multiplied together, the expression is said to be *expanded.*

For example, in the expression $(m - 2n)(2m - n)$, the binomial $m - 2n$ is to be multiplied by the binomial $2m - n$. Performing the multiplication, the product is $2m^2 - 5mn + 2n^2$, which is the expanded form of the expression.

EXAMPLES FOR PRACTICE.

496. Multiply the following:

1. $x^2 + 2xy + y^2$ by $x + y$. Ans. $x^3 + 3x^2y + 3xy^2 + y^3$.
2. $3ab^2m^3 + 4a^2b - 2$ by $a^6b^7m^8$. Ans. $3a^7b^9m^{11} + 4a^8b^8m^8 - 2a^6b^7m^8$.
3. $c^2 - d^2$ by $c^2 + d^2$. Ans. $c^4 - d^4$.
4. $x^4 + x^2y^2 + y^4$ by $x^2 - y^2$. Ans. $x^6 - y^6$.
5. $3a^2 - 7a + 4$ by $2a^2 + 9a - 5$. Ans. $6a^4 + 13a^3 - 70a^2 + 71a - 20$.

Expand the following:

6. $(2a - 3c)(4 - 3a)$. Ans. $8a - 12c - 6a^2 + 9ac$.
7. $(x + 2)(x - 2)(x^2 + 4)$. Ans. $x^4 - 16$.
8. $[x(x^2 - y^2) - 2][x(x^2 + y^2) + 2]$.

NOTE.—The expressions in the brackets reduce to $x^3 - xy^2 - 2$ and $x^3 + xy^2 + 2$. The product of these is $x^6 - x^2y^4 - 4xy^2 - 4$. Ans.

THREE IMPORTANT EXAMPLES.

497. Let a and b be *any* two quantities; we wish to find the forms of the following products:

$$(a + b)^2,\ (a - b)^2, \text{ and } (a + b)(a - b).$$

By actual multiplication we find

$$(a + b)^2 = a^2 + 2ab + b^2. \qquad \textbf{(1.)}$$

$$(a - b)^2 = a^2 - 2ab + b^2. \qquad \textbf{(2.)}$$

$$(a + b)(a - b) = a^2 - b^2. \qquad \textbf{(3.)}$$

Hence:

498. *The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first and the second, plus the square of the second.*

The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first and the second, plus the square of the second.

The product of the sum and difference of two quantities is equal to the difference of their squares.

499. The foregoing statements should be committed to memory, since their use will frequently save tedious calculation. *The student is advised to practice until he is certain that he knows them perfectly.*

EXAMPLE.—Square $3x^2 + 5$.

SOLUTION.—The square of the first term is $3x^2 \times 3x^2 = 9x^4$; twice the product of the terms is $30x^2$; and the square of the last term is 25. Hence, by formula **1**, letting $a = 3x^2$ and $b = 5$,

$$(3x^2 + 5)^2 = 9x^4 + 30x^2 + 25. \quad \text{Ans.}$$

EXAMPLE.—Square $4cd - x$.

SOLUTION.—The square of the first term is $16c^2d^2$; twice the product of the first and the second is $8cdx$; and the square of the last term is x^2. Hence, by formula **2**, letting $a = 4cd$ and $b = x$,

$$(4cd - x)^2 = 16c^2d^2 - 8cdx + x^2. \quad \text{Ans.}$$

EXAMPLE.—Expand $(x^2 + 3)(x^2 - 3)$. (See Art. **495**.)

SOLUTION.—The square of the first term is x^4, and of the second, 9. Hence, by formula **3**, letting $a = x^2$ and $b = 3$,

$$(x^2 + 3)(x^2 - 3) = x^4 - 9. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

500. Square the following:

1. $m + n$. Ans. $m^2 + 2mn + n^2$.
2. $4x + 2$. Ans. $16x^2 + 16x + 4$.
3. $3a - 5b$. Ans. $9a^2 - 30ab + 25b^2$.

Expand the following:

4. $(m + 1)(m - 1)$. Ans. $m^2 - 1$.
5. $(x^2 + y^2)(x^2 - y^2)$. Ans. $x^4 - y^4$.

6. $(4a + 4b^2)(4a - 4b^2)$. Ans. $16a^2 - 16b^4$.

7. Square $2c^2 - c + d$.

NOTE.—First separate $2c^2 - c + d$ into two terms by enclosing $c + d$ in parenthesis; then the expression becomes $2c^2 - (c - d)$, and considering this as a binomial we find the square to be $4c^4 - 4c^2(c - d) + (c - d)^2$.

$$-4c^2(c - d) = -4c^3 + 4c^2d,$$
$$(c - d)^2 = c^2 - 2cd + d^2.$$

Adding these results to $4c^4$, the final result is $4c^4 - 4c^3 + 4c^2d + c^2 - 2cd + d^2$. Ans.

DIVISION.

INTRODUCTORY.

501. When two quantities are given, and we wish to find a third quantity which, if multiplied by one of the first two, will produce the other, the process of finding this third quantity is called **division.** Thus, if the given quantities are ab and a, and we wish to find a quantity which, if multiplied by a, will give ab, we must divide ab by a; our quotient will be b, since $a \times b = ab$. Division is, therefore, the inverse of multiplication.

502. The following laws of division follow directly from the statements of Arts. **486** to **489**:

If the dividend and the divisor have like signs, the quotient will have the plus sign; if they have unlike signs, the quotient will have the minus sign.

The coefficient of the quotient is equal to the coefficient of the dividend divided by the coefficient of the divisor.

The exponent of a letter in the quotient is equal to its exponent in the dividend minus its exponent in the divisor.

503. Let it be required to divide a^2 by a^2. We have to obtain a quotient, which, when multiplied by the divisor a^2, will produce the dividend a^2. The quotient is evidently 1. By Art. **502,** however, we know that $a^2 \div a^2 = a^{2-2} = a^0$. Hence, *any quantity whose exponent is 0 is equal to 1.*

From the foregoing principles, the rules for division are obtained.

DIVISION OF MONOMIALS.

504. Rule.—*Divide the coefficient of the dividend by the coefficient of the divisor and to the quotient annex the letters of the dividend, each with an exponent equal to its exponent in the dividend minus its exponent in the divisor, omitting those letters whose exponents become zero.*

Make the sign of the quotient plus when the dividend and divisor have like signs, and minus when they have unlike signs.

EXAMPLE.—Divide $6a^5b^4c^3$ by $-3a^2bc^3$.

SOLUTION.—The quotient of $6 \div 3$ is 2. The letters to be annexed, and their exponents, are $a^{5-2} = a^3$, and $b^{4-1} = b^3$. The c has an exponent of $3 - 3 = 0$, so that it becomes equal to 1, and is omitted. The sign of the quotient is minus. Hence, $6a^5b^4c^3 \div -3a^2bc^3 = -2a^3b^3$. Ans.

PROOF.— $-3a^2bc^3 \times -2a^3b^3 = 6a^5b^4c^3$.

EXAMPLE.—Divide $-10a^6b^3c^2d$ by $-2ab^3c$.

SOLUTION.— $-10a^6b^3c^2d \div -2ab^3c = 5a^{6-1}b^{3-3}c^{2-1}d = 5a^5cd$. Ans.

EXAMPLES FOR PRACTICE.

505. Divide the following:

1. $12m^3n$ by $4n$. Ans. $3m^3$.
2. $30x^6y^5bc^3$ by $-6x^5y^5c^2$. Ans. $-5xbc$.
3. $-44a^3b^3c^3$ by $-11ab^2c^3$. Ans. $4a^2b$.
4. $-100x^4y^3z^2$ by x^3y^2. Ans. $-100xyz^2$.
5. $75pq^2x^3m^4$ by $75x^3$. Ans. pq^2m^4.

DIVISION OF POLYNOMIALS.

506. When the divisor is a monomial:

Rule.—*Divide each term of the dividend by the divisor, and connect the partial quotients by their proper signs.*

EXAMPLE.—Divide $12a^2b^4 - 9ab^3 + 6a^3b^4$ by $3ab^3$.

SOLUTION.— $3ab^3\overline{)\,12a^2b^4 - 9ab^3 + 6a^3b^4}$

quotient $\quad 4ab \quad -3 \quad +2a^2b$. Ans.

EXAMPLES FOR PRACTICE.

507. Divide the following:

1. $64m^2n^3 - 32mn^2 + 8m^2n$ by $8mn$. Ans. $8mn^2 - 4n + m$.
2. $27x^3y^2z - 9x^3yz^2 - 333x^3y^2z^2$ by $-3x^3yz$.
   Ans. $-9y + 3z + 111yz$.
3. $10(x+y)^2 - 5a(x+y) + 5a^2(x+y)$ by $5(x+y)$.
   Ans. $2(x+y) - a + a^2$.

508. When the divisor is a polynomial:

Rule.—*Arrange both dividend and divisor according to the ascending or descending powers of some letter.*

Divide the first term of the dividend by the first term of the divisor for the first term of the quotient.

Subtract from the dividend the product of the divisor and this term of the quotient.

Treat the remainder as a new dividend, and proceed as before, until there is no remainder, or until the final remainder contains no term which is divisible by the first term of the divisor.

EXAMPLE.—Divide $x^4 + x^3 - 9x^2 - 16x - 4$ by $x^2 + 4x + 4$.

SOLUTION.—

$$
\begin{array}{l}
\hphantom{x^2+4x+4)x^4+x^3-9x^2-16x-4(\;}\textit{quotient.}\\
x^2 + 4x + 4)\,x^4 + x^3 - 9x^2 - 16x - 4\,(x^2 - 3x - 1. \quad \text{Ans.}\\
\hphantom{x^2 + 4x + 4)\,}\underline{x^4 + 4x^3 + 4x^2}\\
\hphantom{x^2 + 4x + 4)\,x^4 +\,} -3x^3 - 13x^2 - 16x\\
\hphantom{x^2 + 4x + 4)\,x^4 +\,} \underline{-3x^3 - 12x^2 - 12x}\\
\hphantom{x^2 + 4x + 4)\,x^4 + x^3 -\,} -x^2 - 4x - 4\\
\hphantom{x^2 + 4x + 4)\,x^4 + x^3 -\,} \underline{-x^2 - 4x - 4}\\
\hphantom{x^2 + 4x + 4)\,x^4 + x^3 - 9x^2\,} 0 \quad 0 \quad 0
\end{array}
$$

The first term x^2 of the divisor is contained in x^4, the first term of the dividend, x^2 times; hence, x^2 is the first term of the quotient. The whole divisor multiplied by this term gives $x^4 + 4x^3 + 4x^2$ as a product, which subtracted from the dividend gives as a remainder, $-3x^3 - 13x^2 - 16x - 4$. It is not necessary here to bring down the -4, since only three terms are required to contain the divisor.

The first term x^2 of the divisor is contained in $-3x^3$, the first term of the new dividend, $-3x$ times. Multiplying the divisor by this new term of the quotient, we have $-3x^3$

$-12x^2 - 12x$. Subtracting this from the first remainder, we obtain $-x^2 - 4x - 4$ for a new remainder, the -4 being brought down from the original dividend. The first term of the divisor is contained in the first term of the new remainder or dividend, -1 times. Multiplying the divisor by this, we get $-x^2 - 4x - 4$, which subtracted from $-x^2 - 4x - 4$, the last remainder, leaves a difference of zero. The work ends here, since there are no more terms in the dividend to be brought down.

EXAMPLE.—Divide $9x^2y^2 + x^4 - 4y^4 - 6x^3y$ by $x^2 + 2y^2 - 3xy$.

SOLUTION.—First arrange the dividend and divisor according to the descending powers of x.

$$\begin{array}{l}
x^2 - 3xy + 2y^2\,)\,x^4 - 6x^3y + 9x^2y^2 - 4y^4\,(\,x^2 - 3xy - 2y^2. \quad \text{Ans.} \\
\qquad\qquad\qquad\quad \underline{x^4 - 3x^3y + 2x^2y^2} \\
\qquad\qquad\qquad\qquad -3x^3y + 7x^2y^2 \qquad\quad -4y^4 \\
\qquad\qquad\qquad\qquad \underline{-3x^3y + 9x^2y^2 - 6xy^3 \qquad\quad} \\
\qquad\qquad\qquad\qquad\qquad\qquad -2x^2y^2 + 6xy^3 - 4y^4 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{-2x^2y^2 + 6xy^3 - 4y^4} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 0 \qquad\quad 0 \qquad\quad 0
\end{array}$$

EXAMPLES FOR PRACTICE.

509. Divide the following:

1. $x^2 - 7x + 12$ by $x - 3$. Ans. $x - 4$.
2. $x^2 + x - 72$ by $x + 9$. Ans. $x - 8$.
3. $2x^3 - x^2 + 3x - 9$ by $2x - 3$. Ans. $x^2 + x + 3$.
4. $x^4 + 11x^2 - 12x - 5x^3 + 6$ by $3 + x^2 - 3x$. Ans. $x^2 - 2x + 2$.
5. $x^4 - 6xy - 9x^2 - y^2$ by $x^2 + y + 3x$. Ans. $x^2 - 3x - y$.
6. $x^6 - 1$ by $x - 1$. Ans. $x^5 + x^4 + x^3 + x^2 + x + 1$.

FACTORING.

510. Factoring is the process of finding the *factors* of a quantity, that is, the quantities or numbers which will divide that quantity without a remainder.

511. Expanding $6a^2b(2b + a)$, we have $12a^2b^2 + 6a^3b$; hence, $6a^2b$ and $(2b + a)$ are the factors of $12a^2b^2 + 6a^3b$.

The monomial factor $6a^2b$ may be further resolved into $3 \times 2 \times a \times a \times b$. In solving examples in factoring, it is not customary to write out the factors of a monomial, since they are generally apparent.

That the student may be able to recognize factors without the labor of actual division, several methods of readily discovering factors are here given.

512. **Equal factors** are those whose terms have the same letters, and whose letters have the same exponents and the same signs. Thus, $5a(2y - x)$ and $5a(2y - x)$ are equal factors of $5a(2y - x) \times 5a(2y - x) = 25a^2(2y - x)^2$; but $5a(2y - x)$ and $-5a(2y - x)$ are unequal factors, since the signs of $5a$ are not the same in both expressions.

513. A product of two equal factors is a **perfect square.** Either of the equal factors of a quantity is called its square root.

514. A product of three equal factors is a **perfect cube.** Any one of the equal factors of a quantity is called its cube root.

515. In factoring, it is important to be able to easily distinguish quantities that are perfect squares and cubes, and to determine their roots. By definition, $9a^2b^2$ is a perfect square because $3ab \times 3ab = 9a^2b^2$, and $3ab$ is its square root. Also, $8a^6$ is a perfect cube because $2a^2 \times 2a^2 \times 2a^2 = 8a^6$, and $2a^2$ is its cube root. In each of these cases the coefficients of the roots are multiplied together, and the exponents added, to produce a perfect power. *Hence, a quantity is a perfect square when its coefficient is a perfect square, and the exponents of all its letters can be divided by 2.* For example, $36x^{10}$, $49b^2c^4d^8$, $16a^6b^{12}$, and 1 are all perfect squares, whose roots are $6x^5$, $7bc^2d^4$, $4a^3b^6$, and 1, respectively. No perfect square, however, can have a minus sign; for, let a = any quantity, $-a \times -a = a^2$, and $a \times a = a^2$. The square root of a^2 may be $-a$, or a, and a square root is often written $\pm a$, read plus or minus a.

A quantity is a perfect cube when its coefficient is a perfect cube, and the exponents of all its letters can be divided by 3. Thus, $27x^{12}$, $-64b^3c^6d^6$, $8a^{18}b^{18}$, and 1 are all perfect cubes, whose roots are $3x^4$, $-4bc^2d^2$, $2a^6b^6$, and 1, respectively. The sign of the cube root is always the same as that of its cube.

CASE I.

516. *When all the terms of an expression are divisible by the same quantity, the expression may be resolved into two factors by dividing it by that quantity.*

EXAMPLE.—Factor $8m^4n^3 - 10m^3n^2y + 2m^2n^2$.

SOLUTION.—We first examine the polynomial to see if one of its terms is contained in each of the others. Beginning with the smallest coefficient, 2, we find it to be contained in each of the others. We next take the literal portion m^2n^2 of this term, and find both of its letters in each of the other terms, with an equal or higher exponent. Dividing through by $2m^2n^2$, we have for our quotient $4m^2n - 5my + 1$. Hence, we have

$$8m^4n^3 - 10m^3n^2y + 2m^2n^2 = 2m^2n^2(4m^2n - 5my + 1). \quad \text{Ans.}$$

EXAMPLE.—Ascertain if $12ab^2c^3 - 18a^3c^2y + 24a^2c^4 - 36a^2bc^5y^2$ has a monomial factor.

SOLUTION.—By inspection we find that the smallest coefficient is not contained in the other three without a remainder. This coefficient, 12, is the product of $3 \times 2 \times 2$. Of these, both 2 and 3 will divide all the coefficients once. Therefore, $2 \times 3 = 6$ is the largest numerical factor. The letters a and c are contained in all the terms; the smallest exponent of a is 1 and of c is 2. The monomial factor is evidently $6ac^2$. Dividing the polynomial by $6ac^2$ the quotient is $2b^2c - 3a^2y + 4ac^2 - 6abc^3y^2$; the factors are $6ac^2$ and $2b^2c - 3a^2y + 4ac^2 - 6abc^3y^2$. Ans.

EXAMPLES FOR PRACTICE.

517. Factor the following expressions:

1. $a^4 + ax$. Ans. $a(a^3 + x)$.
2. $12a^5 - 2a^3 + 4a^4$. Ans. $2a^3(6a^2 - 1 + 2a)$.
3. $30m^4n^2 - 6n^3$. Ans. $6n^2(5m^4 - n)$.
4. $16x^2y^3 - 8x^5 + 8$. Ans. $8(2x^2y^3 - x^5 + 1)$.
5. $4x^3y - 12x^2y^2 + 8xy^3$. Ans. $4xy(x^2 - 3xy + 2y^2)$.
6. $49a^2b^3c^4 - 63a^3b^2c^4 + 7a^4b^2c^3$. Ans. $7a^2b^2c^3(7bc - 9ac + a^2)$.

CASE II.

518. *To factor a trinomial which is a perfect square: Any trinomial is a perfect square when the first and the last term are perfect squares and positive, and the second term is twice the product of their square roots.*

Thus, let a and b represent any two quantities whatever, and we have the general forms of the square as follows:

$$a^2 + 2ab + b^2 = (a + b)(a + b) = (a + b)^2. \qquad (4.)$$

$$a^2 - 2ab + b^2 = (a - b)(a - b) = (a - b)^2. \qquad (5.)$$

These, it will be seen, are simply the inverse of formulas **1** and **2**, Art. **497**. The sign of the second term of the square always determines the sign of the second term of the root, b in this particular case.

519. Since a may represent one quantity and b any other quantity, it is evident that *any* trinomial having the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$ is a perfect square.

Rule.—*Extract the square roots of the first and the last term of the trinomial, and connect the results by the sign of the second term.*

EXAMPLE.—Factor $x^2 + 2xy + y^2$.

SOLUTION.—We first see if the trinomial has the form stated in Art. **518.** The first and the last term we see to be perfect squares, and their roots to be x and y. The second term is also twice the product of the roots x and y, and, since it has the plus sign, the binomial root must be $x + y$. Hence, we have a square of the form $a^2 + 2ab + b^2$, and

$$x^2 + 2xy + y^2 = (x + y)(x + y) = (x + y)^2. \quad \text{Ans.}$$

EXAMPLE.—Factor $16m^4 + 9n^6 - 24m^2n^3$.

SOLUTION.—The first term of the expression is a perfect square, but the last term is not. Inspecting the second term we find it to be the square of $3n^3$, and the third term to be twice the product of $3n^3$ and the square root, $4m^2$, of the first term. Arranging the trinomial so that the first and the last term are perfect squares, we get $16m^4 - 24m^2n^3 + 9n^6$ (a square of the form $a^2 - 2ab + b^2$), and we have $16m^4 + 9n^6 - 24m^2n^3 = 16m^4 - 24m^2n^3 + 9n^6 = (4m^2 - 3n^3)(4m^2 - 3n^3) = (4m^2 - 3n^3)^2$. Ans.

EXAMPLE.—Factor $4x^2 + x^2y^2 + 2x^2y$.

SOLUTION.—Arranging the trinomial so that the first and the last term are perfect squares, we have $4x^2 + 2x^2y + x^2y^2$. Now, although the first and the last term are perfect squares with roots $2x$ and xy, respectively, the second term is only equal to the product of the roots; hence, the trinomial is *not* a perfect square, and can only be factored by Case I. Each term contains x^2, and we have $4x^2 + x^2y^2 + 2x^2y = x^2(4 + y^2 + 2y)$. Ans.

520. *Should two of the terms of a trinomial be perfect squares, and have like signs, and the other term be twice the product of their roots, the trinomial is a perfect square.*

Compare this statement with Art. **518.** Thus, $2ab - a^2 - b^2$, if divided by -1, becomes $-2ab + a^2 + b^2 = a^2 - 2ab + b^2$; hence, $2ab - a^2 - b^2 = -(a^2 - 2ab + b^2) = -(a - b)^2$.

EXAMPLE.—Factor $4pq - 4p^2 - q^2$.

SOLUTION.—Dividing first by -1 we have $-4pq + 4p^2 + q^2 = 4p^2 - 4pq + q^2 = (2p - q)^2$. Hence, $4pq - 4p^2 - q^2 = -(4p^2 - 4pq + q^2) = -(2p - q)^2$. Ans.

EXAMPLE.—Factor $16r^2s^2 + 16r^4 + 4s^4$.

SOLUTION.—The expression contains three squares, but, by careful inspection, we see that the first term is also twice the product of the square roots of the other two. Thus, $16r^2s^2 + 16r^4 + 4s^4 = 16r^4 + 16r^2s^2 + 4s^4 = (4r^2 + 2s^2)^2$. Ans.

EXAMPLES FOR PRACTICE.

521. Factor the following trinomials:

1. $x^2 - 16x + 64$. Ans. $(x - 8)^2$.
2. $n^6 - 26n^3 + 169$. Ans. $(n^3 - 13)^2$.
3. $25x^2 + 70xyz + 49y^2z^2$. Ans. $(5x + 7yz)^2$.
4. $16c^2 + b^2 - 8bc$. Ans. $(4c - b)^2$.
5. $2mx - m^2 - x^2$. Ans. $-(m - x)^2$.
6. $a^2b^4c^6 - 2ab^2c^3 + 1$. Ans. $(ab^2c^3 - 1)^2$.

CASE III.

522. *To factor an expression which is the difference between two perfect squares:*

This case is the inverse of formula **3,** Art. **497,** and may be expressed by the formula

$$a^2 - b^2 = (a + b)(a - b). \qquad (6.)$$

523. Since a may represent one quantity and b any other quantity, it is evident from formula **6** that any expression which is the difference between two perfect squares may be factored by the following

Rule.—*Extract the square roots of the first and the last term. Write the first root plus the second for one factor, and the first root minus the second for the other.*

EXAMPLE.—Factor $9x^8y^6 - 4$.

SOLUTION.—The square roots of the first and the last term are $3x^4y^3$ and 2. The sum of these roots is $3x^4y^3 + 2$, and the second subtracted from the first is $3x^4y^3 - 2$. Hence, by formula **6**, letting $a = 3x^4y^3$ and $b = 2$,

$$9x^8y^6 - 4 = (3x^4y^3 + 2)(3x^4y^3 - 2). \quad \text{Ans.}$$

EXAMPLE.—Factor $(a + b)^2 - m^2n^2$.

SOLUTION.—The square roots of the first and the last term are $a + b$ and mn. The sum of these roots is $a + b + mn$, and the second subtracted from the first is $a + b - mn$. Hence, by formula **6**, letting $a = a + b$ and $b = mn$,

$$(a + b)^2 - m^2n^2 = (a + b + mn)(a + b - mn). \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

524. Factor the following expressions:

1. $a^2 - 16$. Ans. $(a + 4)(a - 4)$.
2. $a^2 - 49c^8$. Ans. $(a + 7c^4)(a - 7c^4)$.
3. $81x^6y^4 - 1$. Ans. $(9x^3y^2 + 1)(9x^3y^2 - 1)$.
4. $(ax + by)^2 - 1$. Ans. $(ax + by + 1)(ax + by - 1)$.
5. $25x^4y^2 - (bx + 1)^2$. Ans. $[5x^2y + (bx + 1)][5x^2y - (bx + 1)] = (5x^2y + bx + 1)(5x^2y - bx - 1)$.
6. $1 - 169x^2y^4z^6$. Ans. $(1 + 13xy^2z^3)(1 - 13xy^2z^3)$.

525. In example 5, the expression $(bx + 1)^2$ should be regarded as a single term; in fact, any number of terms may be regarded as a single term by enclosing them in parenthesis and operating on them as though they were a single letter.

526. When solving any examples requiring the application of the rule in Art. **523**, first ascertain if the numerical coefficients of the two terms are perfect squares; if not, there is no use of examining further.

FRACTIONS.

DEFINITIONS.

527. A **fraction,** in algebra, is considered as an expression indicating division. The sign $\div$ is seldom used, it being more convenient to write the dividend, or quantity to be divided, above a horizontal line, with the divisor below it, in the form of a fraction.

Thus the fraction $\frac{a+b}{c-d}$ means that $a+b$ is to be divided by $c-d$, and is the same as $(a+b) \div (c-d)$. It is read "$a+b$ divided by $c-d$" or "$a+b$ *over* $c-d$." All fractions are read in this way in algebra, except simple numerical fractions, as $\frac{1}{2}$, $\frac{3}{3}\frac{5}{2}$, etc., which are read as in arithmetic.

528. The quantities above and below the line are called the **numerator** and the **denominator,** respectively, as in the case of numerical fractions. They are known as the **terms** of a fraction.

529. Since dividing any quantity by 1 does not change its value, we may write any quantity as a fraction by making the quantity itself the numerator, and 1 the denominator. Thus, $7x^2y$ may be written $\frac{7x^2y}{1}$, and not be altered in value.

530. A **reciprocal** of a quantity is 1 divided by that quantity. It is not necessarily written as a fraction, but when so written has the quantity for the denominator, and 1 for the numerator. Thus, the reciprocal of 5 is $\frac{1}{5}$, but the reciprocal of $\frac{1}{5}$ is $\frac{5}{1} = 5$; the reciprocal of $x^2 + y^2 = \frac{1}{x^2 + y^2}$; and the reciprocal of $\frac{2}{x^2 + y^2} = \frac{x^2 + y^2}{2}$. Hence, the reciprocal of a fraction may be obtained by inverting the fraction.

531. The **three signs of a fraction** are: the sign before the dividing line, which affects the entire fraction; the sign of the numerator; and the sign of the denominator.

When any one of these signs is omitted, it is understood to be plus. *Any two signs of a fraction may be changed without altering its value, but if any one, or all three, be changed, the value of the fraction will be changed from + to − or from − to +.*

When either the numerator or the denominator has more than one term, it should be enclosed in a parenthesis when performing operations affecting it as a whole. The parenthesis may be removed after the operations are completed.

Take the fraction $-\frac{a-b}{c-d}$; placing numerator and denominator in parentheses, we have $-\frac{(a-b)}{(c-d)}$. The signs of the numerator and denominator are each + and that of the fraction −.

Let the quotient of $a - b \div c - d = q$; then:

$$-\frac{+(a-b)}{+(c-d)} = -(+q) = -q.$$
$$-\frac{-(a-b)}{-(c-d)} = -(+q) = -q.$$
$$-\frac{+(a-b)}{-(c-d)} = -(-q) = +q.$$
$$-\frac{-(a-b)}{+(c-d)} = -(-q) = +q.$$
$$+\frac{+(a-b)}{+(c-d)} = +(+q) = +q.$$
$$+\frac{-(a-b)}{-(c-d)} = +(+q) = +q.$$
$$+\frac{+(a-b)}{-(c-d)} = +(-q) = -q.$$
$$+\frac{-(a-b)}{+(c-d)} = +(-q) = -q.$$

REDUCTION OF FRACTIONS.

532. To *reduce* a fraction is to change its form without changing its value. Thus, $\frac{10x}{5}$ and $\frac{20x}{10}$ have different forms, but like values, since $10x \div 5$ and $20x \div 10$ are each equal to $2x$.

The terms of a fraction may both be multiplied, or may both be divided by the same quantity without changing their value.

533. To reduce a fraction to its simplest form:

Rule.—*Resolve each term into its factors, and cancel factors which appear in both.*

534. *In performing all operations on fractions, the student must learn to use a polynomial factor as a single quantity, like a monomial factor.*

This is illustrated in the following examples, where there are polynomial factors in both numerator and denominator that can be canceled.

EXAMPLE.—Reduce $\frac{x^2+2xy+y^2}{x^2-y^2}$ to its simplest form.

SOLUTION.—Factoring both numerator and denominator,

$$\frac{x^2+2xy+y^2}{x^2-y^2}=\frac{(x+y)(x+y)}{(x+y)(x-y)}.$$

Canceling the common factor $x+y$ from both gives, as the result,

$$\frac{\cancel{(x+y)}(x+y)}{\cancel{(x+y)}(x-y)}=\frac{x+y}{x-y}. \quad \text{Ans.}$$

EXAMPLE.—Reduce $\frac{3x^5-6x^4y}{6x^2y^2-12xy^3}$ to its simplest form.

SOLUTION.— $\frac{3x^5-6x^4y}{6x^2y^2-12xy^3}=\frac{3x^4(x-2y)}{6xy^2(x-2y)}$, when factored.

Canceling the common factors, we have as the result,

$$\frac{\overset{x^3}{\cancel{3x^4}}\cancel{(x-2y)}}{\underset{2}{\cancel{6xy^2}}\cancel{(x-2y)}}=\frac{x^3}{2y^2}. \quad \text{Ans.}$$

535. Sometimes the whole numerator is contained in the denominator, or the denominator in the numerator. The numerator or denominator will then reduce to the number 1.

EXAMPLE.—Reduce $\frac{b+3c^2}{2b^2+6bc^2}$ to its simplest form.

SOLUTION.— $\frac{b+3c^2}{2b^2+6bc^2}=\frac{\cancel{b+3c^2}}{2b\cancel{(b+3c^2)}}=\frac{1}{2b}.$ Ans.

Example.—Reduce $\frac{x^6-1}{x^3-1}$ to its simplest form.

Solution.— $\frac{x^6-1}{x^3-1} = \frac{(x^3+1)\cancel{(x^3-1)}}{\cancel{x^3-1}} = \frac{x^3+1}{1} = x^3+1$. **Ans.**
(Art. 529.)

536. From the last example it will be seen that division may sometimes be performed by cancelation. Thus, $\frac{x^6-1}{x^3-1}$ means $(x^6-1) \div (x^3-1)$, and the divisor x^3-1 canceled from the dividend x^6-1 gives the quotient x^3+1.

A factor must be common to each term of the numerator and to each term of the denominator in order to be canceled. Thus, the factor x can not be canceled from $\frac{3ax}{x+4m}$ because it is not common to both terms of the denominator.

EXAMPLES FOR PRACTICE.

537. Reduce the following to their simplest form.

1. $\frac{3a+3b}{a^2-b^2}$. Ans. $\frac{3}{a-b}$.
2. $\frac{x^4-y^4}{x^2-y^2}$. Ans. x^2+y^2.
3. $\frac{54a^3b^5c^2}{72a^2b^2c}$. Ans. $\frac{3ab^3c}{4}$.
4. $\frac{12a^2x^3}{36a^3x^5}$. Ans. $\frac{1}{3ax^2}$.
5. $\frac{n^3-2n^2}{n^2-4n+4}$. Ans. $\frac{n^2}{n-2}$.

538. When fractions are to be added or subtracted, it is necessary to so reduce them that all the denominators will be alike. This is called **reducing them to a common denominator.** See Arithmetic, Arts. **95** to **99.**

539. To reduce fractions to a common denominator:

Rule.—*Resolve each denominator into its factors.*

Take each factor as many times as it occurs in any one denominator, and find the product of these factors.

Divide this product by each of the denominators. Multiply the corresponding numerators by these quotients, for new numerators. Write each new numerator with the common denominator beneath it.

EXAMPLE.—Reduce $\frac{7a}{x+y}$, $\frac{3ab}{x^2-y^2}$, and $\frac{2b}{(x+y)^2}$ to a common denominator.

SOLUTION.—Factoring the denominators, we have $x+y$ not factorable. $x^2-y^2=(x+y)(x-y)$, and $(x+y)^2=(x+y)(x+y)$. Now we have two separate factors, $x+y$ and $x-y$, of which $x+y$ occurs twice in $(x+y)^2$. Hence, our common denominator is $(x+y)(x+y)(x-y)=(x+y)^2(x-y)$. Dividing this product by $x+y$ we have $(x+y)(x-y)=x^2-y^2$ as our quotient. Hence, our first new numerator is $7a(x^2-y^2)$ and the new fraction is $\frac{7a(x^2-y^2)}{(x+y)^2(x-y)}$. Similarly, $\frac{3ab}{x^2-y^2}$ becomes $\frac{3ab(x+y)}{(x+y)^2(x-y)}$, and $\frac{2b}{(x+y)^2}$ becomes $\frac{2b(x-y)}{(x+y)^2(x-y)}$.

The student should note that this denominator can be written in several different ways, and he should not become confused if his work does not always agree with the answer. Besides $(x+y)(x+y)(x-y)$ and $(x+y)^2(x-y)$, it may be written $(x^2-y^2)(x+y)$, $(x^2+2xy+y^2)(x-y)$, or $x^3+x^2y-xy^2-y^3$. These five expressions have exactly the same value. The student should prove this statement by substituting numbers for x and y.

EXAMPLES FOR PRACTICE.

540. Reduce the following to a common denominator:

1. $\frac{3yz}{2x}$, $\frac{4xz}{3y}$, and $\frac{5xy}{4z}$. Ans. $\frac{18y^2z^2}{12xyz}$, $\frac{16x^2z^2}{12xyz}$, and $\frac{15x^2y^2}{12xyz}$.

2. $\frac{x^2y}{10}$, $\frac{xyz}{15}$, and $\frac{7yz^2}{30}$. Ans. $\frac{3x^2y}{30}$, $\frac{2xyz}{30}$, and $\frac{7yz^2}{30}$.

3. $\frac{2}{a^3x^3}$, $\frac{3}{ax^3}$, and $\frac{4}{a^2x}$. Ans. $\frac{2}{a^3x^3}$, $\frac{3a^2}{a^3x^3}$, and $\frac{4ax^2}{a^3x^3}$.

4. $\frac{m+n}{m-n}$, and $\frac{m-n}{m+n}$. Ans. $\frac{m^2+2mn+n^2}{m^2-n^2}$, and $\frac{m^2-2mn+n^2}{m^2-n^2}$.

5. $\frac{2}{x}$, $\frac{3}{2x-1}$, and $\frac{2x-1}{4x^2-1}$.

Ans. $\frac{2(4x^2-1)}{x(4x^2-1)}$, $\frac{3x(2x+1)}{x(4x^2-1)}$, and $\frac{x(2x-1)}{x(4x^2-1)}$.

ADDITION AND SUBTRACTION OF FRACTIONS.

541. To add or subtract fractions:

Rule.—*Reduce the fractions, if necessary, to a common denominator. Add or subtract the numerators, and write the result over the common denominator.*

EXAMPLE.—Find the sum of $\frac{2a-b}{5}$ and $\frac{a+b}{4}$.

SOLUTION.— $\frac{2a-b}{5}$ and $\frac{a+b}{4}$, reduced to a common denominator, become $\frac{4(2a-b)}{20}$ and $\frac{5(a+b)}{20}$, which are equal, respectively, to $\frac{8a-4b}{20}$ and $\frac{5a+5b}{20}$. Adding the numerators, we have $8a-4b+5a+5b=13a+b$. The result written over the common denominator gives as the sum, $\frac{13a+b}{20}$. The work is written as follows:

$$\frac{2a-b}{5}+\frac{a+b}{4}=\frac{8a-4b}{20}+\frac{5a+5b}{20}=$$
$$\frac{8a-4b+5a+5b}{20}=\frac{13a+b}{20}.\quad \text{Ans.}$$

EXAMPLE.—Subtract $\frac{6b-2}{3b}$ from $\frac{4a-1}{2a}$.

SOLUTION.—Reducing the fractions to a common denominator, $\frac{4a-1}{2a}-\frac{6b-2}{3b}=\frac{12ab-3b}{6ab}-\frac{12ab-4a}{6ab}$. Subtracting the second numerator from the first, and writing the result over the common denominator, we have $\frac{12ab-3b}{6ab}-\frac{12ab-4a}{6ab}=\frac{(12ab-3b)-(12ab-4a)}{6ab}=\frac{12ab-3b-12ab+4a}{6ab}$, with the parentheses removed. Combining like terms in the numerator gives as the result $\frac{4a-3b}{6ab}$. Ans.

542. *If, as in the example just given, the numerator of the fraction to be subtracted has more than one term,* care must be taken *to change the sign of every term before combining.* It will usually be convenient to enclose the whole numerator in a parenthesis before combining. The parenthesis may then be removed by the rules of Arts. **482** and **483.**

EXAMPLE.—Simplify $\frac{x^3}{x-1} - \frac{x^2}{x+1} - \frac{x}{x-1} + \frac{1}{x+1}$.

SOLUTION.—Reducing to the common denominator $x^2 - 1$,

$$\frac{x^3}{x-1} - \frac{x^2}{x+1} - \frac{x}{x-1} + \frac{1}{x+1} = \frac{x^4 + x^3}{x^2 - 1} - \frac{x^3 - x^2}{x^2 - 1} - \frac{x^2 + x}{x^2 - 1} + \frac{x - 1}{x^2 - 1}.$$

Adding or subtracting the numerators as required, we have

$$\frac{(x^4 + x^3) - (x^3 - x^2) - (x^2 + x) + (x - 1)}{x^2 - 1},$$

which, with the parentheses removed =

$$\frac{x^4 + x^3 - x^3 + x^2 - x^2 - x + x - 1}{x^2 - 1}.$$

Combining like terms we have as the result

$$\frac{x^4 - 1}{x^2 - 1} = x^2 + 1. \quad \text{Ans.}$$

EXAMPLE.—Simplify $\frac{1}{(x-2)^2} + \frac{1}{2-x}$.

SOLUTION.—If the denominator of the second fraction were written $x - 2$ instead of $2 - x$, $(x-2)^2$ would be the common denominator. By Art. **531**, the signs of the denominator and the sign before the fraction $\frac{1}{2-x}$ may be changed, giving $-\frac{1}{-2+x} = -\frac{1}{x-2}$. (Art. **456.**) Hence, we have $\frac{1}{(x-2)^2} + \frac{1}{2-x} = \frac{1}{(x-2)^2} - \frac{1}{x-2}$, which, when reduced to a common denominator, is equal to

$$\frac{1}{(x-2)^2} - \frac{x-2}{(x-2)^2} = \frac{1-(x-2)}{(x-2)^2} = \frac{1-x+2}{(x-2)^2} = \frac{3-x}{(x-2)^2}. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

543. Simplify the following:

1. $\frac{x}{3} + \frac{x}{4} + \frac{x}{5}$. Ans. $\frac{47x}{60}$.

2. $\frac{4x-3}{5} + \frac{7x+1}{3} + \frac{3x}{2}$. Ans. $\frac{139x-8}{30}$.

3. $\frac{1}{x-y} - \frac{1}{x^2-y^2}$. Ans. $\frac{x+y-1}{x^2-y^2}$.

4. $\frac{a^2+b^2}{2} - \frac{(a+b)^2}{4}$. Ans. $\frac{2(a^2+b^2)-(a^2+2ab+b^2)}{4} = \frac{(a-b)^2}{4}$,

after removing parentheses and combining.

5. $\frac{a^2}{a^2-1} + \frac{a}{a-1} - \frac{a}{a+1}$. Ans. $\frac{a^2+2a}{a^2-1}$.

6. $\frac{4m^2+1}{4m^2} - \frac{3m-1}{12m^3} + \frac{1-12n}{12n}$. Ans. $\frac{n+m^2}{12m^3n}$.

7. $\dfrac{y}{(x+y)^2} + \dfrac{y}{x^2-y^2} - \dfrac{1}{x+y}$.

Ans. $\dfrac{y(x-y)+y(x+y)-(x^2-y^2)}{(x+y)^2(x-y)} = \dfrac{2xy-x^2+y^2}{x^3+x^2y-xy^2-y^3}$.

8. $\dfrac{x}{x+1} + \dfrac{x}{1-x} + \dfrac{3x}{x^2-1}$. Ans. $\dfrac{x}{x^2-1}$.

MULTIPLICATION OF FRACTIONS.

544. Multiplication, in fractions, is the process of finding a fractional part of a fraction. Thus, $\frac{3}{4} \times \frac{1}{2}$ means $\frac{1}{2}$ of $\frac{3}{4}$. One-half of $\frac{3}{4}$ inch, for example, is $\frac{3}{8}$ inch; $\frac{2}{3}$ of $\frac{3}{4}$ inch is $\frac{1}{2}$, or $\frac{6}{12}$, inch. The result in each case is the same as that which would be obtained by finding the product of the numerators and writing it over the product of the denominators.

545. Hence, to multiply fractions:

Rule.—*Multiply the numerators together for the numerator of the product, and the denominators together for the denominator of the product.*

546. Any number of fractions may be multiplied together. The operation may be very much shortened by resolving the terms of the fractions into their factors, and canceling. The product should be reduced to its simplest form.

EXAMPLE.—Find the product of $\frac{6a^2}{5}$, $\frac{2ab}{3c}$, and $\frac{2ac}{b^2}$.

SOLUTION.—The product of the numerators is $6a^2 \times 2ab \times 2ac = 24a^4bc$, and of the denominators, $5 \times 3c \times b^2 = 15b^2c$. Writing $24a^4bc$ over $15b^2c$, we have for the product, $\frac{24a^4bc}{15b^2c} = \frac{8a^4}{5b}$, when reduced to its lowest terms. The work is written as follows:

$$\frac{\overset{2}{\cancel{6}}a^2}{5} \times \frac{2a\cancel{b}}{\cancel{3c}} \times \frac{2a\cancel{c}}{\underset{b}{\cancel{b^2}}} = \frac{8a^4}{5b}. \quad \text{Ans.}$$

EXAMPLE.—Find the product of $\frac{x^2+2x}{(x-1)^2}$, x^2-1, and $\frac{x^2-4x+4}{x^2-4}$.

SOLUTION.—First make $x^2 - 1$ a fraction by writing 1 for its denominator, thus, $\frac{x^2-1}{1}$; then, factoring both terms of each fraction, we have

$$\frac{x^2+2x}{(x-1)^2} \times \frac{x^2-1}{1} \times \frac{x^2-4x+4}{x^2-4} =$$

$$\frac{x\cancel{(x+2)}(x+1)\cancel{(x-1)}\cancel{(x-2)}(x-2)}{\cancel{(x-1)}(x-1)\cancel{(x-2)}\cancel{(x+2)}} = \frac{x(x+1)(x-2)}{x-1}. \text{ Ans.}$$

EXAMPLE.—Find the product of $\frac{1}{a^3} - \frac{4c^2}{a}$, and $\frac{a^3}{1+2ac}$.

SOLUTION.—Performing the subtraction, $\frac{1}{a^3} - \frac{4c^2}{a} = \frac{1-4a^2c^2}{a^3}$.

Multiplying, $\frac{1-4a^2c^2}{a^3} \times \frac{a^3}{1+2ac} = \frac{\overset{1-2ac}{\cancel{(1-4a^2c^2)}}\cancel{a^3}}{\cancel{(1+2ac)}\cancel{a^3}} = 1 - 2ac$. Ans.

EXAMPLES FOR PRACTICE.

547. Multiply the following:

1. $\frac{3a^2bc}{5abc^2}$ by $\frac{10ab^2c}{3abc}$. Ans. $\frac{2ab}{c}$.

2. $\frac{5x^2y}{7x}$ by $21xy$. Ans. $15x^2y^2$.

Find the product of:

3. $\frac{3x^2y}{4xz^2}$, $\frac{5y^2z}{6xy}$, and $\frac{-12x^2}{2xy^2}$. Ans. $-\frac{15x}{4z}$.

4. $\frac{x^2-y^2}{c^3-d^3}$, $\frac{c-d}{(x+y)^2}$, and $\frac{x^3+y^3}{x-y}$. Ans. $\frac{x^2-xy+y^2}{c^2+cd+d^2}$.

5. $\frac{4y}{x} - \frac{16}{xy}$, and $\frac{1}{2y+4}$. Ans. $\frac{2y-4}{xy}$.

6. $\frac{a+b}{2} + \frac{a-b}{4}$, and $\frac{4}{9a^2+6ab+b^2}$. Ans. $\frac{1}{3a+b}$.

DIVISION OF FRACTIONS.

548. Division, in fractions, is the reverse of multiplication, and is the process we use when, given one of two fractions and their product, we are required to find the other. For example, we are required to divide $\frac{a}{4}$ by $\frac{1}{2}$. We wish to find such a fraction that, multiplied by $\frac{1}{2}$, will give $\frac{a}{4}$.

This fraction is $\frac{a}{2}$, for $\frac{a}{2} \times \frac{1}{2} = \frac{a}{4}$. Also $\frac{x}{5} \div \frac{x}{7} = \frac{7}{5}$, since $\frac{7}{5} \times \frac{x}{7} = \frac{x}{5}$. If, in this case, we had inverted the divisor and multiplied, we should have had $\frac{x}{5} \times \frac{7}{x} = \frac{7}{5}$.

549. Hence, to divide by a fraction:

Rule.—*Invert the divisor, and proceed as in multiplication.*

EXAMPLE.—Divide $\frac{3a^2b}{5x^3y}$ by $\frac{9ab^3}{10x^4y^2}$.

SOLUTION.—The divisor inverted $= \frac{10x^4y^2}{9ab^3}$.

$$\text{Hence, } \frac{3a^2b}{5x^3y} \div \frac{9ab^3}{10x^4y^2} = \frac{3a^2b}{5x^3y} \times \frac{10x^4y^2}{9ab^3} = \frac{\cancel{3} \times \overset{2}{\cancel{10}}\overset{a}{\cancel{a^2}}\cancel{b}\overset{x}{\cancel{x^4}}\overset{y}{\cancel{y^2}}}{\cancel{5} \times \underset{3}{\cancel{9}}\cancel{a}\underset{b^2}{\cancel{b^3}}\cancel{x^3}\cancel{y}} = \frac{2axy}{3b^2}. \text{ Ans.}$$

EXAMPLE.—Divide $x^2 + 2x + 1$ by $\frac{x+1}{x-1}$.

SOLUTION.—By Art. **529**, $(x^2 + 2x + 1) \div \frac{x+1}{x-1} = \frac{x^2 + 2x + 1}{1} \times \frac{x-1}{x+1} = \frac{\cancel{(x+1)}\,(x+1)\,(x-1)}{\cancel{x+1}} = x^2 - 1$. Ans.

EXAMPLES FOR PRACTICE.

550. Divide the following:

1. $\frac{9x^3 - 3x^4}{24}$ by $\frac{3x}{8}$. Ans. $\frac{3x^2 - x^3}{3}$.
2. $\frac{ab - bx}{a + z}$ by $\frac{ac - cx}{a + z}$. Ans. $\frac{b}{c}$.
3. $\frac{1 - 8b^2 + 16b^4}{1 + 2b}$ by $\frac{1 - 4b^2}{3a}$. Ans. $3a(1 - 2b)$.
4. $6a^4cd - 6abcd$ by $\frac{6acd}{a^2 + ab + b^2}$. Ans. $a^3 - b^3$.

MIXED QUANTITIES AND COMPLEX FRACTIONS.

551. An **integral expression** is one containing neither fractions nor negative exponents. The expression $a^2 + 2ab$ is integral, but the expressions $a^2 + \frac{1}{2ab}$, $2a^{-2}$, $\frac{3}{a^2 + b}$ are not. The expression $2a^{-2}$ is only another way of writing $\frac{2}{a^2}$.

The use of negative exponents will be explained in subsequent paragraphs.

552. The **integral part** of an expression is that part which, if taken by itself, would be an integral expression.

553. A **mixed quantity** is an expression containing both integral and fractional parts, as $2a^2 - \frac{c+d}{4}$. Considering the integral part, $2a^2$, as a fraction with a denominator 1 (Art. **529**), a mixed quantity becomes simply the indicated addition or subtraction of two fractions; thus $2a^2 - \frac{c+d}{4} = \frac{2a^2}{1} - \frac{c+d}{4}$.

554. A fraction may be reduced to either an entire or mixed quantity by dividing the numerator by the denominator, provided the division be possible. It frequently happens that by performing the indicated division, the fraction will be reduced to a simpler form. The case of reducing a fraction to an entire quantity was taken up in Art. **535.**

EXAMPLE.—Simplify $\frac{4x^2 + 12x - 1}{2x + 3}$.

SOLUTION.—Performing the indicated division,

$$\begin{array}{l} 2x+3)\,4x^2 + 12x - 1\,(2x + 3 - \frac{10}{2x+3}. \quad \text{Ans.} \\ \underline{4x^2 + 6x} \\ 6x - 1 \\ \underline{6x + 9} \\ -10 \end{array}$$

555. Mixed quantities are frequently more convenient to handle as fractions.

To reduce a mixed quantity to a fraction:

Rule.—*Write the integral part with a denominator 1, and perform the indicated addition or subtraction.*

EXAMPLE.—Reduce $x^2 + xy + y^2 - \frac{b}{x-y}$ to a fraction.

SOLUTION.— $x^2 + xy + y^2 - \frac{b}{x-y} = \frac{x^2 + xy + y^2}{1} - \frac{b}{x-y}$; subtracting the second fraction from the first gives

$$\frac{(x^2 + xy + y^2)(x - y) - b}{x - y} = \frac{x^3 - y^3 - b}{x - y}. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

556. Solve the following:

1. Reduce $\frac{a^2c+b^2}{c}$ to a mixed quantity. Ans. $a^2+\frac{b^2}{c}$.

2. Simplify $\frac{x^2+4xy+5y^2-3x}{x+2y}$. Ans. $x+2y-3+\frac{y^2+6y}{x+2y}$.

3. Reduce $x+3-\frac{7x+3}{2x+1}$ to a fraction. Ans. $\frac{2x^2}{2x+1}$.

4. From $3a+\frac{a+b}{d}$ subtract $a-\frac{a-b}{d}$.

Ans. $2a+\frac{2a}{d}=\frac{2a(d+1)}{d}$.

5. Divide $m+n-\frac{2n}{m-n}$ by $m-n-\frac{2n}{m+n}$. Ans. $\frac{m+n}{m-n}$.

SUGGESTION.—First reduce the mixed quantities to fractions.

557. A **complex fraction** is one which contains fractions in one or both of its terms. Thus, $\frac{a+\frac{x}{y}}{a+x}$, $\frac{a-b}{\frac{x}{y}}$, and $\frac{\frac{a}{b}}{\frac{c}{d}}$ are complex fractions.

558. Complex fractions can be reduced by performing the indicated division; thus, $\frac{\frac{5}{8}}{\frac{3}{4}}=\frac{5}{8}\div\frac{3}{4}=\frac{5}{8}\times\frac{4}{3}=\frac{5}{6}$. A much simpler way is to multiply both terms by the least common denominator of the fractions contained. Thus, $\frac{\frac{5}{8}\times 8}{\frac{3}{4}\times 8}=\frac{5}{6}$.

559. Hence, to simplify a complex fraction:

Rule.—*Multiply both terms by the common denominator of the fractional parts.*

EXAMPLE.—Simplify $\frac{\frac{x}{y}-\frac{y}{x}}{\frac{1}{y}-\frac{1}{x}}$.

SOLUTION.—The common denominator of the fractional parts is xy. Multiplying each term by this, we have

$$\frac{\frac{x}{y} \times xy - \frac{y}{x} \times xy}{\frac{1}{y} \times xy - \frac{1}{x} \times xy} = \frac{x^2 - y^2}{x - y} = x + y. \quad \text{Ans.}$$

The multiplication can frequently be performed mentally, without writing the common denominator, at the same time canceling common factors.

EXAMPLE.—Simplify $\dfrac{1}{1 + \dfrac{a}{1 + a + \dfrac{2a^2}{1 - a}}}$.

SOLUTION.—This is the case of a complex fraction in which the denominator is itself a complex fraction.

First, consider the part $\dfrac{a}{1 + a + \dfrac{2a^2}{1 - a}}$.

Multiplying both terms by $1 - a$, we have

$$\frac{a(1 - a)}{(1 + a)(1 - a) + 2a^2} = \frac{a - a^2}{1 - a^2 + 2a^2} = \frac{a - a^2}{1 + a^2}.$$

The fraction thus becomes $\dfrac{1}{1 + \dfrac{a - a^2}{1 + a^2}}$.

Multiplying both terms by $1 + a^2$, the common denominator, we have

$$\frac{1 + a^2}{1 + a^2 + a - a^2} = \frac{1 + a^2}{1 + a}. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

560. Simplify the following:

1. $\dfrac{\frac{3ac^2}{16}}{24}$. Ans. $\dfrac{ac^2}{128}$.

2. $\dfrac{1 + \frac{a}{c}}{c - \frac{a^2}{c}}$. Ans. $\dfrac{1}{c - a}$.

3. $\dfrac{2\frac{7}{8}}{8 - 2x + \frac{x^2}{8}}$. Ans. $\dfrac{23}{(8 - x)^2}$.

SUGGESTION.— $2\frac{7}{8}$ means $2 + \frac{7}{8}$. Hence, for the numerator multiply 2 by the least common denominator 8, and add 7.

4. $\dfrac{1}{x + \dfrac{1}{1 + \dfrac{x + 1}{3 - x}}}$. Ans. $\dfrac{4}{3x + 3}$.

THEORY OF EXPONENTS.

561. An exponent has already been defined in Art. **444.** In addition to positive integral exponents, as there defined, we frequently have to deal with negative and with fractional exponents. Thus, in the quantities a^3, a^{-3}, and $a^{\frac{2}{3}}$, we have the positive integral exponent 3, the negative exponent -3, and the fractional exponent $\frac{2}{3}$. The rules for positive and integral exponents apply also to negative and fractional exponents.

562. Since letters may represent numbers, they may be used for exponents, the same as figures. Thus, a^n means that a is to be taken as many times as a factor as there are units in n, or $a \times a \times a$, etc., to n factors. Such exponents are called **literal exponents.** Fractional, negative, and literal exponents are all read by using the word exponent. Thus, $a^{\frac{2}{3}}$, a^{-4}, $a^{\frac{n}{m}}$, and a^n are read, "a, exponent $\frac{2}{3}$," "a, exponent minus 4," "a, exponent n over m," and "a, exponent n," respectively.

563. A **fractional exponent** is an expression of a root or of a power and a root combined. The numerator of the fraction denotes the power, and the denominator denotes the root. Thus, $\sqrt[3]{a}$ may be written $a^{\frac{1}{3}}$. Suppose we wish to find the cube root of a^6; instead of writing it $\sqrt[3]{a^6} = a^2$, we may write it $a^{\frac{6}{3}} = a^2$. Hence,

The numerator of a fractional exponent denotes a power, and the denominator, a root.

For example, $a^{\frac{1}{2}} = \sqrt{a}$; $c^{\frac{2}{3}} = \sqrt[3]{c^2}$; $x^{\frac{m}{n}} = \sqrt[n]{x^m}$; $2c^{\frac{m}{5}} = 2\sqrt[5]{c^m}$, the exponent $\frac{m}{5}$ applying only to the c.

564. The meaning of **negative exponents** may be illustrated as follows: Let it be required to divide a^4 by a^2. By Art. **502,** we have, $\frac{a^4}{a^2} = a^{4-2} = a^2$; likewise, $a^2 \div a^2 = \frac{a^2}{a^2} = a^{2-2} = a^0$; also, $a^0 \div a^2 = \frac{a^0}{a^2} = a^{0-2} = a^{-2}$, etc. From this, it will be seen that $\frac{a^0}{a^2} = a^{-2}$; but, by Art. **503,** $a^0 = 1$, so

that 1 may be written for a^0, in this expression, thus: $\frac{a^0}{a^2} = \frac{1}{a^2} = a^{-2}$. Hence,

A quantity affected with a negative exponent denotes the reciprocal of the same quantity affected with an equal positive exponent. (Art. **530.**)

565. Since in $\frac{1}{a^2} = a^{-2}$, the a^{-2} changes to a^2 when placed in the denominator, we may state the following principle:

A factor may be changed from the numerator to the denominator, or from the denominator to the numerator, if the sign of its exponent be changed.

For example, $\frac{n^{-3}}{ab} = \frac{1}{abn^3}$; $\frac{n}{ab^{-\frac{1}{2}}} = \frac{nb^{\frac{1}{2}}}{a}$; $\frac{x^{-\frac{3}{5}}}{5y^{-1}} = \frac{y}{5x^{\frac{3}{5}}}$, etc. In the last, the positive exponent 1 of the y is not written.

EXAMPLE.—Express, with positive exponents,

$$a^{-1}b^{-2}c^3 + a^{-2}b^{-\frac{3}{4}}c^{-\frac{1}{3}} + a^3b^{-2}.$$

SOLUTION.—Since these terms may be taken as fractions, with 1 for the denominators, we have, by transferring the letters with negative exponents to the denominators,

$$a^{-1}b^{-2}c^3 + a^{-2}b^{-\frac{3}{4}}c^{-\frac{1}{3}} + a^3b^{-2} = \frac{c^3}{ab^2} + \frac{1}{a^2b^{\frac{3}{4}}c^{\frac{1}{3}}} + \frac{a^3}{b^2}. \text{ Ans.}$$

566. The student must note very carefully that factors of an entire term only can be changed from numerator to denominator, or vice versa, and that when thus changed they become factors of the whole of the other term. Thus, in $\frac{a}{bc^{-2} + d}$, c^{-2} *can not* be transferred to the numerator by merely changing the sign of the exponent. The exponent may, however, be made positive by multiplying both terms by c^2; thus, $\frac{a \times c^2}{(bc^{-2} + d) \times c^2} = \frac{ac^2}{b + c^2d}$. In $\frac{ac^{-2}}{b + d}$, if we transfer the c^{-2}, it becomes $\frac{a}{c^2(b + d)}$, c^2 becoming a factor of the *entire* denominator.

EXAMPLE.—Clear $x^2y^{-2}z^{-1} + \frac{2xy}{y^{-1} - x^3} - \frac{3a^{-1}b^{-2}c^3}{a^2 + b}$ of negative exponents.

SOLUTION.—Treat each term of the expression separately. $x^2y^{-2}z^{-1} = \frac{x^2y^{-2}z^{-1}}{1}$; changing the factors with negative exponents to the denominator, and at the same time changing the signs of the exponents, we have $\frac{x^2}{y^2z}$. In $\frac{2xy}{y^{-1} - x^3}$, y^{-1} is not a factor of the whole denominator, so we must multiply both terms of the fraction by the reciprocal of y^{-1} or y; thus, $\frac{2xy \times y}{(y^{-1} - x^3) \times y} = \frac{2xy^2}{1 - x^3y}$. In $\frac{3a^{-1}b^{-2}c^3}{a^2 + b}$, a^{-1} and b^{-2} are factors of the entire numerator, so we write them as factors of the entire denominator, with the signs of the exponents changed; thus, $\frac{3a^{-1}b^{-2}c^3}{a^2 + b} = \frac{3c^3}{ab^2(a^2 + b)} = \frac{3c^3}{a^3b^2 + ab^3}$. Hence, $x^2y^{-2}z^{-1} + \frac{2xy}{y^{-1} - x^3} - \frac{3a^{-1}b^{-2}c^3}{a^2 + b} = \frac{x^2}{y^2z} + \frac{2xy^2}{1 - x^3y} - \frac{3c^3}{a^3b^2 + ab^3}$. Ans.

EXAMPLE.—Solve the following:

$$a^3 \times a^{-1};\ n \times n^{-\frac{1}{4}};\ 2c^{-\frac{2}{3}} \times \frac{1}{-3\sqrt[3]{c^2}};\ c^{\frac{n}{m}} + c^{\frac{2n}{m}};\ x^2 + \sqrt[5]{x^2}.$$

Write the answers with positive exponents.

SOLUTION.— $a^3 \times a^{-1} = a^{3+(-1)} = a^{3-1} = a^2$. Ans.

$$n \times n^{-\frac{1}{4}} = n^{1+(-\frac{1}{4})} = n^{1-\frac{1}{4}} = n^{\frac{3}{4}}. \quad \text{Ans.}$$

$$2c^{-\frac{2}{3}} \times \frac{1}{-3\sqrt[3]{c^2}} = \frac{2c^{-\frac{2}{3}}}{1} \times -\frac{1}{3\sqrt[3]{c^2}} = \frac{2}{c^{\frac{2}{3}}} \times -\frac{1}{3c^{\frac{2}{3}}} = -\frac{2 \times 1}{c^{\frac{2}{3}} \times 3c^{\frac{2}{3}}} =$$

$$-\frac{2}{3c^{\frac{2}{3}+\frac{2}{3}}} = -\frac{2}{3c^{\frac{4}{3}}}. \quad \text{Ans.}$$

$$c^{\frac{n}{m}} + c^{\frac{2n}{m}} = c^{\frac{n}{m}-\frac{2n}{m}} = c^{\frac{n-2n}{m}} = c^{\frac{-n}{m}} = c^{-\frac{n}{m}} = \frac{1}{c^{\frac{n}{m}}}. \quad \text{Ans.}$$

$$x^2 + \sqrt[5]{x^2} = x^2 + x^{\frac{2}{5}} = x^{2-\frac{2}{5}} = x^{\frac{10}{5}-\frac{2}{5}} = x^{\frac{8}{5}}. \quad \text{Ans.}$$

567. A letter may be raised to any power by multiplying its exponent by the index of the power. Thus, $(x^2)^3 = x^2 \times x^2 \times x^2 = x^{2\times3} = x^6$, $(a^2b^3)^5 = a^{2\times5}b^{3\times5} = a^{10}b^{15}$, $(a^2)^{\frac{1}{2}} = a$, $(a^2)^{-\frac{1}{2}} = a^{-1}$, etc.

EXAMPLE.—Find the values of the following: $(a^{-1})^{-\frac{1}{2}}$; $(cd^{-2})^{\frac{1}{2}}$; $(x^a)^{-b} + (x^{-a})^{-b}$.

SOLUTION.—In the first, multiplying the exponents, we have $-1 \times -\frac{1}{3} = \frac{1}{3}$. Hence, $(a^{-1})^{-\frac{1}{3}} = a^{\frac{1}{3}}$, or $\sqrt[3]{a}$. Ans. In like manner, $(cd^{-2})^{\frac{5}{2}} = c^{\frac{5}{2}}d^{-5}$, Ans., since $1 \times \frac{5}{2} = \frac{5}{2}$, and $-2 \times \frac{5}{2} = -5$.

In the next one, $(x^a)^{-b} = x^{-ab}$ and $(x^{-a})^{-b} = x^{ab}$. Dividing, $x^{-ab} \div x^{ab} = x^{-ab-ab} = x^{-2ab}$. Ans.

568. A root of a letter affected with an exponent is extracted by dividing the exponent by the index of the root. Thus, $\sqrt[4]{y^{12}} = y^{\frac{12}{4}} = y^{12 \div 4} = y^3$. From this principle the following rule may be deduced.

To extract any root of a monomial:

Rule.—*Extract the required root of the numerical coefficient, and divide the exponent of each letter by the index of the root. Make the sign of every even root of a positive quantity $\pm$, and the sign of every odd root of any quantity the same as that of the quantity.*

EXAMPLE.—Find the value of $\sqrt[4]{256a^4b^{12}c^8}$.

SOLUTION.—The 4th root of 256 is 4. The exponent of a in the root is $4 \div 4 = 1$; of b, $12 \div 4 = 3$; and of c, $8 \div 4 = 2$. As this is an even root of a positive quantity, the sign should be $\pm$. Hence, $\sqrt[4]{256a^4b^{12}c^8} = \pm 4ab^3c^2$. Ans.

EXAMPLE.—Find the value of $\sqrt[3]{\dfrac{27m^3x^9}{a^9b^6c^{12}}}$.

SOLUTION.— $\sqrt[3]{27m^3x^9} = 3mx^3$; $\sqrt[3]{a^9b^6c^{12}} = a^3b^2c^4$. The quantity is positive, and, as this is an odd root, its sign must be the same, or positive.

$$\text{Hence, } \sqrt[3]{\frac{27m^3x^9}{a^9b^6c^{12}}} = \frac{3mx^3}{a^3b^2c^4}. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

569. Clear the following of negative exponents:

1. $x^2y^{-2}z^{-\frac{1}{2}}$. Ans. $\dfrac{x^2}{y^2z^{\frac{1}{2}}}$.

2. $3a^{-1}b + \dfrac{2a}{b^{-3}c^{-1}} + c^{-1}$. Ans. $\dfrac{3b}{a} + 2ab^3c + \dfrac{1}{c}$.

3. $\dfrac{4a^{-2}(c+d)}{2c+d}$. Ans. $\dfrac{4(c+d)}{a^2(2c+d)}$.

Express the following without radical signs:

4. $\sqrt[3]{b^{-2}}$. Ans. $(b^{-2})^{\frac{1}{3}}$ or $b^{-\frac{2}{3}}$.

5. $4a\sqrt{a^{-1}b^{-3}}$. Ans. $4aa^{-\frac{1}{2}}b^{-\frac{3}{2}} = 4a^{\frac{1}{2}}b^{-\frac{3}{2}}$.

Find the values of the following:

6. $m^{\frac{1}{2}} \times m^{-\frac{1}{3}}$. Ans. $m^{\frac{1}{6}}$.

7. $2ab^{\frac{1}{2}} \times a^{-\frac{1}{3}}b$. Ans. $2a^{\frac{2}{3}}b^{\frac{3}{2}}$.

8. $c^{\frac{n}{2}} \div \sqrt{c^{-n}}$. Ans. c^n.

9. $2x^{-2} + (x^2)^{-\frac{1}{2}}$. Ans. $2x^{-1}$.

10. $\left(cd^{-\frac{n}{m}}\right)^{2m} \times \sqrt{d^{4n}}$. Ans. c^{2m}.

EQUATIONS.

DEFINITIONS.

570. As defined in Art. **437**, an **equation** is a statement of equality between two expressions, as $x + 6 = 14$.

Every equation has two parts, called the **first** and the **second member.** The first member is the part on the left of the sign of equality, and the second member the part on the right of that sign. In $x + 6 = 14$, $x + 6$ is the first member, and 14 is the second member.

571. Equations usually consist of **known** and **unknown quantities**; that is, of quantities whose values are given, and of quantities whose values are not given, but are to be found. Thus, in $x + 6 = 14$, 6 and 14 are known quantities, and x is unknown; but since by the statement of the equation, $x + 6$ must equal 14, x must have such a value that when added to 6 the sum will be 14. Hence, the value of x is fixed for this particular case, and in a similar manner the value of a single unknown quantity in any equation is fixed by the relations that it bears to the known quantities, and this value can usually be found.

572. **To solve an equation** is to find the value of the unknown quantity. This is done by a series of **transformations** by which the first member becomes the unknown quantity, and the second member becomes a known quantity, which is, therefore, the value of the unknown quantity.

TRANSFORMATIONS.

573. In transforming an equation, the equality of its members must be preserved; otherwise the existing relations between the known and unknown quantities will be destroyed. Transformations are based upon the following principles:

574. In any equation:

I. The same quantity may be added to both members. For example, if 2 be added to both members of $x^2 = 16$, the members of the resulting equation, $x^2 + 2 = 18$, will be equal.

II. The same quantity may be subtracted from both members. Thus, if $x^2 = 16$, then $x^2 - 2 = 14$.

III. Both members may be multiplied or both divided by the same quantity. Thus, if $x^2 = 16$, then $2x^2 = 32$ and $\frac{x^2}{2} = 8$.

IV. Both members may be raised to the same power. Thus, if $x^2 = 16$, then $x^4 = 256$.

V. Like roots of both members may be extracted. Thus, if $x^2 = 16$, then $x = 4$.

A little thought will show that none of these operations will destroy the quality of the members. In the equation $16 = 16$, for example, by I, $16 + 2 = 16 + 2$; by II, $16 - 2 = 16 - 2$; by III, $16 \times 2 = 16 \times 2$, etc. It is to be observed, however, that after any transformation, the *members* do not equal their original values.

575. To **transpose** a term in an equation is to change it from one member of an equation to the other. *A term my be transposed to the other member of an equation, if its sign be changed.* Thus, in the equation $2x + 5 = 13$, let it be required to transpose $+ 5$ to the second member; changing its sign, we have $2x = 13 - 5$, or $2x = 8$. For, subtract 5 from both members and we have $2x + 5 - 5 = 13 - 5$, or $2x - 8$. Suppose we had $2x - 5 = 13$; changing the sign of $- 5$, and placing it in the second member, we have $2x = 18$; for $2x - 5 + 5 = 13 + 5$, or $2x = 18$.

576. *When the same term appears in both members of an equation, with the same sign, it may be dropped from each.* This is called **cancelation.** Thus, in $x + a = 16 + a$ we may cancel a, and have $x = 16$, for subtracting a from both members $x + a - a = 16 + a - a$; hence, $x = 16$. In $x - a = 16 - a$, $x - a + a = 16 - a + a$, and $x = 16$. But, in $x - a = 16 + a$, the a's will not cancel, since they have different signs.

577. Changing Signs.—It is sometimes desirable to change the sign of a term in one of the members of an equation. This may be effected by multiplying both members by -1. This gives the same result as changing the signs of all the terms of both members; thus, $x + a + 3 = a - x - 7$ may be changed to $-x - a - 3 = -a + x + 7$. According to Art. **574,** III, this transformation does not destroy the equality of the members.

578. Clearing of fractions is usually necessary before performing any operations.

Rule.—*To clear an equation of fractions, multiply each term of the equation by the common denominator of all the fractions.*

EXAMPLE.—Clear of fractions $x + \frac{x}{2} + \frac{3x}{4} - \frac{2x}{6} = 10$.

SOLUTION.—The common denominator of all the fractions is 12; multiplying each term by 12, we have

$$12x + 6x + 9x - 4x = 120.$$

EXAMPLE.—Clear of fractions $\frac{2x}{x+2} = \frac{1}{2} - \frac{3x+2}{x^2-4}$.

SOLUTION.—The common denominator is $2(x^2 - 4)$; multiplying through by this we get $4x(x-2) = (x^2 - 4) - 2(3x + 2)$; removing the parentheses this becomes $4x^2 - 8x = x^2 - 4 - 6x - 4$.

Where a fraction is preceded by a minus sign, care must be taken to change the sign of every term of the numerator when clearing of fractions. See Art. **542.**

EXAMPLES FOR PRACTICE.

579. Clear the following equations of fractions:

1. $x + \frac{3x}{4} + \frac{5}{7} = 16 - \frac{2}{x}$. Ans. $28x^2 + 21x^2 + 20x = 448x - 56$.

2. $\frac{x}{4} - \frac{x-3}{2} = \frac{a}{6}$. Ans. $3x - 6x + 18 = 2a$.

3. $\frac{x}{a-b} - x = \frac{a-b}{a+b} - 1$.
 Ans. $ax + bx - a^2x + b^2x = a^2 - 2ab + b^2 - a^2 + b^2$.

4. $\frac{1}{a-b} = \frac{x}{a-b} - \frac{a+b}{x}$. Ans. $x = x^2 - a^2 + b^2$.

SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY.

580. Simple equations, when reduced to their simplest form, contain the first power only, of the unknown quantity. When there is but one unknown quantity, it is usually represented by x. Numbers and the *first* letters of the alphabet are used for known quantities.

581. To solve simple equations with one unknown quantity:

Rule.—*Clear of fractions. Remove all signs of aggregation. Transpose all terms containing the unknown quantity to the first member; all others to the second member. Combine the first member into one term, and simplify the second member. Divide both members by the coefficient of the unknown quantity.*

In some cases, this order of operations may be changed to advantage.

582. To verify the result, substitute the value of the unknown quantity in the original equation, which should then reduce so that both members will be alike. When this occurs the equation is said to be **satisfied.**

EXAMPLE.—Solve $2x + 5 = 25 - 3x$.

SOLUTION.—Transposing the unknown quantities to the first member, we have $2x + 3x + 5 = 25$.

Transposing the known quantities to the second member, we have

$$2x + 3x = 25 - 5.$$

Combining like terms, $5x = 20.$

Dividing by 5, $x = 4.$ Ans.

Now, if the substituting of 4 for x will satisfy the original equation we know our answer is correct. Thus, substituting 4 for x, we have

$$2 \times 4 + 5 = 25 - 3 \times 4.$$
$$8 + 5 = 25 - 12.$$
$$13 = 13.$$

Hence the equation is satisfied, and our result is correct.

EXAMPLE.—Solve $16 - x - \{7x - [8x - (9x - \overline{3x - 6x})]\} = 0.$

SOLUTION.—Removing the symbols of aggregation (Art. **484**),

$$16 - x - 7x + 8x - 9x + 3x - 6x = 0.$$

Transposing 16 to the second member,

$$-x - 7x + 8x - 9x + 3x - 6x = -16.$$

Combining like terms, $-12x = -16.$

Dividing both members by -12,

$$x = \tfrac{16}{12} = 1\tfrac{1}{3}. \quad \text{Ans.}$$

EXAMPLE.—Solve $\frac{2x + 2}{2} + \frac{1}{4} = \frac{8 - 6x}{5} + \frac{2(6x + 7)}{8}.$

SOLUTION.—Simplifying the first and the last term of the equation it becomes,

$$x + 1 + \frac{1}{4} = \frac{8 - 6x}{5} + \frac{6x + 7}{4}.$$

Clearing of fractions we have,

$$20x + 20 + 5 = 32 - 24x + 30x + 35.$$

Transposing, $20x + 24x - 30x = 32 + 35 - 20 - 5.$

Combining, $14x = 42.$

Dividing by 14, $x = 3.$

EXAMPLE.—Solve $\frac{3}{1 - x} - \frac{2}{1 + x} + \frac{1}{1 - x^2} = 0.$

SOLUTION.—Clearing of fractions by multiplying by $1 - x^2$,

$$3(1 + x) - 2(1 - x) + 1 = 0.$$
$$3 + 3x - 2 + 2x + 1 = 0.$$

Uniting terms, $5x = -2.$

$x = -.4.$ Ans.

NOTE.— 0 multiplied or divided by any number $= 0$.

583. Literal equations are those in which the known quantities are represented partially or wholly by letters. In solving such equations, we can not always combine the unknown quantities into one term.

EXAMPLE.—Solve $2ax - 3b = x + c - 3ax$.

SOLUTION.—Transposing the terms containing the unknown quantities to the first member and the remaining terms to the second member, and combining like terms,

$$5ax - x = 3b + c.$$

Factoring $5ax - x$ to bring x alone in the first member,

$$(5a - 1)x = 3b + c.$$

The coefficient of x is now $5a - 1$, this being considered as one quantity.

Dividing by $5a - 1$, $\quad x = \dfrac{3b + c}{5a - 1}$. Ans.

PROOF.—Since the original equation is equivalent to $5ax - x = 3b + c$, it will be sufficient to satisfy this equation. Hence, substituting the value of x,

$$\frac{5a(3b + c)}{5a - 1} - \frac{3b + c}{5a - 1} = 3b + c, \text{ or } \frac{(5a - 1)(3b + c)}{5a - 1} = 3b + c.$$

Canceling the $5a - 1$, $3b + c = 3b + c$.

EXAMPLE.—Solve $\dfrac{3x + 1}{x + 1} = \dfrac{3bx - 2a + c}{b(x + 1) - a}$.

SOLUTION.—Clearing of fractions,

$$(3x + 1)[b(x + 1) - a] = (x + 1)(3bx - 2a + c), \text{ or}$$
$$3bx(x + 1) - 3ax + b(x + 1) - a = 3bx(x + 1) - (2a - c)(x + 1).$$

Canceling $3bx(x + 1)$ from both members,

$$-3ax + bx + b - a = -2ax + cx - 2a + c.$$

Transposing and uniting terms,

$$-ax + bx - cx = -a - b + c.$$

Changing signs and factoring,

$$(a - b + c)x = a + b - c.$$
$$x = \frac{a + b - c}{a - b + c}.$$

EXAMPLES FOR PRACTICE.

584. Solve the following:

1. $16 - 3x = 13 - 6x$. Ans. $x = -1$.
2. $3(4x - 5) + 6 = 1 + 2x$. Ans. $x = 1$.
3. $6(5 - 2x) = 6 - 2(x - 2)$. Ans. $x = 2$.

4. $\dfrac{2x}{3} - \dfrac{4x}{3} = 5 - \dfrac{3x}{4}$. Ans. $x = 60$.

5. $\dfrac{x+1}{3} - \dfrac{x+4}{5} = 16 - \dfrac{x+3}{4}$. Ans. $x = 41$.

6. $\dfrac{x}{3} - \dfrac{x^2 - 5x}{3x - 7} = \dfrac{2}{3}$. Ans. $x = -7$.

7. $\dfrac{5 - 2x}{x + 1} - \dfrac{3 - 2x}{x + 4} = 0$. Ans. $x = 4\frac{1}{4}$.

8. $2x - 4a = 3ax + a^2 - a^2x$. Ans. $x = \dfrac{a^2 + 4a}{a^2 - 3a + 2}$.

9. $\dfrac{ax + 2x}{5a} - \dfrac{a^2 + 4a + 4}{4b} = 0$. Ans. $x = \dfrac{5a^2 + 10a}{4b}$.

SUGGESTION.—Transposing the second term to the second member,

$$\frac{ax + 2x}{5a} = \frac{a^2 + 4a + 4}{4b} = \frac{(a + 2)^2}{4b}.$$

Multiplying both sides by $5a$,

$$ax + 2x = \frac{5a(a + 2)^2}{4b}.$$

Solving for x,

$$x = \frac{5a(a + 2)^2}{(a + 2)4b} = \frac{5a(a + 2)}{4b} = \frac{5a^2 + 10a}{4b}.$$

10. $\dfrac{a(c^2 + x^2)}{cx} = ab + \dfrac{ax}{c}$. Ans. $x = \dfrac{c}{b}$.

PROBLEMS LEADING TO SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY.

585. There are two steps in the solution of problems by algebra:

First.—The relations which exist between the known and the unknown quantities, that is, between those whose values are given in the problem and those whose values are required, must be stated by one or more equations. This is called the **statement** of the problem.

Second.—The resulting equation or equations must be solved, giving the values of the required quantities.

It will thus be seen that by the algebraic method, the answer to a problem is used in the solution and operated upon as though it were a known quantity, which is one great advantage over the arithmetical method.

586. The ability to state a problem by means of an equation depends upon the ingenuity of the operator and his ability to reason, rather than upon his knowledge of algebra. No definite rule can be given for making the statement, but in general, where there is only one unknown quantity in a problem:

Decide what quantity it is whose value is to be found. This will be the unknown quantity, or the answer. Then represent the unknown quantity by x and form an equation that will indicate the relations between the known and the unknown quantities as stated in the problem.

The equation will also indicate the operations that would be performed in proving the statement made in the problem were the answer known. Hence, the equation may often be formed by noticing what operations would be performed upon the answer in proving.

587. The following examples are illustrations of the statement and the solution of algebraic problems; they should be studied carefully.

EXAMPLE.—Find such a number that, when 14 is added to its double, the sum shall be 30.

SOLUTION.—The quantity whose value is required is the number itself. As this is the unknown quantity, let $x =$ the number, whence $2x$ must be double the number. Now the problem states that when 14 is added to double the number the sum will be 30. In other words, when 14 is added to $2x$, the sum will be 30. Hence, the statement of the problem in the form of an equation is,

$$2x + 14 = 30;$$

whence, solving, $x = 8$. Ans.

EXAMPLE.—Find a number which, when multiplied by 4, will exceed 40 as much as it is now below 40.

SOLUTION.—Let $x =$ the required number, which, when multiplied by 4, becomes $4x$. According to the conditions of the problem, the amount by which 4 times the required number, or $4x$, exceeds 40 is equal to the amount that the number itself, or x, is below 40.

But $4x - 40$ is the amount by which $4x$ exceeds 40, and $40 - x$ is the amount by which x is below 40.

Hence, by the conditions, we have the statement,

$$4x - 40 = 40 - x.$$

Transposing and uniting, $5x = 80$,

or $x = 16$. Ans.

EXAMPLE.—Two loads of brick together weigh 4,000 lb.; but if 500 lb. be transferred from the smaller to the larger load, the latter will weigh 7 times as much as the former. How much does each load weigh?

SOLUTION.—If the weights of the two loads were known and it was desired to prove the correctness of the example, we should add 500 lb. to the weight of the larger load and subtract 500 lb. from the weight of the smaller load, as stated in the example. The larger load should then weigh 7 times as much as the smaller. To obtain the equation the same thing is done by letting $x =$ the weight of one load, whence $4,000 - x$ is equal to the weight of the other load.

Let $x =$ the weight of the smaller load.

Then, $4,000 - x =$ the weight of the larger load.

Also, $x - 500 =$ the weight of the smaller load after transferring 500 lb.

And $4,000 - x + 500 =$ the weight of the larger load after transferring 500 lb.

By the conditions, the larger load now weighs 7 times as much as the smaller.

Hence, $7(x - 500) = 4,000 - x + 500.$

Solving, $7x - 3,500 = 4,500 - x,$

or $8x = 8,000;$

whence, $x = 1,000$ lb. = weight of smaller load,
and $4,000 - x = 3,000$ lb. = weight of larger load. } Ans.

PROOF.— $1,000 - 500 = 500 =$ weight of the smaller load, and $3,000 + 500 = 3,500 =$ weight of the larger load after the 500 pounds have been transferred; $3,500 \div 500 = 7$.

Until the student has obtained considerable proficiency in solving problems of this kind, it is a good plan to prove all problems.

EXAMPLE.—The circumference of the fore wheel of a carriage is 10 feet, and of the rear wheel, 12 feet. What distance has the carriage traveled, when the fore wheel has made 8 more turns than the hind wheel?

SOLUTION.—In this example the distance traveled is not known, but is required to be found. Suppose that the distance is known, and that it is equal to x feet, and that we wish to see whether the statement is true that the fore wheel makes 8 more revolutions than the rear wheel in passing over x feet. The number of revolutions of the fore wheel is evidently $\frac{x}{10}$, and of the rear wheel, $\frac{x}{12}$. The example states that the difference between them is equal to 8.

Hence, $\dfrac{x}{10} - \dfrac{x}{12} = 8.$ (1)

Solving for x, $12x - 10x = 960,$

or $2x = 960$

and $x = 480$ feet. Ans.

PROOF.— $\dfrac{480}{10} = 48 =$ revolutions of fore wheel.

$\dfrac{480}{12} = 40 =$ revolutions of hind wheel.

$48 - 40 = 8.$ Compare this proof with (1).

EXAMPLE.—A water cistern connected with three pipes can be filled by one of them in 80 minutes, by another in 200 minutes, and by the third in 300 minutes. In what time will the cistern be filled when all three pipes are open at once?

SOLUTION.—Here the unknown quantity is the number of minutes required to fill the cistern by all three pipes together. Supposing this to be x minutes, the example may be proved by noticing that the sum of the quantities of water flowing through each pipe separately in a given length of time, as 1 minute, must be equal to the quantity flowing through all three together in the same length of time. According to the problem, the quantity discharged by the first pipe in 1 minute would be $\frac{1}{80}$, by the second $\frac{1}{200}$, and by the third $\frac{1}{300}$ of the contents of the cistern. In like manner the quantity discharged by all three at once in 1 minute would be $\dfrac{1}{x}$. Then, if the example is stated correctly, we must have

$$\frac{1}{80} + \frac{1}{200} + \frac{1}{300} = \frac{1}{x}.$$

Clearing of fractions,

$x(30 + 12 + 8) = 2,400,$

or $50x = 2,400;$

whence, $x = 48$ minutes. Ans.

EXAMPLE.—A man rows a boat a certain distance *with* the tide, at the rate of $6\frac{2}{3}$ miles an hour, and returns at the rate of $3\frac{1}{3}$ miles an hour, *against* a tide half as strong. If the man is pulling at a uniform rate, what is the velocity of the stronger tide?

SOLUTION.—If the following statement is not clear, the student should reason it out for himself in a manner similar to that used in the last three examples.

Let $x =$ number of miles per hour that the stronger tide is running; then $\dfrac{x}{2} =$ number of miles per hour that the weaker tide is running.

Hence, $6\frac{2}{3} - x$ and $3\frac{1}{3} + \dfrac{x}{2}$ are expressions for the rate at which the

man is pulling. But, as he is pulling at a constant rate all the time. these expressions must be equal. Hence,

$$6\tfrac{2}{3} - x = 3\tfrac{1}{3} + \frac{x}{2},$$

$$\text{or} \quad \frac{20}{3} - x = \frac{10}{3} + \frac{x}{2}.$$

Clearing of fractions, $40 - 6x = 20 + 3x$,

or $-9x = -20$;

whence, $x = 2\frac{2}{9}$ miles per hour. Ans.

EXAMPLES FOR PRACTICE.

588. Solve the following examples:

1. The greater of two numbers is four times the lesser number, and their sum is 400; what are the numbers? Ans. 80 and 320.

2. A farmer has 108 animals, consisting of horses, sheep, and cows, He has four times as many cows as horses, lacking 8, and five times as many sheep as horses, lacking 4; how many has he of each kind?

Ans. { 12 horses. 40 cows. 56 sheep.

3. A can do a piece of work in 8 days, and B can do it in 10 days; in what time can they do it working together? Ans. $4\frac{4}{9}$ days.

4. Find five consecutive numbers whose sum is 150.

Ans. 28, 29, 30, 31, and 32.

5. A boat whose rate of sailing is 6 miles per hour in still water moves down a stream which flows at the rate of 3 miles per hour, and returns, making the round trip in 8 hours; how far did it go down the stream? Ans. 18 miles.

THE TRIGONOMETRIC FUNCTIONS.

DEFINITIONS.

589. Plane trigonometry treats of the solution of plane triangles.

Every triangle has six parts—three angles and three sides. When three of these parts are given, if one part at least is a side, the other three can be found. This process of finding the unknown parts of a triangle is the **solution** of the triangle.

590. The **complement** of an angle is the difference between 90° and the angle. Thus, the complement of an angle of 35° is an angle of 55°, because 90° − 35° = 55°. In a right-angled triangle, the right angle is 90°; since the sum of the three angles of the triangle is 180°, the sum of the two acute angles is 180° − 90° = 90°. Therefore, each acute angle of a right-angled triangle is the complement of the other acute angle.

591. The **supplement** of an angle is the difference between 180° and the angle. Thus, the supplement of an angle of 35° is an angle of 145°, because 180° − 35° = 145°.

592. The solution of a triangle is accomplished by means of the **trigonometric functions.** These functions are the ratios of the sides of a right-angled triangle; the most important of these functions are the *sine*, *cosine*, *tangent*, and *cotangent*. These are abbreviated to *sin*, *cos*, *tan*, and *cot*.

593. In the right-angled triangle $A\ B\ C$, Fig. 63, the sides a, b, and c are opposite, respectively, to the angles A, B, and C. The hypotenuse is c, and C is the right angle. The short side b is adjacent to angle A and opposite angle B; the short side a is opposite to angle A and adjacent to angle B.

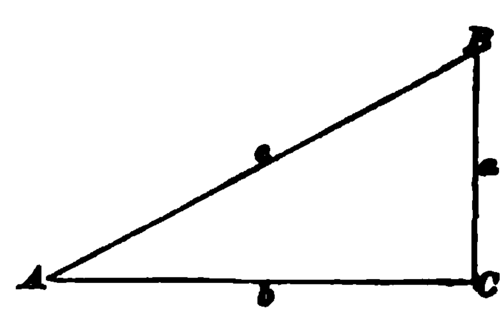


FIG. 63.

Then the trigonometric functions are defined as follows:

594. The **sine** of an angle is the quotient of the opposite side divided by the hypotenuse. Thus, in Fig. 63 $\sin A = \frac{a}{c}$.

595. The **cosine** of an angle is the quotient of the adjacent side divided by the hypotenuse. Thus, $\cos A = \frac{b}{c}$.

596. The **tangent** of an angle is the quotient of the opposite side divided by the adjacent side. Thus, $\tan A = \frac{a}{b}$.

597. The **cotangent** of an angle is the quotient of the adjacent side divided by the opposite side. Thus, $\cot A = \frac{b}{a}$.

598. From the definitions of the functions, we have: $\sin B = \frac{b}{c}$; $\cos B = \frac{a}{c}$; $\tan B = \frac{b}{a}$; and $\cot B = \frac{a}{b}$. Comparing the functions of angle B with those of angle A, $\sin B = \cos A$ $\left(\text{since each is equal to } \frac{b}{c}\right)$, $\cos B = \sin A$, $\tan B = \cot A$, and $\cot B = \tan A$. It has been shown (Art. **590**) that angle A is the complement of angle B. Therefore, the sine of an angle is equal to the cosine of its complement, and the tangent of an angle is equal to the cotangent of its complement. For example, $\sin 36° = \cos(90° - 36°) = \cos 54°$; $\tan 28° = \cot 62°$, etc.

TRIGONOMETRIC TABLES.

599. Every angular function has a different value for each of the angles between 0° and 90°. The numerical values of these functions are called natural sines, cosines, etc., and are given in the tables of Natural Sines, Cosines, Tangents, and Cotangents. In many tables, both natural and logarithmic functions are given. The student should not attempt to use the latter until he thoroughly understands logarithms. The table of natural functions, and its use, will now be explained.

600. Given an Angle, to Find Its Functions.

EXAMPLE.—Let it be required to find the sine, cosine, and tangent of an angle of 37° 24′.

SOLUTION.—Look in the table of **natural sines** along the tops of the pages and find 37°. The left-hand column is marked (′), meaning that the minutes are to be sought in that column, and begin with 0, 1, 2, 3, etc., up to 60. Glancing *down* this column until 24 is found, find opposite this 24 in the column marked *sine*, and headed 37°, the number .60738; then .60738 = sin 37° 24′. In exactly the same manner, find in

the column marked *cosine* and headed 37°, the number .79441, which corresponds to cos 37° 24′; or cos 37° 24′ = .79441. So, also, find in the column marked *tangent* and headed 37°, and opposite 24′, the number .76456; hence, tan 37° 24′ = .76456.

601. In most of the tables published, the angles run only from 0° to 45° at the top of the page; to find an angle greater than 45°, *look at the bottom of the page and glance upwards*, **using the extreme right-hand column to find minutes,** which begin with 0 at the bottom and run upwards, 1, 2, 3, etc., up to 60.

EXAMPLE.—Find the sine, cosine, and tangent of 77° 43′.

SOLUTION.—Since this angle is greater than 45°, look in the tables along the bottom of the page, until the column marked 77° is found. Glancing up the column of minutes on the *right*, until 43′ is found, find opposite 43′ in the column marked *sine* (and 77°) at the bottom, the number .97711; this is the sine of 77° 43′, or sin 77° 43′ = .97711. Similarly, in the column marked *cosine*, find opposite 43′ in the right-hand column, the number .21275; this is the cosine of 77° 43′, or cos 77° 43′ = .21275. So, also, find that 4.59283 is the tangent of 77° 43′, or tan 77° 43′ = 4.59283.

602. Let it be required to find the sine of 14° 22′ 26″.

EXPLANATION.—The sine of 14° 22′ 26″ lies between sin 14° 22′ and sin 14° 23′. Sin 14° 22′ = .24813; sin 14° 23′ = .24841; difference = .00028. Neglect for the moment the fact that the functions are decimal fractions; then the difference between sin 14° 22′ and sin 14° 23′, that is, the difference between 24841 and 24813, is 28, or, corresponding to a difference of 1′ in the angle, there is a difference of 28 in the sine. Now, since 1′ = 60″, the difference between sin 14° 22′ and sin 14° 22′ 26″, that is, the difference corresponding to a difference of 26″ in the angle, must be $\frac{26}{60} \times 28 = 12.1$. Since .1 is less than .5, omit it, and we have 12 as the difference to be added to 24813. $24813 + 12 = 24825$; taking account now of the fact that the function is a decimal fraction, sin 14° 22′ 26″ = .24825.

In all work with the tables it will always be found most convenient to neglect for the moment the decimal point,

consider the difference a whole number, and afterwards locate the decimal point in its proper position.

603. Reference to the table of functions shows that, as the angles increase in magnitude, the sines and tangents increase, while the *cosines and cotangents decrease.* In the above example, therefore, had it been required to find the *cosine* of 14° 22′ 26″, the correction for the 26″ would have been *subtracted* from the cosine of 14° 22′, instead of being added to it.

EXAMPLE.—Find the sine, cosine, and cotangent of 56° 43′ 17″.

SOLUTION.—Sin 56° 43′ = .83597. Sin 56° 44′ = .83613. Since 56° 43′ 17″ is greater than 56° 43′ and less than 56° 44′, the value of the sine of the angle lies between .83597 and .83613; neglecting the decimal character of the functions the difference = 83613 − 83597 = 16; multiplying this by the fraction $\frac{17}{60}$, $16 \times \frac{17}{60} = 4.53$. Since .53 exceeds .5, we take 5 as the difference to be *added* to 83597. Adding, 83597 + 5 = 83602; hence, sin 56° 43′ 17″ = .83602. Ans.

Cos 56° 43′ = .54878; cos 56° 44′ = .54854; expressed as a whole number the difference = 54878 − 54854 = 24, and $24 \times \frac{17}{60} = 7$, nearly. Now, since the *cosine* is desired, we must *subtract* this correction from 54878; subtracting, 54878 − 7 = 54871. Hence, cos 56° 43′ 17″ = .54871. Ans.

Cot 56° 43′ = .65646; cot 56° 44′ = .65604; difference = 65646 − 65604 = 42, and $42 \times \frac{17}{60} = 12$, nearly. Now, since the cotangent decreases as the angle increases, we must *subtract* this correction from 65646; thus, 65646 − 12 = 65634. Hence, cot 56° 43′ 17″ = .65634. Ans.

604. Given the Function, to Find the Corresponding Angle.—This is the reverse of the process for finding the function of the angle. If the angle corresponding to the given function is an exact number of degrees and minutes, the function will be found in the table. In such a case, we have simply to find the given function, and take the degrees from the end of the column, and the minutes at the end of the row. If the name of the function is at the top of the column, the number of degrees will be found there also, and the minutes at the left. If the name is at the bottom, the degrees will be at the bottom, and the minutes at the right.

CASE I.

605. *The function is found exactly in the table.*

EXAMPLE.—Find the angle whose sine is .24982.

SOLUTION.—In table of sines, we find .24982 in the column under 14°. Since the name of the function is at the top of the column, we take the number of degrees from the top, and the minutes from the left; thus, 14° at the top, and 28′ at the left. Hence, the angle whose sine is .24982 is 14° 28′. Ans.

EXAMPLE.—Find the angle whose cotangent is .68557.

SOLUTION.—In the table of tangents, we find .68557 under 34°. The name of the function is, however, at the bottom of the page, and we must take the degrees, 55°, from that end of the column. The minutes, 34′, are found in the right-hand column. Hence, the angle whose cotangent is .68557 is 55° 34′. Ans.

CASE II.

606. *The function is not found exactly in the table.*

Let it be required to find the angle whose sine is .42531.

EXPLANATION.—Referring to the table of sines, this number is found to lie between .42525, the sine of 25° 10′, and .42552, the sine of 25° 11′. Neglecting decimals, the difference between these two sines $= 42552 - 42525 = 27$; the difference between 42525, corresponding to the sine of 25° 10′, and 42531, corresponding to the sine of the given angle $= 42531 - 42525 = 6$. Since 27 is the difference for 1′, a difference of 6 corresponds to $\frac{6}{27}$ of 1′; hence, the angle whose sine $= .42531 = 25° \; 10\frac{6}{27}'$.

Since $1' = 60''$, $\frac{6}{27}$ of a minute $= \frac{6}{27} \times 60 = 13.3''$. Therefore, the angle whose sine is $.42531 = 25° \; 10' \; 13.3''$.

The given function should always be compared with the function of the *angle next lower*, and the correction in seconds should be added to that angle. In the case of the sine and tangent, we take the difference between the given function and the next smaller function appearing in the tables; but with the cosine and cotangent, we take the difference between the given function and the next larger function which appears in the tables.

EXAMPLE.—Find the angle whose cosine is .27052.

SOLUTION.—Looking in the table of cosines, the given function is found to belong to an angle greater than 45° and, hence, must be sought for in the columns marked *cosine* at the bottom of the page. It is found between the numbers .27060 = cos 74° 18′ and .27032 = cos 74° 19′. The difference between the two is .27060 − .27032 = .00028, or 28, neglecting decimals. The cosine of the *smaller angle*, or 74° 18′, is .27060, and the difference between this and the given cosine is .27060 − .27052 = .00008, or 8, neglecting decimals.

Hence, $\frac{8}{28} \times 60 = 17.14''$, nearly, and the angle whose cosine is .27052 = 74° 18′ 17.14″, or cos 74° 18′ 17.14″ = .27052. Ans.

EXAMPLE.—Find the angle whose tangent is 2.15841.

SOLUTION.— 2.15841 falls between 2.15760 = tan 65° 8′, and 2.15925 = tan 65° 9′. The difference = 2.15925 − 2.15760 = .00165, or 165, considered as a whole number. 2.15841 − 2.15760 = 81, neglecting the decimal. $\frac{81}{165} \times 60 = 29.5''$, nearly, and the angle whose tangent is 2.15841 = 65° 8′ 29.5″, or tan 65° 8′ 29.5″ = 2.15841.

EXAMPLES FOR PRACTICE.

607. Solve the following examples:

1. Find the (*a*) sine, (*b*) cosine, and (*c*) tangent of 48° 17′.

Ans. (*a*) .74644. (*b*) .66545. (*c*) 1.12172.

2. Find the (*a*) sine, (*b*) cosine, and (*c*) cotangent of 13° 11′ 6″.

Ans. (*a*) .22810. (*b*) .97364. (*c*) 4.26855.

3. Find the (*a*) sine, (*b*) cosine, and (*c*) tangent of 72° 0′ 1.8″.

Ans. (*a*) .95106. (*b*) .30901. (*c*) 3.07777.

4. (*a*) Of what angle is .26489 the sine, (*b*) of what is it the cosine, and (*c*) of what is it the cotangent?

Ans. (*a*) 15° 21′ 37.2″. (*b*) 74° 38′ 22.8″. (*c*) 75° 9′ 49″.

5. (*a*) Of what angle is .68800 the sine, (*b*) of what the cosine, and (*c*) of what the tangent?

Ans. (*a*) 43° 28′ 20″. (*b*) 46° 31′ 40″. (*c*) 34° 31′ 40.5″.

SOLUTION OF TRIANGLES.

RIGHT-ANGLED TRIANGLES.

608. When any side and one of the acute angles of a right-angled triangle (also called *right triangle*) are given, the remaining sides and angles may be found; also, when any two sides are given, the remaining parts may be found.

609. The following relations between the sides and angles of a right triangle are derived directly from the definitions of the trigonometric functions :

I. *Side opposite an angle* = *hypotenuse* × *sine of angle.*

II. *Side adjacent* = *hypotenuse* × *cosine.*

III. *Side opposite* = *side adjacent* × *tangent.*

IV. *Side adjacent* = *side opposite* × *cotangent.*

V. *Hypotenuse* $= \dfrac{\textit{side opposite}}{\textit{sine}}$.

VI. *Hypotenuse* $= \dfrac{\textit{side adjacent}}{\textit{cosine}}$.

These relations are sufficient to find the sides. The angles may be found from the sides by the relations given in Arts. **594** to **597.** To show the application of these relations to the solution of triangles, a number of examples are given.

610. Case I.—*The hypotenuse and an acute angle being given, to find the remaining parts :*

EXAMPLE.—In Fig. 64, the hypotenuse AB of the right-angled triangle ACB is 24 feet and the angle A is 29° 31′; to find the sides AC and BC and the angle B.

NOTE.—When working examples of this kind, construct the figure, and mark the known parts. This is a great help in solving the example. Hence, in the figure draw the angle A to represent an angle of 29° 31′, and complete the right-angled triangle ACB, right-angled at C, as shown. Mark the angle A and the hypotenuse, as is done in the figure.

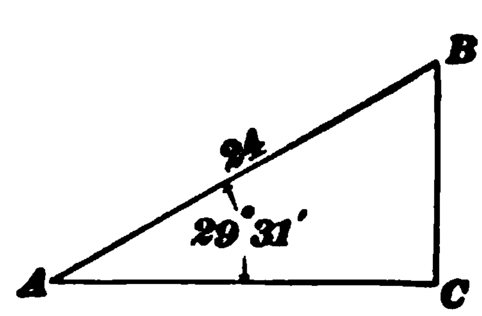


FIG. 64.

SOLUTION.—Angle $B = 90° - 29° 31' = 60° 29'$. $A C$ is the side adjacent to angle A. Hence, from relation II, Art. **609,** $A C$, or side adjacent = hypotenuse × cosine = 24 × cos 29° 31′. In the table of Natural Cosines we find the cosine of 29° 31′ to be .87021. Therefore, $A C = 24 \times .87021 = 20.89$ feet, nearly.

To find the side $B C$, we use relation I, Art. **609.** $B C$, or side opposite = hypotenuse × sine = 24 × sin 29° 31′. The sine of 29° 31′ is .49268; hence, $B C = 24 \times .49268 = 11.82$ feet, nearly.

$$\text{Ans.}\begin{cases}\text{Angle } B = 60° 29'.\\ \text{Side } A C = 20.89 \text{ ft.}\\ \text{Side } B C = 11.82 \text{ ft.}\end{cases}$$

611. Case II.—*An acute angle and one of the short sides being given, to determine the remaining parts:*

EXAMPLE.—In Fig. 64, suppose the side $A C$ to be 75 feet and the angle A to be 32° 24′. The sides $B C$, $A B$, and the angle B are required.

SOLUTION.—Angle $B = 90° - 32° 24' = 57° 36'$. To find $B C$, we have relation III, Art. **609,** $B C$, side opposite = side adjacent × tangent = 75 × tan 32° 24′. Referring to the table of Natural Tangents, the tangent of 32° 24′ = .63462. $B C = 75 \times .63462 = 47.6$ ft., nearly. To find the hypotenuse $A B$, we use relation VI, Art. **609,** hypotenuse $= \dfrac{\text{side adjacent}}{\text{cosine}}$. $A B = \dfrac{A C}{\cos A} = \dfrac{75}{\cos 32° 24'} = \dfrac{75}{.84433} = 88.81$ ft.

$$\text{Ans.}\begin{cases}\text{Angle } B = 57° 36'.\\ \text{Side } B C = 47.6 \text{ ft.}\\ \text{Side } A B = 88.81 \text{ ft.}\end{cases}$$

612. Case III.—*Two sides being given, to find the third side and the acute angles:*

EXAMPLE.—In the right-angled triangle $A B C$, Fig. 65, right-angled at C, $A C = 18$ and $B C = 15$; to find $A B$ and the angles A and B.

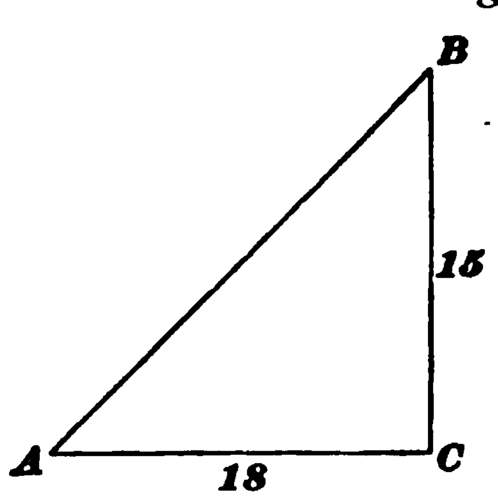


FIG. 65.

SOLUTION.—According to the definition of the tangent, Art. **596,**

$$\text{tangent } A = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{15}{18} = .83333.$$

Looking in the table of Tangents, the angle whose tangent is nearest to .83333 is 39° 48′. Hence, angle $A = 39° 48'$, nearly. Angle $B = 90° - 39° 48' = 50° 12'$.

To find the hypotenuse, we use relation V, Art. **609,**

$$A B, \text{ hypotenuse} = \frac{\text{side opposite}}{\text{sine}} = \frac{B C}{\sin A}.$$

The sine of 39° 48′ is .64011. Hence, $A\,B = \dfrac{15}{.64011} = 23.43$.

Ans. $\begin{cases} \text{Angle } A = 39°\ 48'. \\ \text{Angle } B = 50°\ 12'. \\ \text{Side } A\,B = 23.43. \end{cases}$

Example.—In the right-angled triangle $A\,B\,C$, Fig. 66, right-angled at C, $A\,C = .024967$ mile and $A\,B = .04792$ mile; to find the other parts.

Solution.—According to the definition of the cosine, Art. **595**,

$$\cos A = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{.024967}{.04792} = .52101.$$

Referring to the table, the angle whose cosine is .52101 is 58° 36′. Therefore, angle $A = 58°\ 36'$. Angle $B = 90° - 58°\ 36' = 31°\ 24'$.

To find side $B\,C$, relation III, Art. **609**, is used.

Side opposite A = side adjacent × tan A, or $B\,C = A\,C \times \tan 58°\ 36' = .024967 \times 1.63826 = .0409$ mile.

Ans. $\begin{cases} \text{Angle } A = 58°\ 36'. \\ \text{Angle } B = 31°\ 24'. \\ B\,C = .0409 \text{ mile.} \end{cases}$

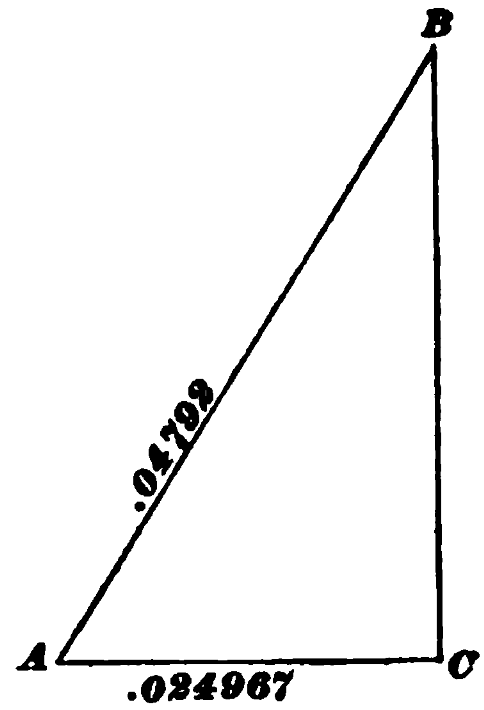


Fig. 66.

Example.—In the right-angled triangle $A\,B\,C$, Fig. 67, right-angled at C, $A\,B = 308$ feet and $B\,C = 234$ feet; to find the other parts.

Solution.—$\sin A = \dfrac{\text{side opposite}}{\text{hypotenuse}} = \dfrac{234}{308} = .75974$. (Art. **594.**)

The angle whose sine is .75974 = 49° 26½′, nearly, = angle A. Referring to the table of Sines, the sine of 49° 26′ is .75965, and the sine of 49° 27′ is .75984. Since the value obtained, .75974, lies nearly half way between these values, the angle lies midway between the above angles, and is 49° 26½′. Angle $B = 90° - 49°\ 26\tfrac{1}{2}' = 40°\ 33\tfrac{1}{2}'$.

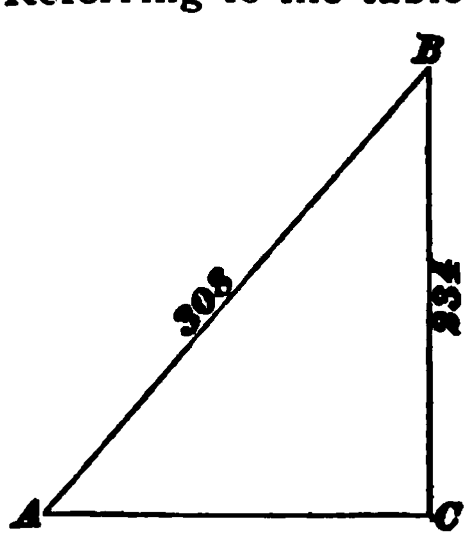


Fig. 67.

To find $A\,C$, use relation IV, Art. **609.**

Side adjacent A = side opposite × cot A, or $A\,C = 234 \times .85586 = 200.27$ feet.

Ans. $\begin{cases} \text{Angle } A = 49°\ 26\tfrac{1}{2}'. \\ \text{Angle } B = 40°\ 33\tfrac{1}{2}'. \\ A\,C = 200.27 \text{ ft.} \end{cases}$

EXAMPLES FOR PRACTICE.

613. Solve the following examples:

1. In a right triangle $A\,B\,C$, right-angled at C, the hypotenuse $A\,B = 40$ inches and angle $A = 28°\ 14'\ 14''$. Solve the triangle.

Ans. $\begin{cases} \text{Angle } B = 61°\ 45'\ 46''. \\ A\,C = 35.24 \text{ in.} \\ B\,C = 18.92 \text{ in.} \end{cases}$

2. In a right triangle $A\,B\,C$, right-angled at C, the side $B\,C = 10$ feet 4 inches. If angle $A = 26° 59' 6''$, what do the other parts equal?

$$\text{Ans.} \begin{cases} \text{Angle } B = 63° \; 0' \; 54''. \\ A\,B = 22 \text{ ft. } 9\tfrac{1}{4} \text{ in., nearly.} \\ A\,C = 20 \text{ ft. } 3\tfrac{1}{4} \text{ in., nearly.} \end{cases}$$

3. In a right triangle $A\,B\,C$, right-angled at C, the hypotenuse $A\,B = 60$ feet and the side $A\,C = 22$ feet. Solve the triangle.

$$\text{Ans.} \begin{cases} \text{Angle } A = 68° \; 29' \; 22.2''. \\ \text{Angle } B = 21° \; 30' \; 37.8''. \\ B\,C = 55.82 \text{ ft.} \end{cases}$$

4. In a right triangle $A\,B\,C$, right-angled at C, side $A\,C = .364$ foot and side $B\,C = .216$ foot. Solve the triangle.

$$\text{Ans.} \begin{cases} \text{Angle } A = 30° \; 41' \; 7.5''. \\ \text{Angle } B = 59° \; 18' \; 52.5''. \\ A\,B = .423 \text{ ft.} \end{cases}$$

OBLIQUE-ANGLED PLANE TRIANGLES.

614. We will give here the method of solving *any* oblique-angled plane triangle, when (1) *two sides and an angle opposite one of them* are given, and (2) when *two angles and a side opposite one of them* are given. Let the student bear in mind, however, that he may use this method of solution only when he knows the general form of the triangle of which he desires the values of some of the parts. This is necessary because in some cases *two solutions* are possible, resulting in the determinations of triangles of quite different forms. We do not think it necessary to go so deeply into the explanation of this point that the student can detect the cases in which two solutions are possible.

615. The solution of the triangle depends upon the following principle: *In any triangle, the sides are proportional to the sines of the opposite angles.* Thus, referring to Fig. 68, the following proportions are true:

$$a : b = \sin A : \sin B.$$
$$a : c = \sin A : \sin C.$$
$$b : c = \sin B : \sin C.$$

The method of solving the triangle is shown in the following examples:

616. Case I.—*Two sides and an angle opposite one of them are given:*

EXAMPLE.—In the triangle $A\,B\,C$, Fig. 68, having given the side $a = 1,686$ feet, the side $b = 960$ feet and the angle $A = 33°\ 35'$; to find the angle B.

SOLUTION.—We have

$a : b = \sin A : \sin B$.

Substituting the known values, we have

$1,686 : 960 = \sin$ of $33°\ 35' : \sin B$.

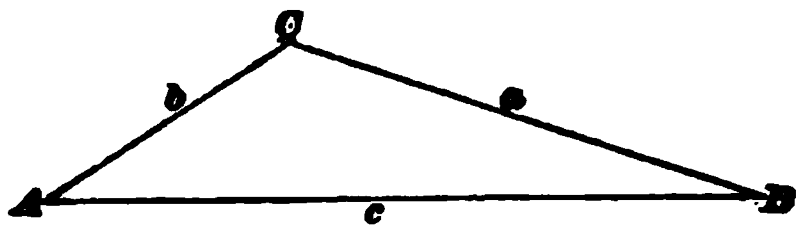


FIG. 68.

From the table of Natural Sines we get .55315 as the sine of 33° 35′. Substituting this value in the proportion and solving, we find that .31496 is the sine of the angle B, which, by consulting the table of Natural Sines, we find to correspond nearly with the angle 18° 22′. Ans.

617. Case II.—*Two angles and a side opposite one of them are given:*

EXAMPLE.—In the triangle $A\,B\,C$, given the angle $A = 33°\ 35'$, the angle $B = 18°\ 22'$, and the side $a = 1,686$ feet; to find the side b.

SOLUTION.—We have $\sin A : \sin B = a : b$. Substituting known values, we get

$$.55315 : .31509 = 1686 : b.$$

Solving the proportion, we find $b = 960$ feet. Ans.

MECHANICS.

(PART 1.)

MATTER AND ITS PROPERTIES.

1799. Matter is anything that occupies space. It is the substance of which all bodies are composed. Matter is composed of *molecules* and *atoms.*

1800. A **molecule** is the smallest portion of matter than can exist without changing its nature.

1801. An **atom** is an indivisible portion of matter.

Atoms unite to form molecules, and a collection of molecules form a mass or body.

A drop of water may be divided and subdivided, until each particle is so small that it can only be seen by the most powerful microscope, but each particle will still be water. Now, imagine the division to be carried on still farther, until a limit is reached beyond which it is impossible to go without changing the nature of the particle. The particle of water is now so small that, if it be divided again, it will cease to be water, and will be something else; we call this particle a *molecule.*

If a molecule of water be divided, it will yield two atoms of hydrogen gas, and one of oxygen gas. If a molecule of sulphuric acid be divided, it will yield two atoms of hydrogen, one of sulphur, and four of oxygen.

It has been calculated that the diameter of a molecule is larger than $\frac{1}{1250000000}$ of an inch, and smaller than $\frac{1}{500000000}$ of an inch.

1802. Bodies are composed of collections of molecules. Matter exists in three conditions or forms: *solid*, *liquid*, and *gaseous.*

1803. A **solid body** is one whose molecules change their relative positions with great difficulty; as iron, wood, stone, etc.

1804. A **liquid body** is one whose molecules tend to change their relative positions easily. Liquids readily adapt themselves to the vessel which contains them, and their upper surface always tends to become perfectly level. Water, mercury, molasses, etc., are liquids.

1805. A **gaseous body,** or gas, is one whose molecules tend to separate from one another; as air, oxygen, hydrogen, etc.

Gaseous bodies are sometimes called **aeriform** (air-like) **bodies.** They are divided into two classes—the so-called "*permanent*" *gases* and *vapors.*

A **permanent gas** is one which remains a gas at ordinary temperatures and pressures.

A **vapor** is a body which, at ordinary temperatures, is a liquid or solid, but, when heat is applied, becomes a gas, as steam.

1806. One body may be in all three states; as, for example, mercury, which at ordinary temperatures is a liquid, becomes a solid (freezes) at 40° below zero, and a vapor (gas) at 600° above zero. By means of great cold, all gases, even hydrogen, have been liquefied, and some solidified.

By means of heat, all solids have been liquefied, and a great many vaporized. It is probable that, if we had the means of producing sufficiently great extremes of heat and cold, all solids might be converted into gases, and all gases into solids.

1807. Every portion of matter possesses certain qualities called *properties.* Properties of matter are divided into two classes, *general* and *special.*

General properties of matter are those which are common to all bodies. They are as follows: *Extension*, *impenetrability*, *weight*, *indestructibility*, *inertia*, *mobility*,

divisibility, *porosity*, *compressibility*, *expansibility*, and *elasticity*.

1808. Extension is the property of occupying space. Since all bodies must occupy space, it follows that extension is a general property.

By **impenetrability** we mean that no two bodies can occupy exactly the same space at the same time.

1809. Weight is the measure of the earth's attraction upon a body. All bodies have weight. In former times it was supposed that gases had no weight, since, if unconfined, they tend to move away from the earth, but, nevertheless, they will finally reach a point beyond which they can not go, being held in suspension by the earth's attraction. Weight is measured by comparison with a standard. The standard is a bar of platinum owned and kept by the Government; it weighs one pound.

1810. Inertia means that a body can not put itself in motion nor bring itself to rest. To do either, it must be acted upon by some force.

1811. Mobility means that a body can be changed in position by some force acting upon it.

1812. Divisibility is that property of matter which indicates that a body may be separated into parts.

1813. Porosity is that property of matter which indicates that there is space between the molecules of a body. Molecules of a body are supposed to be spherical, and, hence, there is space between them, as there would be between peaches in a basket. The molecules of water are larger than those of salt; so that when salt is dissolved in water, its molecules wedge themselves between the molecules of the water, and unless too much salt is added, the water will occupy no more space than it did before. This does not prove that water is penetrable, for the molecules of salt occupy the space that the molecules of water did not.

Water has been forced through iron by pressure, thus proving that iron is porous.

1814. Compressibility is that property of matter which indicates that the molecules of a body may be crowded nearer together, so as to occupy a smaller space.

1815. Expansibility is that property of matter which indicates that the molecules of a body may be forced apart, so as to occupy a greater space.

1816. Elasticity is that property of matter which indicates that if a body be distorted within certain limits, it will resume its original form when the distorting force is removed. Glass, ivory, and steel are very elastic.

1817. Indestructibility indicates that matter can never be destroyed. A body may undergo thousands of changes; be resolved into its molecules, and its molecuies into atoms, which may unite with other atoms to form other molecules and bodies entirely different from the original body, but the same number of atoms remain. The whole number of atoms in the universe is exactly the same now as it was millions of years ago, and will always be the same. *Matter is indestructible.*

1818. Special properties are those which are not possessed by all bodies. Some of the most important are as follows: *Hardness*, *tenacity*, *brittleness*, *malleability*, and *ductility*.

1819. Hardness is that property of matter which indicates that some bodies may scratch other bodies. Fluids and gases do not possess hardness. The diamond is the hardest of all substances.

1820. Tenacity is that property of matter which indicates that some bodies resist a force tending to pull them apart. Steel is very tenacious.

1821. Brittleness is that property of matter which indicates that some bodies are easily broken; as glass, crockery, etc.

1822. Malleability is that property of matter which indicates that some bodies may be hammered or rolled into sheets. Gold is the most malleable of all substances.

1823. Ductility is that property of matter which indicates that some bodies may be drawn into wire. Platinum is the most ductile of substances.

MOTION AND VELOCITY.

1824. Motion is the opposite of rest, and indicates a changing of position in relation to some object. If a large stone is rolled down hill, it is in motion in relation to the hill.

If a person is on a railway-train, and walks in the opposite direction from that in which the train is moving, and with the same speed, he will be in motion as regards the train, but at rest with respect to the earth, since, until he gets to the end of the train, he will be directly over the spot at which he was when he started to walk.

1825. The **path** of a body in motion is the line described by its *central point*. No matter how irregular the shape of the body may be, nor how many turns and twists it may make, the line which indicates the direction of the center of the body for every instant that it was in motion is the path of the body.

1826. Velocity is rate of motion. It is measured by a unit of space passed over in a unit of time. When equal spaces are passed over in equal times, the velocity is said to be **uniform.** In all other cases, it is **variable.**

If the fly-wheel of an engine keeps up a constant speed of a certain number of revolutions per minute, the velocity of any point is uniform. A railway-train having a constant speed of 40 miles per hour moves 40 miles every hour, or $\frac{40}{60} = \frac{2}{3}$ of a mile every minute, and since equal spaces are passed over in equal times, the velocity is uniform.

1827. To find the uniform velocity which a body must have to pass over a certain distance or space in a given time :

Rule.—*Divide the distance by the time.*

Let s = distance traveled by moving body;
v = uniform velocity of body;
t = the time.

Then, $$v = \frac{s}{t}. \qquad \textbf{(90.)}$$

EXAMPLE.—The piston of a steam-engine travels 3,000 feet in 5 minutes; what is its velocity in feet per minute?

SOLUTION.—Here 3,000 feet is the distance, and 5 minutes is the time. Applying formula **90**,

$$v = \frac{s}{t} = \frac{3,000}{5} = 600 \text{ feet per minute. Ans.}$$

CAUTION.—Before applying the above or any of the succeeding rules, care must be taken to reduce the values given to the denominations required in the answer. Thus, had the velocity been required to have been in feet per second instead of feet per minute in the above example, the 5 minutes should first be reduced to seconds before dividing. The operation would then have been 5 min. $= 5 \times 60 = 300$ sec. Then, according to the formula,

$$v = 3,000 \div 300 = 10 \text{ ft. per sec. Ans.}$$

Had the velocity been required in inches per second, it would have been necessary to reduce the 3,000 feet to inches and the 5 minutes to seconds before dividing. Thus, 3,000 ft. $\times$ 12 = 36,000 in. 5 min. $\times$ 60 = 300 sec. Now, applying the formula,

$$v = \frac{36,000}{300} = 120 \text{ in. per sec. Ans.}$$

EXAMPLE.—A railroad-train travels 50 miles in $1\frac{1}{2}$ hours; what is its average velocity in feet per second?

SOLUTION.—Reducing the miles to feet and the hours to seconds, 50 miles $\times$ 5,280 = 264,000 ft. $1\frac{1}{2}$ hours $\times 60 \times 60$ = 5,400 sec. Applying formula **90**,

$$v = \frac{264,000}{5,400} = 48\frac{8}{9} \text{ ft. per sec. Ans.}$$

1828. To find the distance which a body would travel in a given time with a given velocity:

Rule.—*Multiply the velocity by the time,*

or $$s = v\,t, \qquad \textbf{(91.)}$$

EXAMPLE.—The velocity of sound in still air is 1,092 feet per second; how many miles will it travel in 16 seconds?

SOLUTION.—Reducing the 1,092 ft. to miles, the velocity is

$$\frac{1,092}{5,280} \text{ mile per second.}$$

Applying formula **91**,

$$s = v\,t = \frac{1,092}{5,280} \times 16 = 3.31 \text{ miles, nearly. Ans.}$$

EXAMPLE.—The piston speed of an engine is 11 ft. per sec.; how many miles does the piston travel in 1 hour and 15 minutes?

SOLUTION.— 1 hour and 15 minutes reduced to seconds $= 4,500$ seconds $=$ the time. 11 feet reduced to miles $= \frac{11}{5,280}$ mile $=$ velocity in miles per second. Applying the formula,

$$s = \frac{11}{5,280} \times 4,500 = 9.375 \text{ miles. Ans.}$$

1829. To find the time it will take a body to move through a given distance with a given uniform velocity:

Rule.—*Divide the distance, or space passed over, by the velocity.*

$$t = \frac{s}{v}. \qquad (\mathbf{92.})$$

EXAMPLE.—Suppose that the radius of the crank of a steam-engine is 15 inches and that the shaft makes 120 revolutions per minute, how long will it take the crank-pin to travel 18,849.6 feet?

SOLUTION.—Since the radius, or distance from the center of the shaft to the center of the crank-pin, is 15 in., the diameter of the circle it moves in is 15 in. $\times 2 = 30$ in. $= 2.5$ ft. The circumference of this circle is $2.5 \times 3.1416 = 7.854$ ft. $7.854 \times 120 = 942.48$ ft., distance that the crank-pin travels in one minute $=$ velocity in feet per minute. Applying the formula,

$$t = \frac{s}{v} = \frac{18,849.6}{942.48} = 20 \text{ min. Ans.}$$

EXAMPLE.—A point on the rim of an engine fly-wheel travels at the rate of 150 feet per second; how long will it take to travel 45,000 feet?

SOLUTION.—Using formula **92**,

$$t = \frac{45,000}{150} = 300 \text{ sec.} = 5 \text{ min. Ans.}$$

EXAMPLES FOR PRACTICE.

1. A locomotive has drivers 80 inches in diameter. If they make 293 revolutions per minute, what is the velocity of the train in (*a*) feet per second? (*b*) miles per hour? Ans. { (*a*) 102.277 ft. per sec. (*b*) 69.734 mi. per hr.

2. Assuming the velocity of steam as it enters the cylinder to be 900 feet per second, how far could it travel, if unobstructed, during the time the fly-wheel of an engine revolved 7 times, if the number of revolutions per minute were 120? Ans. 3,150 ft.

3. The average speed of the piston of an engine is 528 feet per minute; how long will it take the piston to travel 4 miles? Ans. 40 min.

4. A speed of 40 miles per hour equals how many feet per second? Ans. $58\frac{2}{3}$ ft.

5. The earth turns around once in 24 hours. If the diameter be taken as 8,000 miles, what is the velocity of a point on the earth in miles per minute? Ans. $17.45\frac{1}{3}$ mi. per min.

6. The stroke of an engine is 28 inches. If the engine makes 11,400 strokes per hour, (*a*) what is its speed in feet per minute? (*b*) How far will this piston travel in 11 minutes? Ans. { (*a*) $443\frac{1}{3}$ ft. per min. (*b*) 4,876 ft. 8 in.

FORCE.

NEWTON'S LAWS OF MOTION.

1830. A **force** is that which produces, or tends to produce or destroy, motion. Forces are called by various names, according to the effects which they produce upon a body, as *attraction*, *repulsion*, *cohesion*, *adhesion*, *accelerating* force, *retarding* force, *resisting* force, etc., but all are equivalent to a push or pull, according to the direction in which they act upon a body.

1831. That the effect of a force upon a body may be compared with another force, it is necessary that three conditions be fulfilled in regard to both bodies. They are as follows:

1. *The point of application, or point at which the force acts upon the body, must be known.*

2. *The direction of the force, or, what is the same thing, the straight line along which the force tends to move the point of application, must be known.*

3. *The magnitude or value of the force, when compared with a given standard, must be known.*

The unit of magnitude of forces will always be taken as one pound, and all forces will be spoken of as a certain number of pounds.

1832. In practice, force is always regarded as a pressure; that is, a force may always be replaced by an equivalent weight. Thus, a force of 20 lb. acting upon a body is regarded as a pressure of 20 lb. produced by a weight of 20 lb. The tendency of a force is always to produce motion in the direction in which it acts. The resistance may be too great to cause motion, but it *always tends* to produce it.

1833. The fundamental principles of the relations between force and motion were first stated by Sir Isaac Newton. They are called "Newton's Three Laws of Motion," and are as follows:

I. *All bodies continue in a state of rest, or of uniform motion in a straight line, unless acted upon by some external force that compels a change.*

II. *Every motion or change of motion is proportional to the acting force, and takes place in the direction of the straight line along which the force acts.*

III. *To every action there is always opposed an equal and contrary reaction.*

1834. In the *first law of motion* it is stated that a body once set in motion by any force, no matter how small, will move forever in a straight line, and always with the same velocity, unless acted upon by some other force which compels a change. It is not possible to actually verify this law, on account of the earth's attraction for all bodies, but from astronomical observations, we are certain that the law is true. This law is often called *the law of inertia.*

1835. The word **inertia** is so abused that a full understanding of its meaning is necessary. Inertia is not a force, although it is often so called. If a force acts upon a body and puts it in motion, the effect of the force is stored in the body, and a second body, in stopping the first, will receive a blow equal in every respect to the original force, assuming that there has been no resistance of any kind to the motion of the first body.

It is dangerous for a person to jump from a fast-moving train, for the reason that, since his body has the same velocity as the train, it has the same force stored in it that would cause a body of the same weight to take the same velocity as the train, and the effect of a sudden stoppage is the same as the effect of a blow necessary to give the person that velocity.

By "bracing" himself and jumping in the same direction that the train is moving, and running, he brings himself gradually to rest, and thus reduces the danger. If a body is at rest, it must be acted upon by a force in order to be put in motion, and no matter how great the force may be, it can not be *instantly* put in motion.

The resistance thus offered to being put in motion is commonly, but erroneously, called the *resistance of inertia.* It should be called the *resistance due to inertia.*

1836. From the *second law*, it is seen that, if two or more forces act upon a body, their final effect upon the body will be in proportion to their magnitude and to the directions in which they act. Thus, if the wind is blowing due west, with a velocity of 50 miles per hour, and a ball is thrown due north with the same velocity, or 50 miles per hour, the wind will carry the ball just as far west as the force of the throw carried it north, and the combined effect will be to cause it to move northwest. The amount of departure from due north will be proportional to the force of the wind, and independent of the velocity due to the force of the throw.

1837. In Fig. 587, a ball e is supported in a cup, the bottom of which is attached to the lever o in such a manner that a movement of o will swing the bottom horizontally and allow the ball to drop. Another ball b rests in a horizontal groove that is provided with a slit in the bottom. A swinging arm is actuated by the spring d in such a manner that, when drawn back as shown and then released, it will strike the lever o and the ball b at the same time. This gives b an impulse in a horizontal direction and swings o so as to allow e to fall.

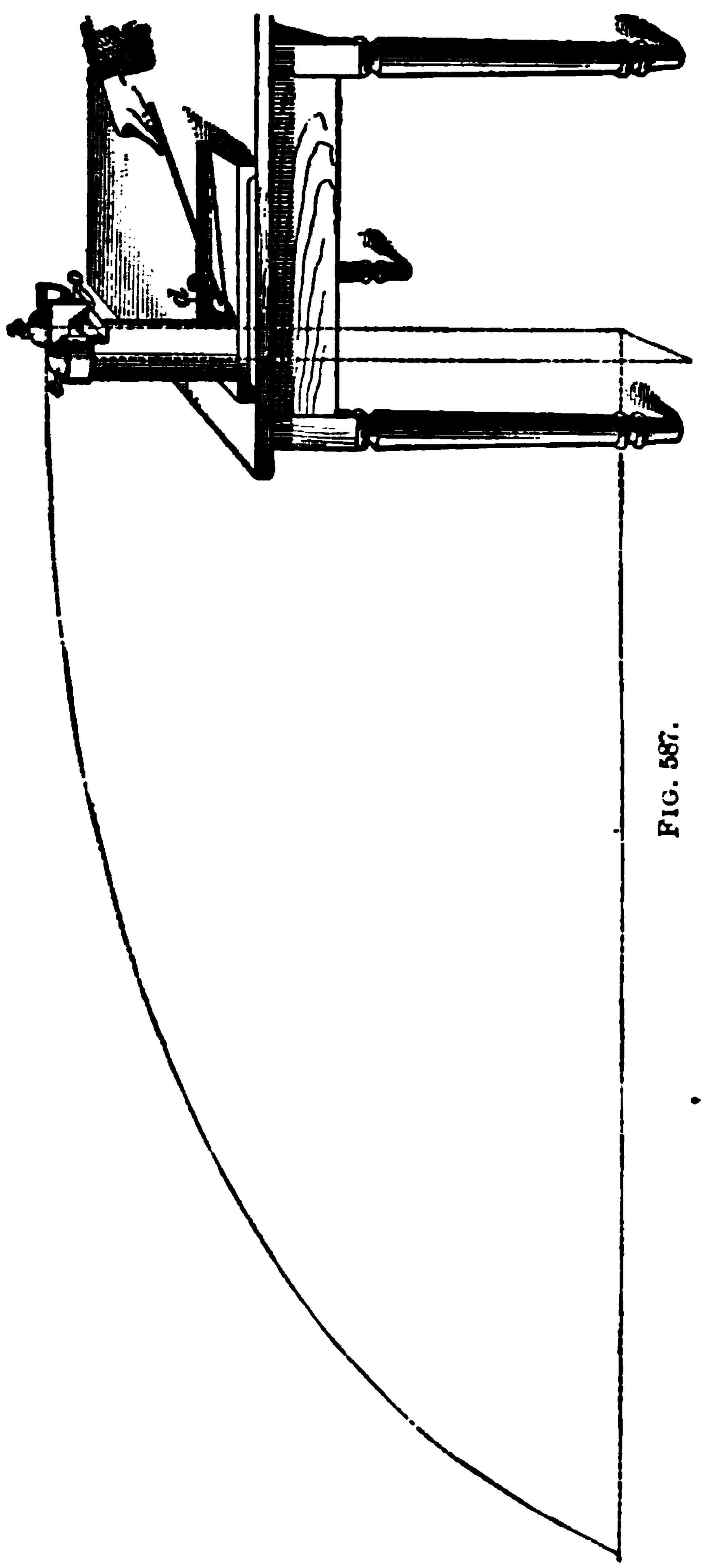


Fig. 587.

On trying the experiment, it is found that b follows a path shown by the curved dotted line, and reaches the floor at the same instant as e, which drops vertically. This shows that the force which gave the first ball its horizontal movement had no effect on the vertical force which compelled both balls to fall to the floor, the vertical force producing the same effect as if the horizontal force had not acted. The second law may also be stated as follows: *A force has the same effect in producing motion, whether it acts upon a body at*

rest or in motion, and whether it acts alone or with other forces.

1838. The *third law* states that action and reaction are equal and opposite. A man can not lift himself by his boot-straps, for the reason that he presses downwards with the same force that he pulls upwards; the downward reaction equals the upward action, and is opposite to it.

In springing from a boat, we must exercise caution, or the reaction will drive the boat from the shore. When we jump from the ground, we tend to push the earth from us, while the earth reacts and pushes us from it.

EXAMPLE.—Two men pull on a rope in opposite directions, each exerting a force of 100 pounds; what is the force which the rope resists?

SOLUTION.—Imagine the rope to be fastened to a tree, and one man to pull with a force of 100 pounds. The rope evidently resists 100 pounds. According to Newton's third law, the reaction of the tree is also 100 pounds. Now, suppose the rope to be slackened, but that one end is still fastened to the tree, and the second man to take hold of the rope near the tree, and pull with a force of 100 pounds, the first man pulling as before. The resistance of the rope is 100 pounds, as before, since the second man merely takes the place of the tree. *He is obliged to exert a force of 100 pounds to keep the rope from slipping through his fingers.* If the rope be passed around the tree and each man pulls an end with a force of 100 pounds in the same and parallel directions, the stress in the rope is 100 pounds, as before, but the tree must resist the pull of both men, or 200 pounds.

1839. A **force** may be represented by a line; thus, in Fig. 588, let A be the *point of application* of the force; let the length of the line A B represent its *magnitude*, and let the arrow-head indicate the *direction* in which the force acts; then the line A B fulfils the three conditions (see Art. **1831**), and the force is fully represented.

FIG. 588.

CENTER OF GRAVITY.

1840. The **center of gravity** of a body is that point at which the body may be balanced, or it is the point at which the whole weight of a body may be considered as concentrated.

1841. In a moving body, the line described by its center of gravity is always taken as the path of the body. In finding the distance that a body has moved, the distance that the center of gravity has moved is taken.

The definition of the center of gravity of a body may be applied to a system of bodies, if they are considered as being connected at their centers of gravity.

1842. If w and W, Fig. 589, be two bodies of known weights, their center of gravity will be at C. The point C may be readily determined, as follows:

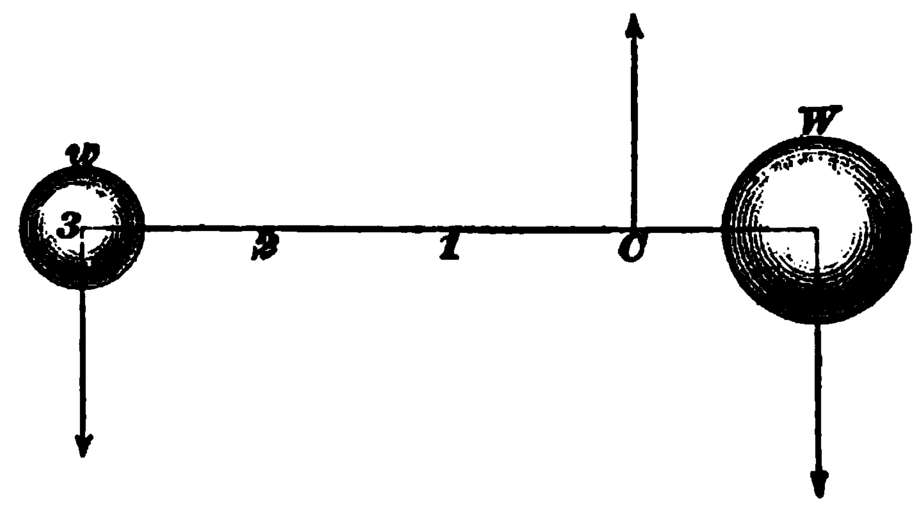


FIG. 589.

Rule.—*The distance of the common center of gravity from the center of gravity of the large weight is equal to the weight of the smaller body multiplied by the distance between the centers of gravity of the two bodies, and this product divided by the sum of the weights of the two bodies.*

Let w = weight of smaller body;
W = weight of larger body;
l = distance between centers of gravity of the two bodies;
l_1 = distance from the center of gravity of the two to the center of gravity of the larger body.

Then, $$l_1 = \frac{w\,l}{W + w}. \qquad \textbf{(93.)}$$

EXAMPLE.—In Fig. 589, $w = 10$ pounds, $W = 30$ pounds, and the distance between their centers of gravity is 36 inches; where is the center of gravity of both bodies situated?

SOLUTION.—Applying formula **93,**

$$l_1 = \frac{10 \times 36}{30 + 10} = 9 \text{ in.} =$$

distance of center of gravity from center of large weight. Ans.

1843. It is now very easy to extend this principle, to find the center of gravity of any number of bodies when their weights and the distances apart of their centers of gravity are known, by the following rule:

Rule.—*Find the center of gravity of two of the bodies, as* W_1 *and* W_4, *in Fig. 590, at* C_1. *Assume that the weight of both bodies is concentrated at* C_1, *and find the center of gravity of this combined weight* C_1, *and the weight of* W_2 *to be at* C_2; *then, find that the center of gravity of the combined weights of* W_1, W_4, *and* W_2 *(concentrated at* C_2) *and* W_3 *to be at* C, *and* C *will be the center of gravity of the four bodies.*

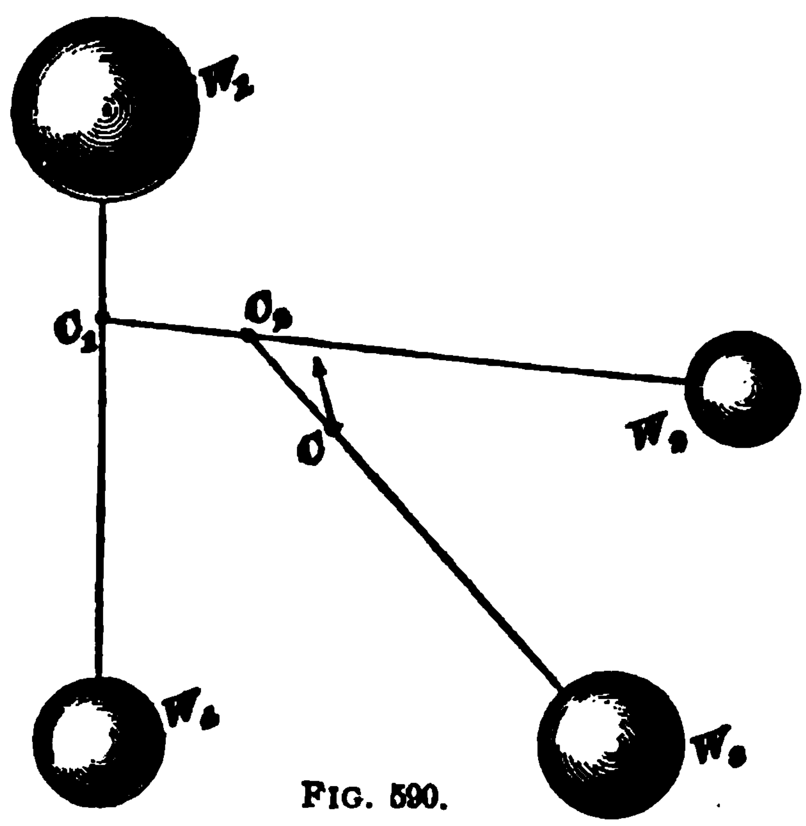


FIG. 590.

1844. To find the center of gravity of **any parallelogram:**

Rule.—*Draw the two diagonals, Fig. 591, and their point of intersection* C *will be the center of gravity.*

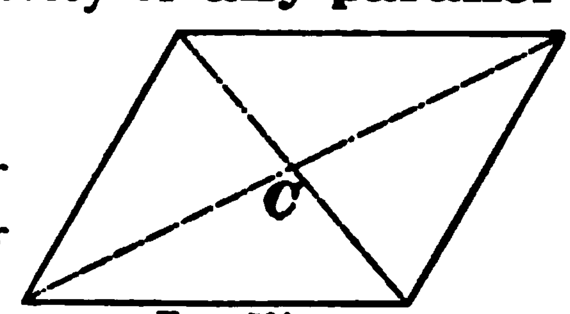


FIG. 591.

1845. To find the center of gravity of a **triangle,** as $A\ B\ C$, Fig. 592:

Rule.—*From any vertex, as* A, *draw a line to the middle point* D *of the opposite side* $B\ C$. *From one of the other vertexes, as* C, *draw a line to* F, *the middle point of the opposite side* $A\ B$; *the point of intersection* O *of these two lines is the center of gravity.*

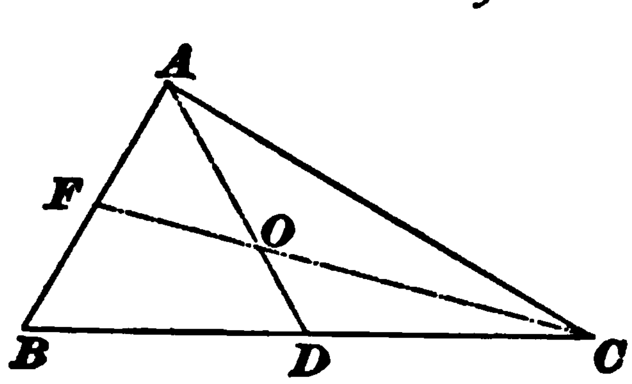


FIG. 592.

It is also true that the distance $D\ O = \frac{1}{3}\ D\ A$, and that

$F\,O = \frac{1}{3}\,F\,C$, and the center of gravity could have been found by drawing from any vertex a line to the middle point of the opposite side, and measuring back from that side $\frac{1}{3}$ of the length of the line.

1846. The center of gravity of **any regular plane figure** is the same as the center of the inscribed or circumscribed circle.

1847. To find the center of gravity of **any irregular plane figure** but of uniform thickness throughout, divide one of the parallel surfaces into triangles, parallelograms,

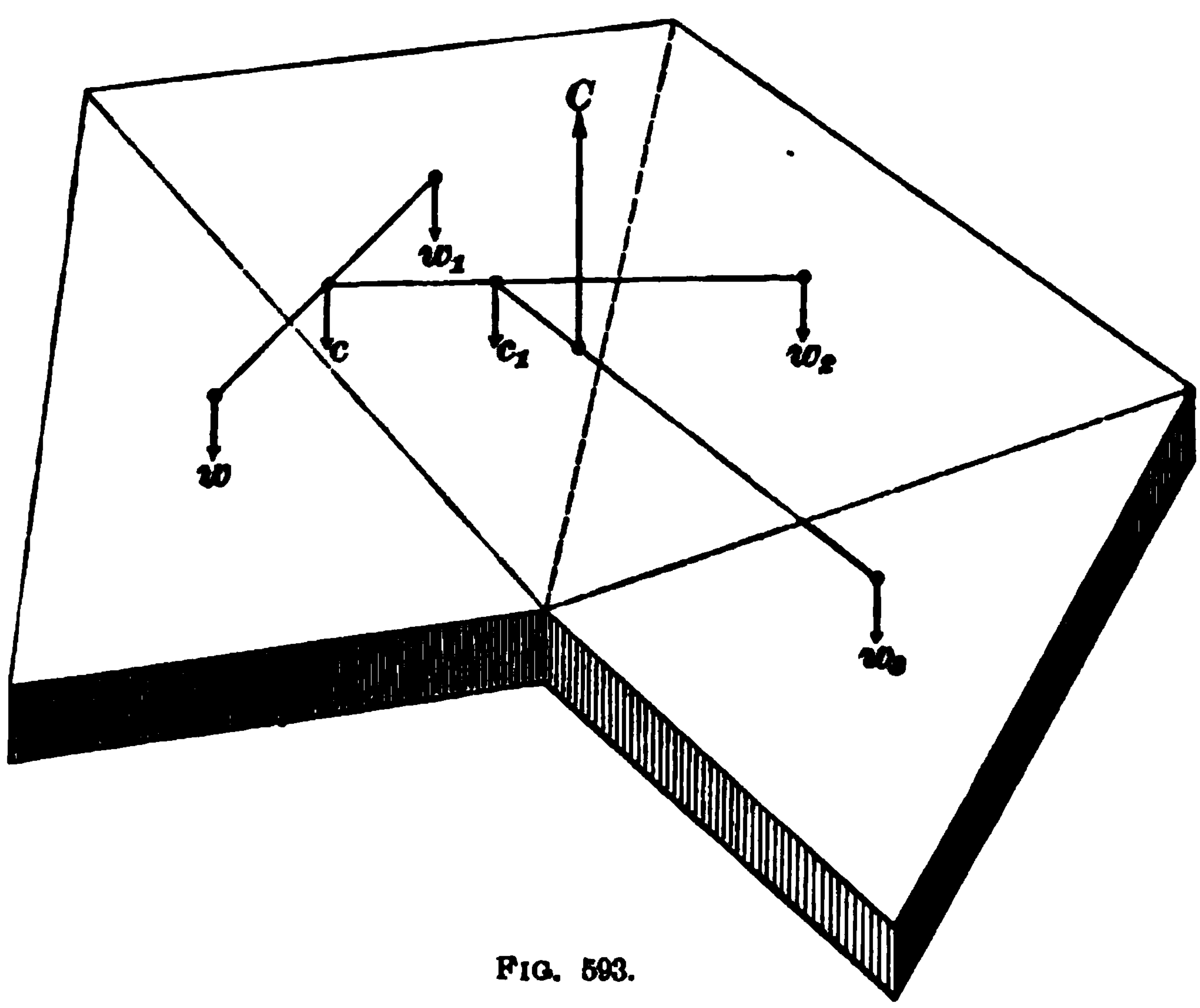


FIG. 593.

circles, ellipses, etc., according to the shape of the figure; find the area and center of gravity of each part separately, and combine the centers of gravity thus found as in the case of more than two bodies whose weights were known by the rule of Art. **1843**, except that the area of each part is used instead of their weights. See Fig. 593.

EXAMPLE.—Suppose that the two balls shown in Fig. 589 are 5 inches and 10 inches in diameter, and weigh 10 pounds and 80 pounds, respectively. If the distance between their centers is 40 inches, and

they are connected by a steel rod 1 inch in diameter, where is the center of gravity, taking the weight of a cubic inch of steel as .283 pound?

SOLUTION.—The length of the rod $= 40 - \frac{5}{2} - \frac{10}{2} = 32\frac{1}{2}$ in. Its volume is $1^2 \times .7854 \times 32\frac{1}{2} = 25.53$ cu. in. $25.53 \times .283 = 7.22$ lb. The rod being straight, its center of gravity is in the middle at a distance of $\frac{32.5}{2} + \frac{5}{2} = 18\frac{3}{4}$ in. from the center of the smaller weight and $\frac{32.5}{2} + \frac{10}{2} = 21\frac{1}{4}$ in. from the center of the larger weight. Now, assuming the weight of the rod to be concentrated at its center of gravity, we have three weights of 10, 7.22, and 80 lb., all in a straight line, and the distances between them given, to find the center of gravity, or balancing-point, of the combination. We will first find the center of gravity of the two smaller weights by formula **93.**

$$l_1 = \frac{7.22 \times 18\frac{3}{4}}{10 + 7.22} = 7.86 \text{ in.} =$$

distance from the center of the 10-lb. weight. Considering both of the smaller weights to be concentrated at this point, we find the center of gravity of this combined weight and the large weight by the same formula:

$$40 - 7.86 = 32.14 \text{ in.} =$$

distance between the center of gravity of the two small weights and the center of gravity of the 80-lb. weight. Applying formula **93,**

$$l_1 = \frac{17.22 \times 32.14}{80 + 17.22} = 5.693 \text{ in.} =$$

distance from the center of the 80-lb. weight. Ans.

1848. Center of Gravity of a Solid.—In a body free to move, the center of gravity will lie in a vertical plumb-line

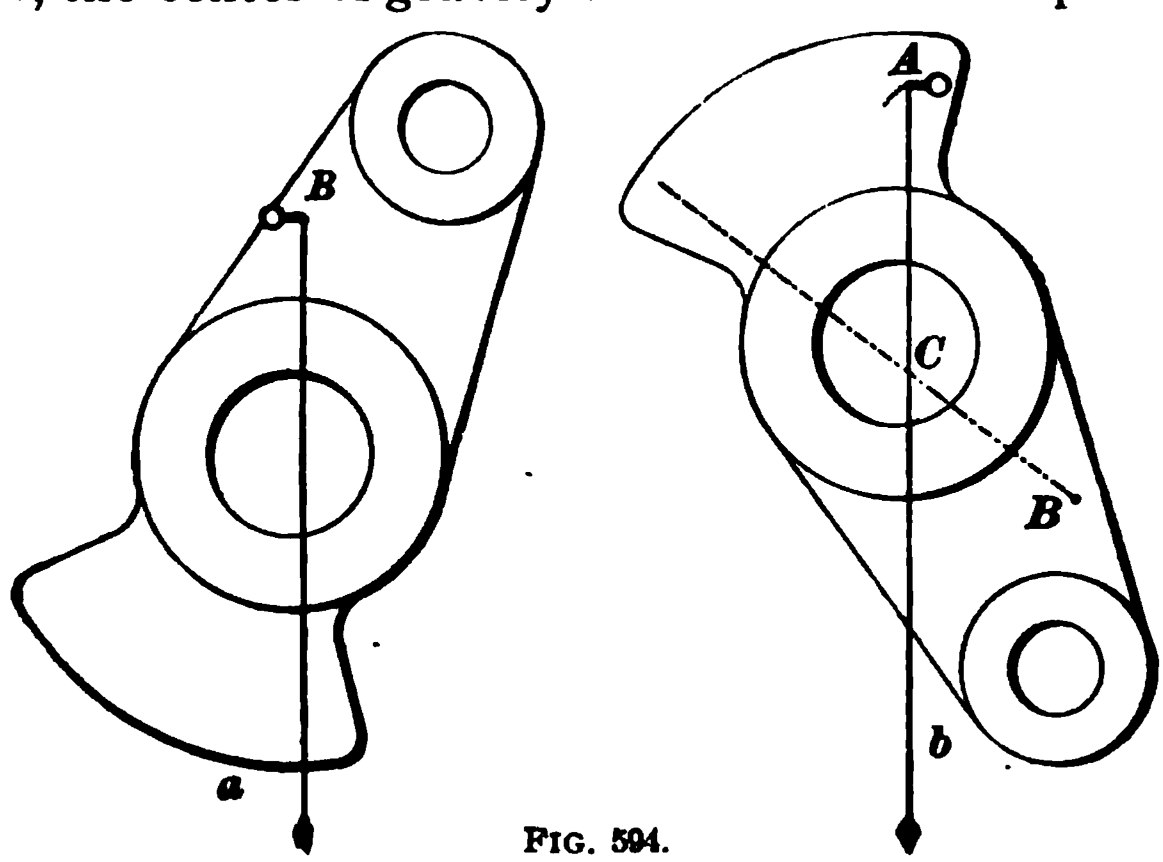


FIG. 594.

drawn through the point of support. Therefore, to find the position of the center of gravity of an irregular solid, as the crank, Fig. 594, suspend it at some point, as *B*, so that it will move freely. Drop a plumb-line from the point of suspension, and mark its direction. Suspend the body at another point, as *A*, and repeat the process. The intersection *C* of the two lines will be directly over the center of gravity.

Since the center of gravity depends wholly upon the shape and weight of a body, it may be without the body; for example, the center of gravity of a circular ring is the same as the center of the circumference of the ring.

EXAMPLES FOR PRACTICE.

1. A spherical shell has a wrought-iron handle attached to it. The shell is 10 inches in diameter, and weighs 20 pounds. The handle is $1\frac{1}{4}$ inches in diameter, and the distance from the center of the shell to the end of the handle is 4 feet. Where is the center of gravity? Take the weight of a cubic inch of wrought iron as .278 pound.

Ans. 13.612 in. from center of shell.

2. The distance between the centers of two bodies is 51 inches. The weights of the bodies being 20 and 73 pounds, where is the center of gravity? Ans. 10.968 in. from the center of large weight.

3. A hollow engine piston weighs 275 pounds, and is $3\frac{1}{4}$ inches thick. Assuming the piston-rod to be straight throughout its entire length, and to weigh 140 pounds, at what point will the piston and rod balance, if the length of the rod is 73 inches from the face of the piston? Consider the weight of the piston to be concentrated at its center.

Ans. 11.15 in., nearly, from face of piston.

4. Weights of 5, 9, and 12 pounds lie in one straight line, in the order named. Distance from the 5-pound weight to the 9-pound weight is 22 inches, and from the 9-pound weight to the 12-pound weight is 18 inches. Where is the center of gravity?

Ans. 13.923 in. from 12-pound weight.

SIMPLE MACHINES.

THE LEVER AND WHEEL AND AXLE.

1849. A **lever** is a bar capable of being turned about a pivot, or point, as in Figs. 595 to 597.

The object *W* to be lifted is called the **weight**; the force used *P* is called the **power**; and the point or pivot *F* is called the **fulcrum**.

That part of the lever between the weight and the fulcrum, or Fb, is called the **weight arm**, and the part be-

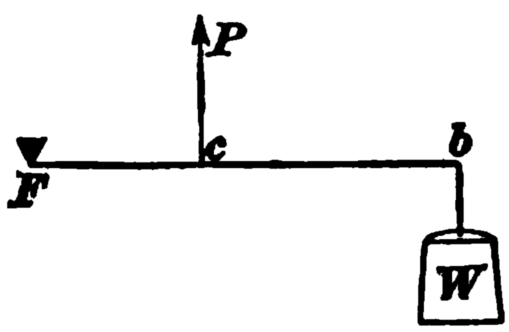


FIG. 595.

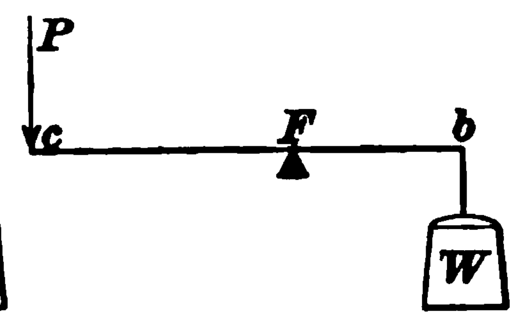


FIG. 596.

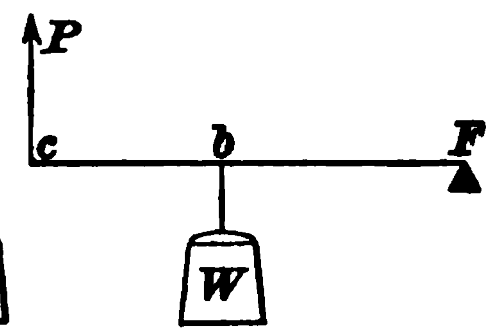


FIG. 597.

tween the power and the fulcrum, or Fc, is called the **power arm.**

1850. In order that the lever shall be in equilibrium (balance), *the power multiplied by the power arm must equal the weight multiplied by the weight arm;* that is, $P \times Fc$ must equal $W \times Fb$.

If F be taken as the center of a circle, and arcs be described through b and c, it will be seen that, if the weight arm is moved through a certain angle, the power arm will move through the same angle. Since, in the same or equal angles, the lengths of the arcs are proportional to the radii with which they were described, it is seen that the power arm is proportional to the distance through which the power moves, and the weight arm is proportional to the distance through which the weight moves.

Hence, instead of writing $P \times Fc = W \times Fb$, we might have written it $P \times$ distance through which P moves $= W \times$ distance through which W moves. This is the general law of all machines, and can be applied to any mechanism, from the simple lever up to the most complicated arrangement. Stated in the form of a rule, it is as follows:

Rule.—*The power multiplied by the distance through which it moves equals the weight multiplied by the distance through which it moves.*

In the above rule, it will be noticed that there are four requirements necessary for a complete knowledge of the lever, viz., the power (or force), the weight, the power arm (or distance through which the power moves), and the weight

arm (or distance through which the weight moves). If any three are given, the fourth may be found by letting x represent the requirement which is to be found, and multiplying the power by the power arm and the weight by the weight arm; then, dividing the product of the two known numbers by the number by which x is multiplied, the result will be the requirement which was to be found.

EXAMPLE.—If the weight arm of a lever is 6 inches long, and the power arm is 4 feet long, how great a weight can be raised by a force of 20 pounds at the end of the power arm?

SOLUTION.—In this example, the weight is unknown; hence, representing it by x, we have, after reducing the 4 ft. to inches, $20 \times 48 = 960$ = power multiplied by the power arm, and $x \times 6$ = weight multiplied by the weight arm. Dividing the 960 by 6, the result is 160 lb., the weight. Ans.

If the distance through which the power or weight moved had been given instead of the power arm or weight arm, and it were required to find the power or weight, the process would have been exactly the same, using the given distance instead of the power arm or weight arm.

EXAMPLE.—If, in the above example, the weight had moved $2\frac{1}{2}$ inches, how far would the power have moved?

SOLUTION.—In this example, the distance through which the power moves is required. Let x represent the distance. Then, $20 \times x$ = distance multiplied by power, and $2\frac{1}{2} \times 160 = 400$ = distance multiplied by the weight. Hence, $x = \frac{400}{20} = 20$ in. = distance through which the power arm moves. Ans.

The ratio between the weights and the power is $160 \div 20 = 8$. The ratio between the distance through which the weight moves and the distance through which the power moves is $2\frac{1}{2} \div 20 = \frac{1}{8}$. This shows that while a force of 1 pound can raise a weight of 8 pounds, the 1-pound weight must move through 8 times the distance that the 8-pound weight does. It will also be noticed that the ratio of the lengths of the two arms of the lever is also 8, since $48 \div 6 = 8$.

1851. The law which governs the straight lever also governs the bent lever, but care must be taken to determine

the true lengths of the lever arms, which are in every case *the perpendicular distances from the fulcrum to the line of direction of the weight or power.*

Thus, in Figs. 598 to 601, Fc in each case represents the

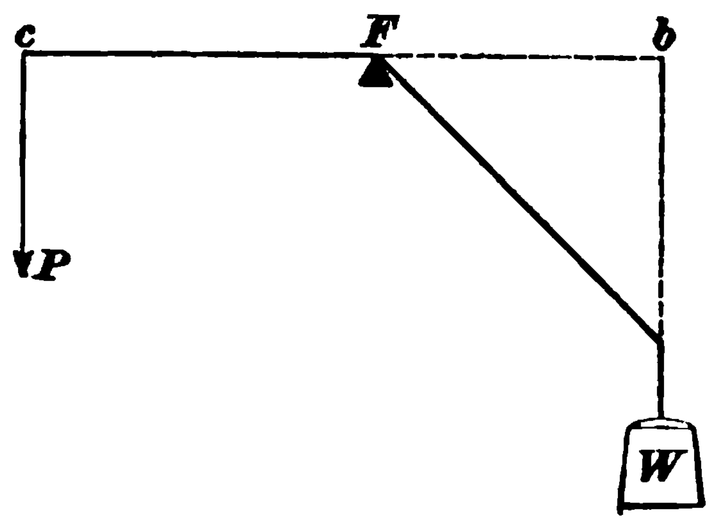


FIG. 598.

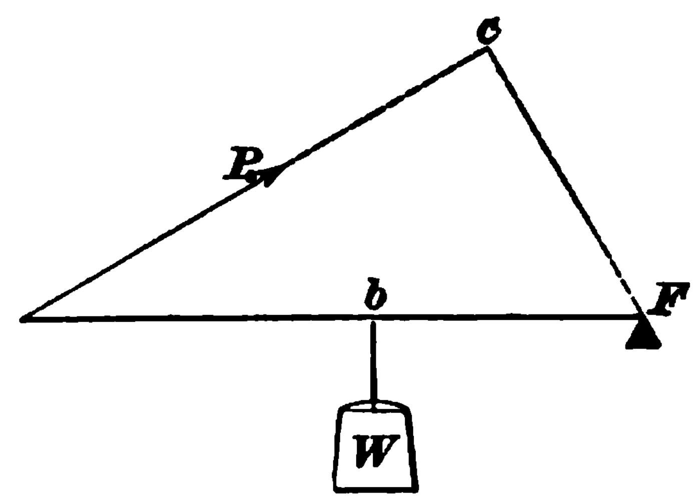


FIG. 599.

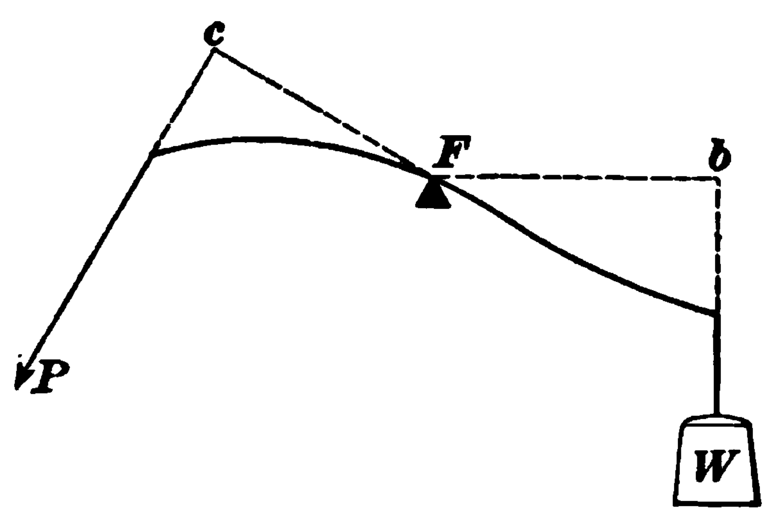


FIG. 600.

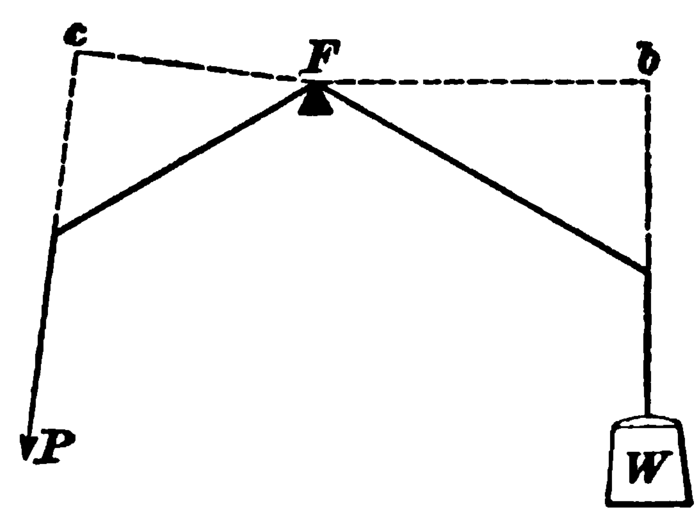


FIG. 601.

power arm, and Fb the weight arm. The following formula applies to any lever, straight or bent:

Let P = power;

W = weight;

a = perpendicular distance of line of direction of power from fulcrum = power arm;

b = perpendicular distance of line of direction of weight from fulcrum = weight arm.

Then, $$Pa = Wb. \qquad \textbf{(94.)}$$

1852. A **compound lever** is a series of single levers arranged in such a manner that when a power is applied to the first, it is communicated to the second, and from this to the third, and so on.

Fig. 602 shows a compound lever. It will be seen that, when a power is applied to the first lever at P, it will be communicated to the second lever at P, from this to the third lever at P, and thus raise the weight W.

The weight which the power of the first lever could raise acts as the power of the second, and the weight which this could raise through the second lever acts as the power of the third lever, and so on, no matter how many single levers make up the compound lever.

In this case, as in every other, the power multiplied by the distance through which it moves equals the weight multiplied by the distance through which it moves.

Hence, if we move the P end of the lever, say 4 inches,

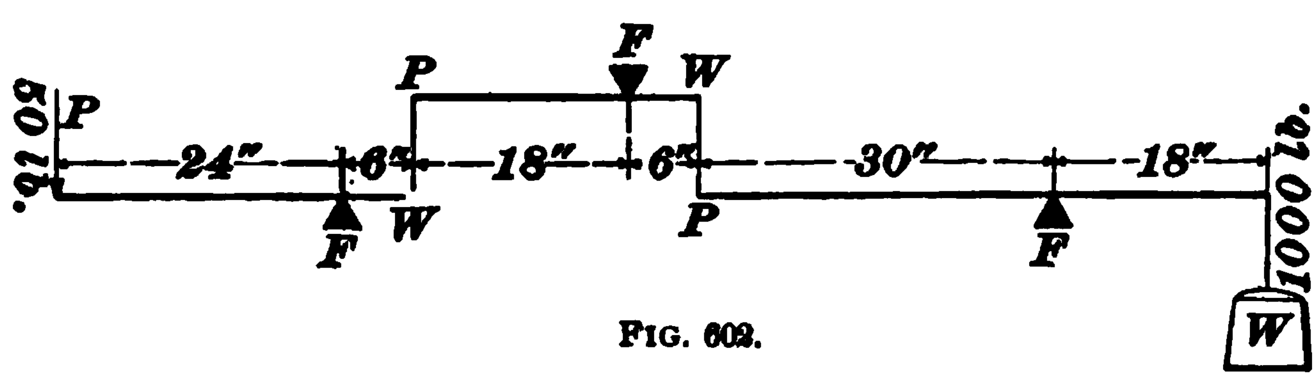


FIG. 602.

and the W end moves $\frac{1}{5}$ of an inch, we know that the ratio between P and W is the same as the ratio between 4 and $\frac{1}{5}$, that is, 1 to 20, and, hence, that 10 pounds at P would balance 200 pounds at W, without measuring the lengths of the different lever arms. If the lengths of the lever arms are known, the ratio between P and W may be readily obtained from the following rule:

Rule.—*The continued product of the power and each power arm equals the continued product of the weight and each weight arm.*

Let $a_1, a_2, a_3 \ldots\ldots$ = power arms of compound lever;
$b_1, b_2, b_3 \ldots\ldots$ = weight arms of compound lever.

Then,

$$P \times a_1 \times a_2 \times a_3 \times \ldots\ldots = W \times b_1 \times b_2 \times b_3 \times \ldots\ldots \quad \textbf{(95.)}$$

EXAMPLE.—If, in Fig. 602, PF = 24 inches, 18 inches, and 30 inches, respectively, and WF = 6 inches, 6 inches, and 18 inches, respectively, how great a force at P would it require to raise 1,000 pounds at W? What is the ratio between W and P?

SOLUTION.—Let x represent the power; then, according to formula **95**, $x \times 24 \times 18 \times 30 = 12{,}960\,x$. $1{,}000 \times 6 \times 6 \times 18 = 648{,}000$.

$$x = \frac{648{,}000}{12{,}960} = 50 \text{ lb. Ans.}$$

$$1{,}000 \div 50 = 20 = \text{ratio of } W \text{ to } P. \text{ Ans.}$$

1853. The **wheel and axle** consists of *two cylinders of different diameters, rigidly connected*, so that they turn together about a common axis, as in Fig. 603. Then, as before, $P \times$ distance through which it moves $= W \times$ distance through which it moves; and, since these distances are proportional to the radii of the power cylinder and weight cylinder, $P \times F\,c = W \times F\,b$.

It is not necessary that an entire wheel be used; an arm, projection, radius, or anything which the power causes to revolve in a circle may be considered as the wheel. Consequently, if it is desired to hoist a weight with a windlass, Fig. 604, the power is applied to the handle of the crank, and the distance between the center line of the crank handle and the axis of the drum corresponds to the radius of the wheel.

FIG. 604.

EXAMPLE.—If the distance between the center line of the handle and the axis of the drum, in Fig. 604, is 18 inches, and the diameter of the drum

is 6 inches, what force will be required at P to raise a load of 300 pounds?

SOLUTION.—$P \times (18 \times 2) = 300 \times 6$, or $P = 50$. Ans.

EXAMPLES FOR PRACTICE.

1. The lever of a safety-valve is of the form shown in Fig. 595, where the force is applied at a point between the fulcrum and the weight lifted. If the distance from the fulcrum to the valve is $5\frac{1}{2}$ inches, and from the fulcrum to the weight is 42 inches, what total force is necessary to raise the valve, the weight being 78 pounds and the weight of valve and lever being neglected? Ans. 595.64 lb.

2. If, in Fig. 602, $PF = 10$, 12, 14, and 16 inches, respectively, and $WF = 2$, 3, 4, and 5 inches, respectively, (*a*) how great a weight can a force of 20 pounds raise? (*b*) What is the ratio between W and P? (*c*) If P moves 4 inches, how far will W move?

Ans. (*a*) 4,480 lb. (*b*) 224. (*c*) $\frac{1}{56}$ in.

3. A windlass is used to hoist a weight. If the diameter of the drum on which the rope winds is 4 inches, and the distance from the center of the handle to the axis of the drum is 14 inches, how great a weight can a force of 32 pounds applied to the handle raise?

Ans. 224 lb.

PULLEYS AND GEARS.

FIXED AND MOVABLE PULLEYS.

1854. A **pulley** is a wheel turning on an axle, over which a cord, chain, or band is passed in order to transmit the force through the cord, chain, or band.

Pulleys are often used for hoisting or raising loads, in which case the frame which supports the axle of the pulley is called the **block.**

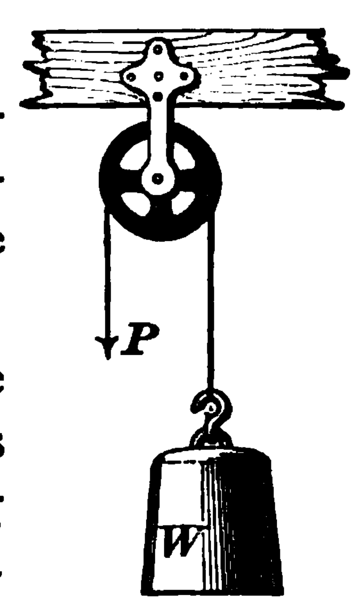


FIG. 605.

1855. A **fixed pulley** is one whose block is not movable (see Fig. 605). In this case, if the weight W be lifted by pulling down P, the other end of the cord W will evidently move the same distance upwards that P moves downwards; hence, P must equal W.

1856. A **movable pulley** is one whose block is movable, as in Fig. 606. One end of the cord is fastened to the

beam, and the weight is suspended from the pulley, the other end of the cord being drawn up by the application of a force

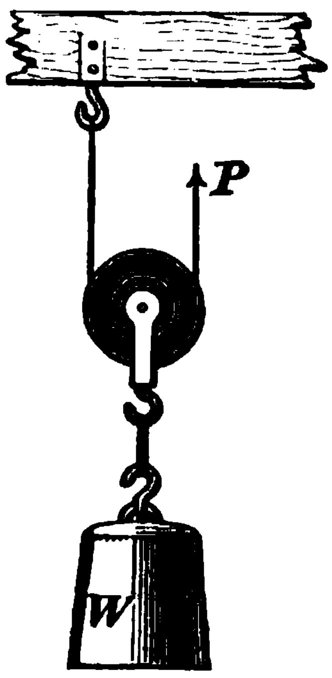


FIG. 606.

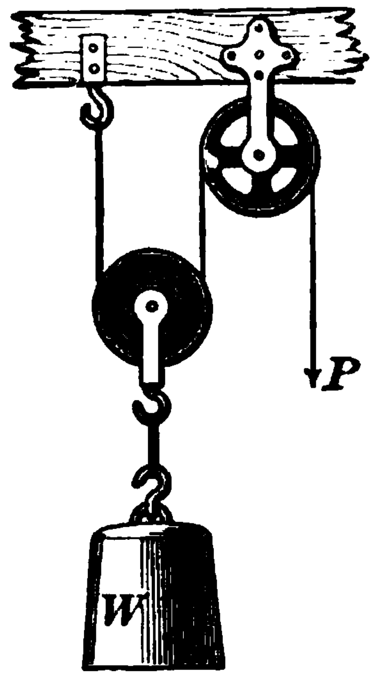


FIG. 607.

P. A little consideration will show that if *P* moves through a certain distance, say 1 foot, *W* will move through *half* that distance, or 6 inches; hence, a pull of 1 pound at *P* will lift 2 pounds at *W*.

The same would also be true if the free end of the cord were passed over a *fixed pulley*, as in Fig. 607, in which case the fixed pulley merely changes the direction in which *P* acts, so that a weight of 1 pound hung on the free end of the cord will balance 2 pounds hung from the *movable pulley*.

1857. A combination of pulleys, as shown in Fig. 608, is sometimes used. In this case, there are three movable and three fixed pulleys, and the amount of movement of *W*, owing to a certain movement of *P*, is readily found.

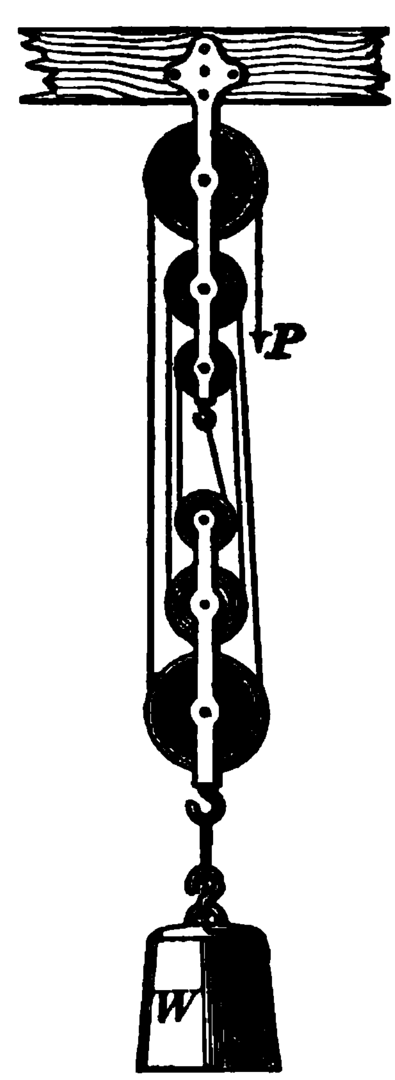


FIG. 608.

It will be noticed that there are *six parts* of the rope, not counting the free end; hence, if the movable block be lifted 1 foot, *P* remaining in the same position, there will be 1 foot of slack in each of the six parts of the rope, or *six feet* in all. Therefore, *P* would have to move 6 feet in order to take up this slack, or *P* moves 6 times as far as *W*. Hence, 1 pound at *P* will support 6 pounds at *W*, since the *power multiplied by the distance through which it moves equals the weight multiplied by the distance through which it moves.* It will also be noticed that there are three movable pulleys, and that $3 \times 2 = 6$.

1858. Law of Combination of Pulleys.—*In any combination of pulleys where one continuous rope is used, a load on the free end will balance a weight on the movable block as many times as great as itself as there are parts of the rope supporting the load—not counting the free end.*

The above law is good, whether the pulleys are side by side, as in the ordinary block and tackle, or whether they are arranged as in the figure.

EXAMPLE.—In a block and tackle having five movable pulleys, how great a force must be applied to the free end of the rope to raise 1,250 pounds?

SOLUTION.—Since there are five movable pulleys, there must be 10 parts of the rope to support them. Hence, according to the above law, a force applied to the free end will support a load 10 times as great as itself, or the force $= \frac{1,250}{10} = 125$ lb. Ans.

PULLEYS FOR TRANSMISSION OF POWER.

1859. Pulleys for the transmission of power by belts may be divided into two principal classes: (1) The solid

FIG. 609.

FIG. 610.

pulley shown in Fig. 609, in which the hub, arms, and rim are one entire casting. (2) The split pulley shown in Fig. 610, which is cast in halves.

This last style of pulley is more readily placed upon and removed from the shaft than the solid pulley. Pulleys are generally cast in halves or parts when they are more than 6 feet in diameter; this is done on account of shrinkage strain in large pulley castings, which renders them liable to crack as a result of unequal cooling of the metal.

1860. Crowning.—In Fig. 611 is shown a section of the rim of a pulley that has crowning, or, in other words, whose diameter is larger at the center of the face than at its edges. This is done to prevent the belt from running off the pulley. The amount of crowning given to pulleys varies from $\frac{3}{16}$ to $\frac{1}{2}$ an inch per foot of width of the pulley face.

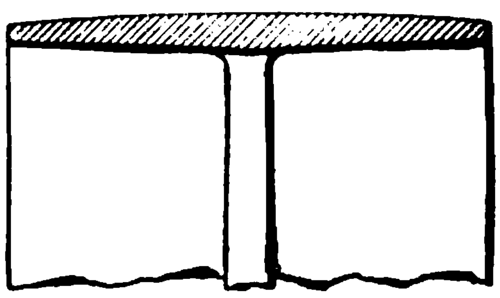


FIG. 611.

1861. Balanced Pulleys.—All pulleys which rotate at high speeds should be balanced. If they are not, the centrifugal force which is generated by the pulley's rotation is greater on one side than on the other, and it will cause the pulley shaft to vibrate and shake. Pulleys should run true, so that the strain or tension of the belt is equal at all parts of the revolution, thus making the transmitting power equal. The smoother the surface of a pulley, the greater is its driving power.

The transmitting power of a pulley can be increased by covering the face of the pulley with a leather or rubber band; this increases the driving power about one-quarter.

1862. The pulley that imparts motion to the belt is called the **driver**; that which receives the motion is called the **driven.**

The revolutions of any two pulleys over which a belt is run vary in an inverse proportion to their diameters; consequently, if a pulley of 20 inches in diameter is driven by one of 10 inches in diameter, the 20-inch pulley will make one revolution while the 10-inch pulley makes two revolutions, or they are in the ratio of 2 to 1.

1863. To find the diameter of the driving pulley, when the diameter of the driven pulley and the number of revolutions per minute of each are given:

Rule.—*The diameter of the driving pulley equals the product of the diameter and number of revolutions of the driven pulley, divided by the number of revolutions of the driving pulley.*

Let D = diameter of the driver;
d = diameter of the driven;
N = number of revolutions of the driver;
n = number of revolutions of the driven.

NOTE.—The words revolutions per minute are frequently abbreviated to R. P. M.

Then, $$D = \frac{d\,n}{N}. \qquad \textbf{(96.)}$$

EXAMPLE.—The driving pulley makes 100 revolutions per minute; the driven pulley makes 75 revolutions per minute, and is 18 inches in diameter; what is the diameter of the driving pulley?

SOLUTION.—Substituting in formula **96**, we have

$$D = \frac{18 \times 75}{100} = 13\tfrac{1}{2} \text{ in. Ans.}$$

1864. The diameter and number of revolutions per minute of the driving pulley being given, to find the diameter of the driven pulley, which must make a given number of revolutions per minute:

Rule.—*The diameter of the driven pulley equals the product of the diameter and number of revolutions of the driving pulley, divided by the number of revolutions of the driven pulley.*

That is, $$d = \frac{D\,N}{n}. \qquad \textbf{(97.)}$$

EXAMPLE.—The diameter of the driver is $13\frac{1}{2}$ inches, and makes 100 revolutions per minute; what must be the diameter of the driven to make 75 revolutions per minute?

SOLUTION.—Substituting in formula **97**, we have

$$d = \frac{13\tfrac{1}{2} \times 100}{75} = 18 \text{ in. Ans.}$$

1865. To find the number of revolutions per minute of the driven pulley, its diameter and the diameter and num ber of revolutions per minute of the driving pulley being given:

Rule.—*The number of revolutions of the driven pulley is equal to the product of the diameter and number of revolutions of the driver, divided by the diameter of the driven pulley.*

That is, $$n = \frac{DN}{d}. \qquad \textbf{(98.)}$$

EXAMPLE.—The driver is 13½ inches in diameter, and makes 100 revolutions per minute; how many revolutions will the driven make in one minute, if it is 18 inches in diameter?

SOLUTION.—Substituting in formula **98**, we have

$$n = \frac{13\tfrac{1}{2} \times 100}{18} = 75 \text{ R. P. M.} \quad \text{Ans.}$$

1866. To find the number of revolutions per minute of the driving pulley, its diameter and the diameter and number of revolutions per minute of the driven pulley being given:

Rule.—*The number of revolutions of the driving pulley is equal to the product of the diameter and number of revolutions of the driven pulley, divided by the diameter of the driving pulley.*

That is, $$N = \frac{dn}{D}. \qquad \textbf{(99.)}$$

EXAMPLE.—The driven pulley is 18 inches in diameter, and makes 75 revolutions per minute; how many revolutions will the driver make in one minute, if it is 13½ inches in diameter?

SOLUTION.—Substituting in formula **99**, we have

$$N = \frac{18 \times 75}{13\tfrac{1}{2}} = 100 \text{ R. P. M.} \quad \text{Ans.}$$

WHEELWORK.

1867. Wheelwork.—A combination of wheels and axles, as in Fig. 612, is called a **train.** The wheel in a train, to which motion is imparted from a wheel on another shaft by such means as a belt or gearing, is called the **driven wheel** or **follower;** the wheel which imparts the motion is called the **driver.**

It will be seen that the wheel and axle bears the same

relation to the train that the simple lever does to the compound lever. Letting D_1, D_2, D_3, etc., represent the diam-

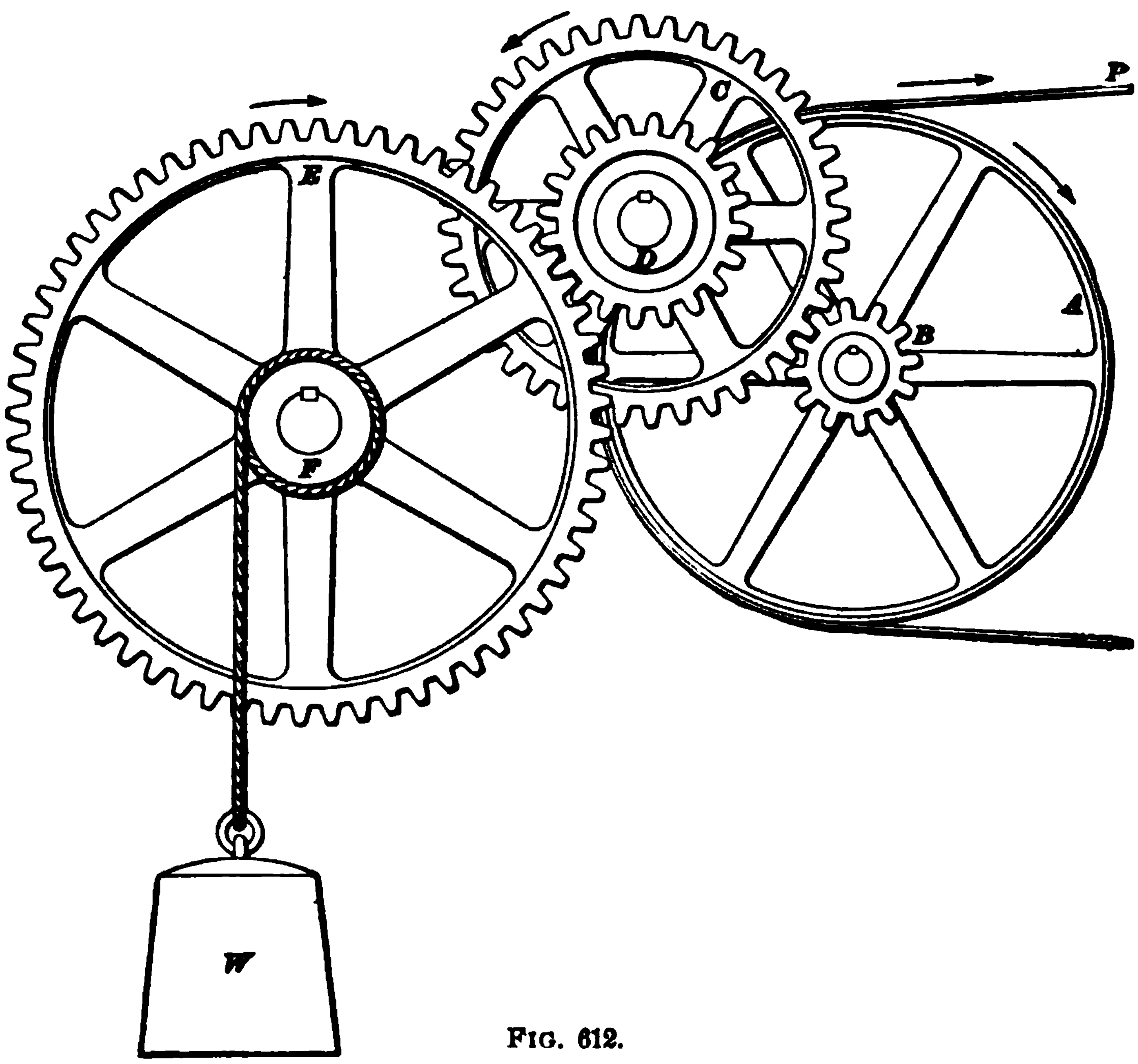


FIG. 612.

eters of the driven wheels and d_1, d_2, d_3, etc., the diameters of the drivers, we have the following

Rule.—*The continued product of the power and the radii of the driven wheels equals the continued product of the weight, the radius of the drum that moves the weight, and the radii of the drivers.*

This rule gives rise to the following formulas:

$$P = \frac{W \times d_1 \times d_2 \times d_3 \times \ldots\ldots}{D_1 \times D_2 \times D_3 \times \ldots\ldots}. \qquad (100.)$$

$$W = \frac{P \times D_1 \times D_2 \times D_3 \times \ldots\ldots}{d_1 \times d_2 \times d_3 \times \ldots\ldots}. \qquad (101.)$$

EXAMPLE.—The radius of the pulley A is 20 inches, of C, 15 inches, and of E, 24 inches; and the radius of the drum F is 4 inches, of the pinion D, 5 inches, and of the pinion B, 4 inches. How great a weight will a force of 1 pound at P raise?

SOLUTION.—Using formula **101**, we have

$$W = \frac{1 \times 20 \times 15 \times 24}{4 \times 5 \times 4} = \frac{7,200}{80} = 90 \text{ lb. Ans.}$$

If the weight W were raised 1 inch, P would fall 90 inches, or P would have to move 90 inches to raise W 1 inch. *Whenever there is a gain in power without a corresponding increase in the initial force, there is a loss in speed;* this is true of any machine.

In the last example, if P were to move the entire 90 inches in one second, W would move only 1 inch in one second.

1868. Instead of using the diameter or radius of a gear, the number of teeth may be used when computing the weight which can be raised, or the velocity, as in the last example.

EXAMPLE.—Assume that the radius of the pulley A, Fig. 612, is 40 inches, and that of F is 12 inches. The number of teeth in B is 9; in c, 27; in D, 12, and in E, 36. If the weight to be lifted is 1,800 pounds, how great a force at P is it necessary to apply to the belt?

SOLUTION.—Let P represent the force (power): then, by formula **100**,

$$P = \frac{1,800 \times 12 \times 9 \times 12}{40 \times 27 \times 36} = \frac{2,332,800}{38,880} = 60 \text{ lb. Ans.}$$

GEAR-WHEELS.

1869. A wheel that is provided with teeth to mesh with similar teeth upon another wheel is called a **gear-wheel,** or **gear.** In Fig. 613 is shown a **spur-gear.** On spur-gears, the teeth are always parallel to the axis of the wheel, or to its shaft.

FIG. 613.

1870. In Fig. 614 is shown a pair of **bevel-gears** in mesh, of which one is smaller than the other. When both are of the same diameter, they are called **miter-gears.**

In Fig. 615 is shown a pair

of miter-gears in mesh. It is obvious that the angle which the teeth of these gears make with the axis of the shaft must be 45°.

1871. In Fig. 616 is shown a revolving screw, or worm, as it is called, in gear; it is used to transmit motion from one shaft to another at right angles to it.

Fig. 614.

As the worm is nothing else than a screw, each revolution given to the worm will rotate the worm-wheel a distance

Fig. 615.

equal to its pitch; consequently, if there are 40 teeth in the worm-wheel, a single-threaded worm will have to make 40 revolutions in order to turn the wheel once.

1872. In Fig. 617 is shown a section of a rack and pinion, both having epicycloidal teeth. The arc *C C* represents part of the **pitch circle**; it is on the pitch circle that all the teeth are laid out. The diameter of a gear or worm-wheel is always taken as the diameter of this circle, unless otherwise specially stated as "diameter over all," or "diameter at the root," etc.

The **pitch** of the teeth of the gear-wheel is the distance from the edge of one tooth to the corresponding edge of the following tooth measured on the pitch circle; it is marked *pitch* in the figure.

FIG. 616.

The length of the tooth of a gear-wheel is .7 of its pitch, .4 of it, called the **root,** being below or within the pitch circle, and .3 of it, called the **addendum,** being above or

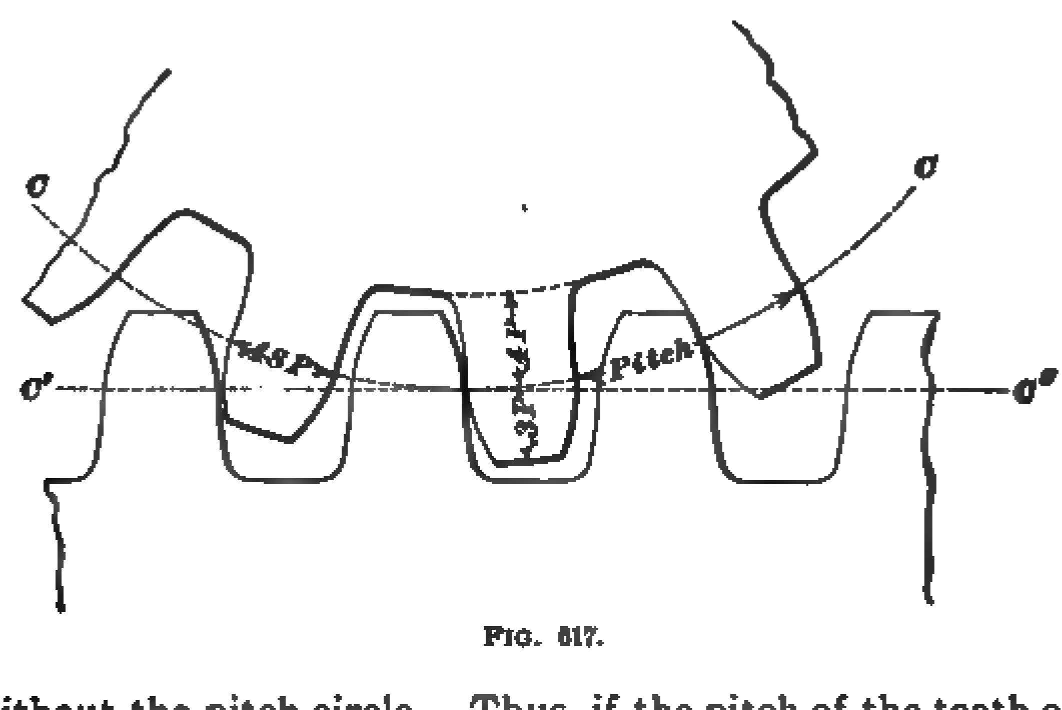


FIG. 617.

without the pitch circle. Thus, if the pitch of the teeth of a gear-wheel is 2 inches, the length of the teeth below the pitch circle is $2 \times .4 = .8$ of an inch; and the length of

the teeth above the pitch circle is $2 \times .3 = .6$ of an inch. Consequently, we have only to multiply the pitch by .4 to obtain the length of the teeth below the pitch circle, and by .3 to obtain the length of the teeth above the pitch circle. The thickness of the teeth of a cast gear-wheel equals $.48 \times P$, that is, .48 of the pitch; therefore, the thickness of the above teeth is $.48 \times 2$, or .96 of an inch.

A rack may be considered as a gear-wheel rolled out so as to make the pitch circle a straight line, as $C' C''$. The teeth of racks are proportioned by the same rules as those of gear-wheels.

1873. For the purpose of calculating the pitch, diameter, number of teeth, etc., of gear-wheels, the following rules are given:

To find the pitch diameter of a gear-wheel in inches, when the pitch and number of teeth are given:

Rule.—*The pitch diameter is equal to the product of the pitch and number of teeth, divided by 3.1416.*

Let $P =$ pitch;
$T =$ number of teeth;
$D =$ pitch diameter of the wheel.

Then, $$D = \frac{P\,T}{3.1416}. \qquad \textbf{(102.)}$$

EXAMPLE.—What is the diameter of the pitch circle of a gear-wheel which has 75 teeth, and whose pitch is 1.675 inches?

SOLUTION.—Substituting in formula **102**, we have

$$D = \frac{1.675 \times 75}{3.1416} = 40 \text{ in. Ans.}$$

1874. To find the number of teeth in a gear-wheel, when the diameter and pitch are given:

Rule.—*The number of teeth is equal to the product of 3.1416 and the diameter, divided by the pitch.*

That is, $$T = \frac{3.1416\,D}{P}. \qquad \textbf{(103.)}$$

EXAMPLE.—The diameter of a gear-wheel is 40 inches, and the pitch of the teeth is 1.675 inches; how many teeth are there in the wheel?

SOLUTION.—Substituting in formula **103,** we have

$$T = \frac{3.1416 \times 40}{1.675} = 75 \text{ teeth. Ans.}$$

1875. To find the pitch of a gear-wheel, when the diameter and the number of teeth are given:

Rule.—*The pitch of the teeth is equal to the product of 3.1416 and the diameter, divided by the number of teeth.*

That is, $$P = \frac{3.1416\,D}{T}. \qquad \textbf{(104.)}$$

EXAMPLE.—The diameter of a gear-wheel is 40 inches, and it has 75 teeth; what is the pitch of the teeth?

SOLUTION.—Applying formula **104,** we have

$$P = \frac{3.1416 \times 40}{75} = 1.675 \text{ in. Ans.}$$

1876. The forms of teeth used in ordinary practice are the epicycloidal and involute.

Fig. 617 shows the epicycloidal form, which is composed of two different curves; namely, that curve from the pitch

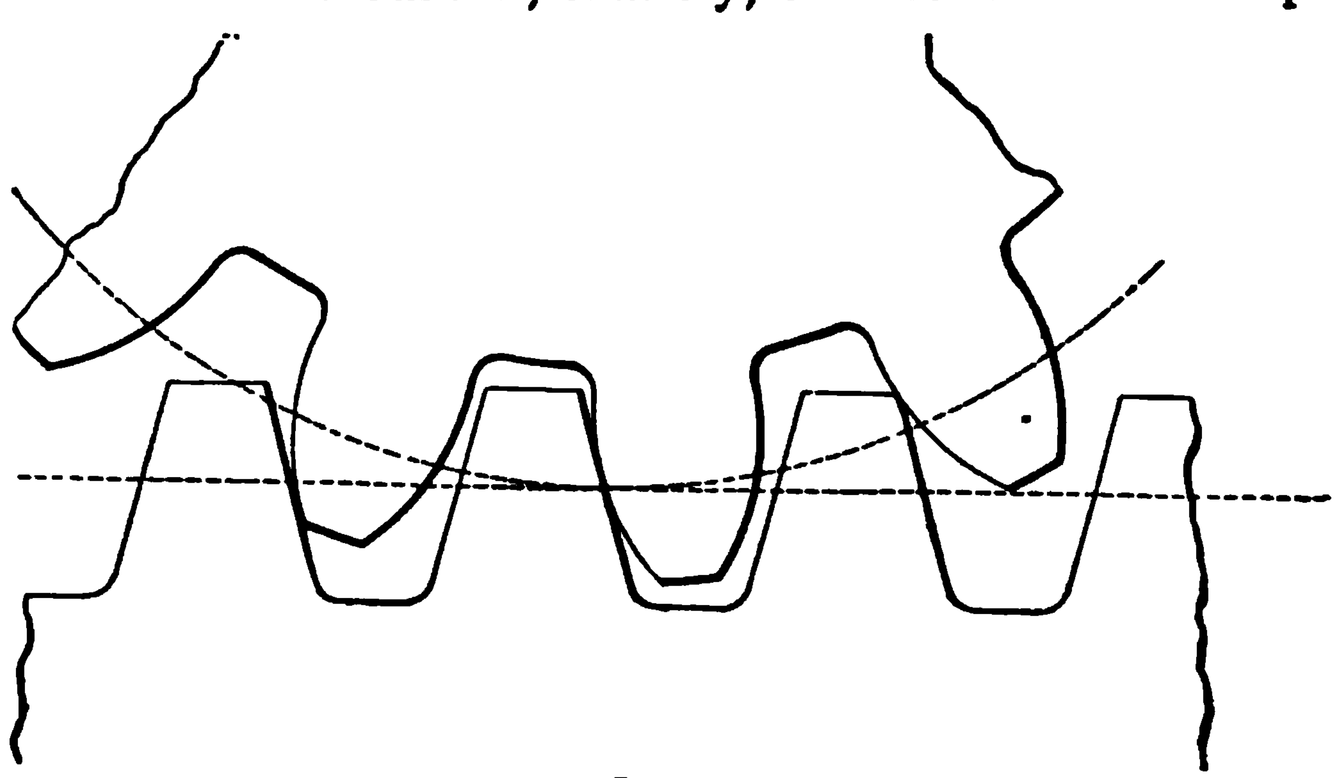


FIG. 618.

circle to the top of the tooth is an epicycloid, and that from the pitch circle to the bottom of the tooth is a hypocycloid.

In gear-wheels where this form of tooth is employed, their pitch circles must run tangent to one another.

1877. In Fig. 618 is shown the involute form of teeth, or teeth having but one curve. The outlines of the teeth shown in the rack are formed of straight lines.

Involute teeth have two great advantages over epicycloidal teeth: (1) They are stronger for the same pitch, as they are thicker at the root. (2) They may be spread apart so that their pitch circles do not run tangent to one another without practically affecting the perfect action of the teeth.

1878. To calculate the number of teeth or speed of one of two gear-wheels which are to gear together:

Let R = number of revolutions per minute of the driver;
r = number of revolutions per minute of the driven;
T = number of teeth in the driver;
t = number of teeth in the driven.

Rule.—*The number of teeth in the driver equals the product of the number of teeth and number of revolutions of the driven, divided by the number of revolutions of the driver.*

That is, $$T = \frac{t\,r}{R}. \qquad \textbf{(105.)}$$

EXAMPLE.—The driven has 27 teeth, and will make 66 revolutions per minute; if the driver makes 99 revolutions per minute, how many teeth are there in the driver?

SOLUTION.—Substituting in formula **105**, we have

$$T = \frac{27 \times 66}{99} = 18 \text{ teeth. Ans.}$$

1879. The number of revolutions per minute of the driver and driven, and the number of teeth in the driver being given, to find the number of teeth in the driven:

Rule.—*The number of teeth in the driven is equal to the product of the number of teeth and revolutions per minute of the driver, divided by the number of revolutions per minute of the driven.*

That is, $$t = \frac{T\,R}{r}. \qquad \textbf{(106.)}$$

EXAMPLE.—The driver has 24 teeth, and makes 99 revolutions per minute, and the driven must make 66 revolutions per minute; how many teeth must there be in the driven?

SOLUTION.—Substituting in formula **106**, we have

$$t = \frac{24 \times 99}{66} = 36 \text{ teeth. Ans.}$$

1880. The number of teeth in the driver and driven, and the number of revolutions per minute of the driver being given, to find the number of revolutions per minute of the driven:

Rule.—*The number of revolutions per minute of the driven is equal to the product of the number of teeth and number of revolutions of the driver, divided by the number of teeth of the driven.*

That is, $$r = \frac{TR}{t}. \qquad \textbf{(107.)}$$

EXAMPLE.—There are 18 teeth in the driver, and it makes 60 revolutions per minute; how many revolutions per minute will the driven make if it has 30 teeth?

SOLUTION.—Applying formula **107**, we have

$$r = \frac{18 \times 60}{30} = 36 \text{ R. P. M. Ans.}$$

1881. The number of teeth in the driver and driven, and the number of revolutions per minute of the driven being given, to find the number of revolutions per minute of the driver:

Rule.—*The number of revolutions of the driver is equal to the product of the number of teeth and revolutions of the driven, divided by the number of teeth of the driver.*

That is, $$R = \frac{tr}{T}. \qquad \textbf{(108.)}$$

EXAMPLE.—If there are 42 teeth in the driven, and if it makes 66 revolutions per minute, how many revolutions per minute will the driver make if it has 18 teeth?

SOLUTION.—Using formula **108**, we have

$$R = \frac{42 \times 66}{18} = 154 \text{ R. P. M. Ans.}$$

EXAMPLE.—In Fig. 619, the crank-shaft makes 60 revolutions per minute; the governor pulley is 4 inches in diameter, and the bevel-

gear on the governor pulley-shaft has 19 teeth; the bevel-gear which meshes with it and drives the governor has 30 teeth. The governor is to make 95 revolutions per minute; what should be the size of the pulley on the crank-shaft?

SOLUTION.—First determine the number of revolutions of the 4-inch pulley in order that the governor shall turn 95 times per minute. Applying formula **108**, we have $R = \frac{30 \times 95}{19} = 150$ revolutions of gear on pulley-shaft = revolutions of governor pulley. Now, applying formula **96**, we have $D = \frac{4 \times 150}{60} = 10$ in. = diameter of the pulley on the crank-shaft. Ans.

EXAMPLE.—In Fig. 619, the fly-wheel is 8 feet in diameter and drives a 5-foot pulley on the main shaft. A 14-inch pulley on the main shaft drives a 16-inch pulley on the countershaft. A 12-inch pulley on the

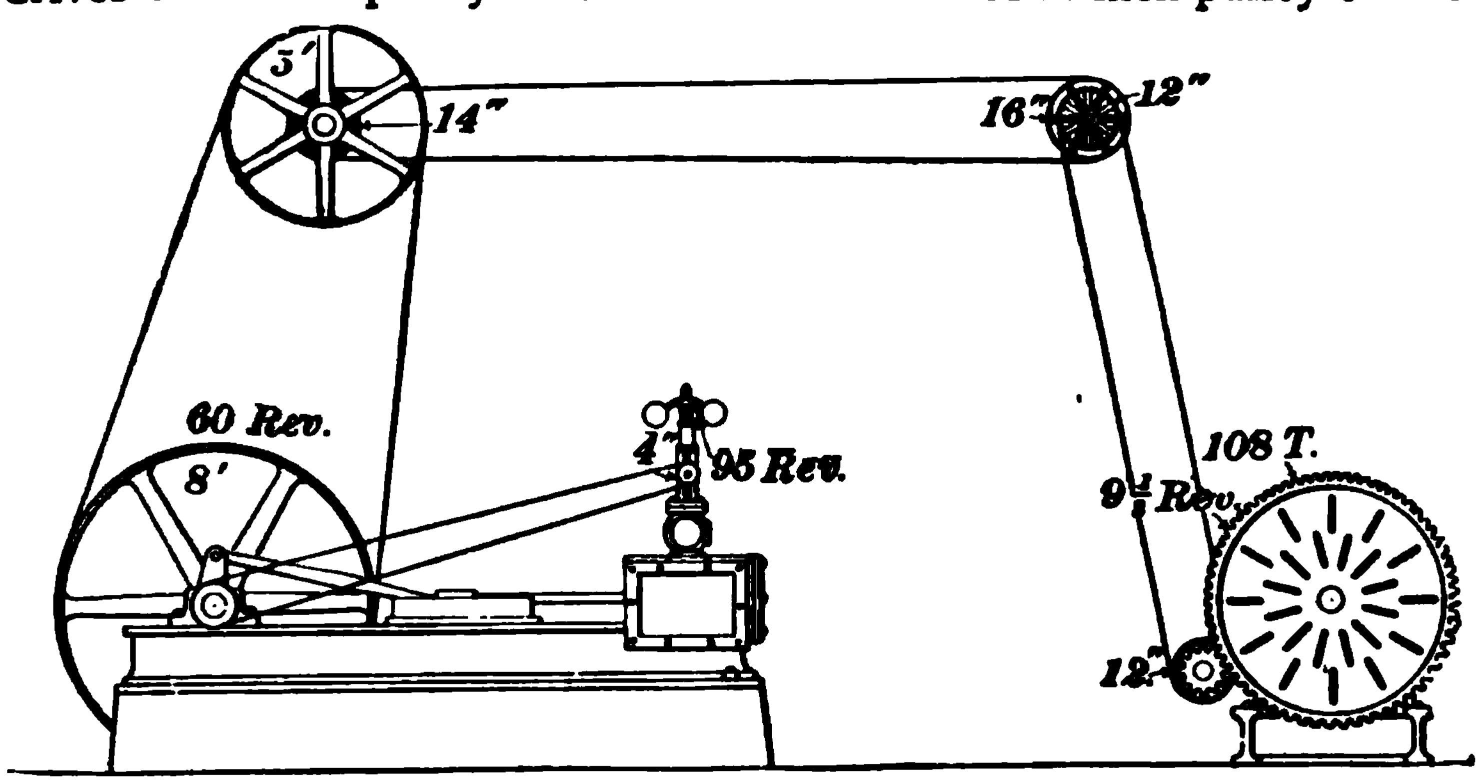


FIG. 619.

countershaft drives a 12-inch pulley on a shaft on which is a pinion that meshes into a large gear, attached to the face-plate of a large lathe, which has 108 teeth. How many teeth must the pinion have in order that the face-plate may make 9¼ revolutions per minute?

SOLUTION.—Applying formula **98**, to find the revolutions per minute of the main shaft, $n = \frac{8 \times 60}{5} = 96$ R. P. M. Applying formula **98** again to find the revolutions of the countershaft, $n = \frac{14 \times 96}{16} = 84$ R. P. M.; and again to find revolutions of the pulley which turns the small gear, $n = \frac{12 \times 84}{12} = 84$ R. P. M. Applying formula **105**, we have $T = \frac{108 \times 9\frac{1}{4}}{84} = 12$ teeth in pinion or driver. Ans.

1882. Horsepower of Gears.—To find the horsepower which can be safely transmitted by gears whose face, or breadth of tooth, is from $2\frac{1}{2}$ to 3 times their pitch:

Rule.—*The horsepower which can be safely transmitted equals the continued product of the square of the pitch, the velocity in feet per minute, and .01.*

Let p = the pitch;
s = circumferential speed of a point on the pitch circle in feet per minute.

Then, $$\text{H. P.} = .01\, s\, p^2. \qquad \textbf{(109.)}$$

EXAMPLE.—What horsepower can be safely transmitted by a gear whose pitch diameter is 66.84 in., pitch $1\frac{3}{4}$ in., and which makes 60 R. P. M.?

SOLUTION.—The velocity which is to be used when applying formula **108** is the circumferential speed of a point on the pitch circle. Hence, $66.84 \times 3.1416 = 209.98$ in. = circumference of pitch circle = $\frac{209.98}{12}$ ft. $\frac{209.98}{12} \times 60 = 1{,}049.9$ = velocity in ft. per min.

Now, applying formula **109**, H. P. $= .01 \times 1{,}049.9 \times 1.75^2 = 32.15$ horsepower. Ans.

1883. When measuring bevel-gears, the diameter of the largest pitch circle should be taken, as D, Fig. 620.

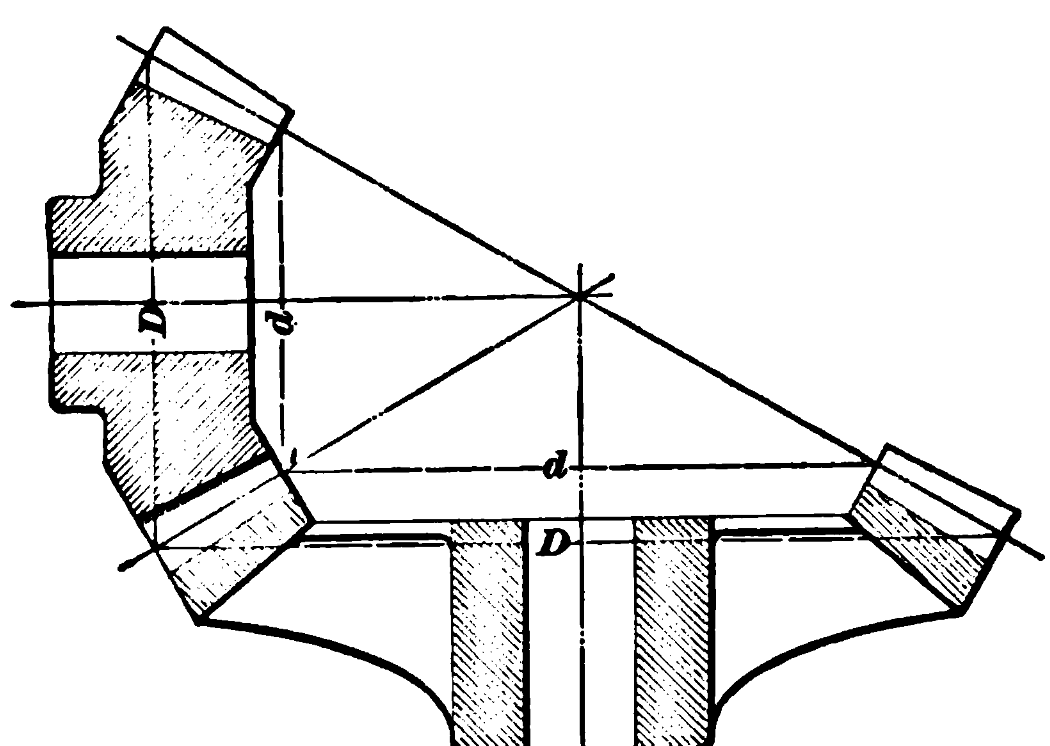


FIG. 620.

When calculating their horsepower, use the small or inner diameter, as d, Fig. 620. Either diameter may be used when calculating the revolutions per minute or number of

teeth by formulas **102** to **108**, but if the inner or outer diameter of one gear be used, the corresponding diameter of the other gear which meshes with it must also be used.

EXAMPLES FOR PRACTICE.

1. The driving pulley makes 110 R. P. M., and is 21 inches in diameter; what should be the size of the driven in order to make 385 R. P. M.? Ans. 6 in.

2. The main shaft of a certain shop makes 120 R. P. M. It is desired to have the countershaft make 150 R. P. M. There are on hand pulleys of 16, 24, 28, 35, and 38 inches in diameter. Can two of these be used, or must a new pulley be ordered?

Ans. Use the 28-inch and the 35-inch pulley.

3. The pinion (driver) makes 174 R. P. M. and follower makes 24 R. P. M.; how many teeth must the pinion have if the follower has 87 teeth? Ans. 12 teeth.

4. If an engine fly-wheel is 66 inches in diameter and makes 160 R. P. M., what must be the diameter of the pulley on the main shaft to make 128 R. P. M.? Ans. 82½ in.

5. What is the pitch diameter of a gear whose pitch is 1¼ inches and has 28 teeth? Ans. 11.14 in.

6. How many teeth are there in a gear whose pitch is .7854 inch and which is 23 inches in diameter? Ans. 92 teeth.

7. What is the pitch of a gear whose diameter is 20.372 inches and which has 128 teeth? Ans. ½ in.

8. In a train of gears the drivers have 16, 30, 24, and 18 teeth, respectively; the followers have 12, 24, 36, and 40 teeth, respectively. If the first driver makes 80 R. P. M., how many R. P. M. will the last follower make? Ans. 40 R. P. M.

9. What horsepower can be safely transmitted by a gear whose pitch is 2½", pitch diameter 44.66", and which makes 80 R. P. M.?

Ans. 42.24 H. P.

THE INCLINED PLANE AND WEDGE.

1884. An **inclined plane** is a slope, or a flat surface, making an angle with a horizontal line.

Three cases may arise in practice with the inclined plane:

1. When the power acts parallel to the plane, as in Fig. 621.

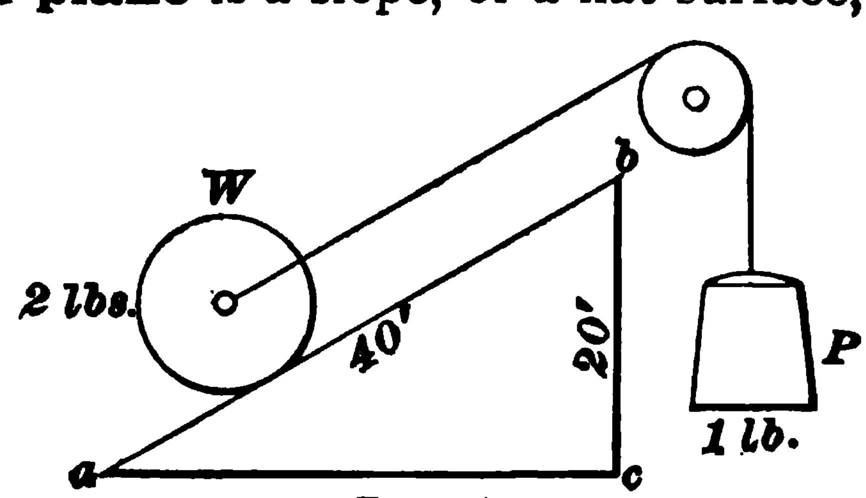


FIG. 621.

2. When the power acts parallel to the base, as in Fig. 622.

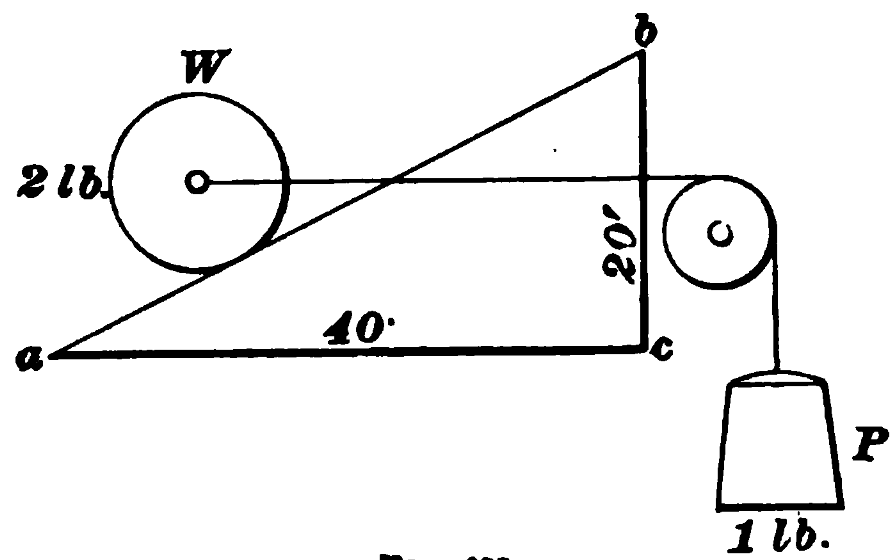


FIG. 622.

3. When the power acts at an angle to the plane, or to the base, as in Fig. 623.

1885. In Fig. 621, the relation existing between the power and the weight is easily found. The weight ascends a distance equal to $c\ b$, or the height of the inclined plane, while the power descends through a distance equal to $a\ b$, or the length of the inclined plane. Therefore, the *power multiplied by the length of the inclined plane equals the weight multiplied by the height of the inclined plane.* Hence, if the length $a\ b = 40$ feet, and the height $c\ b = 20$ feet, $W \times 20 = P \times 40$, or 1 pound at P will balance 2 pounds at W.

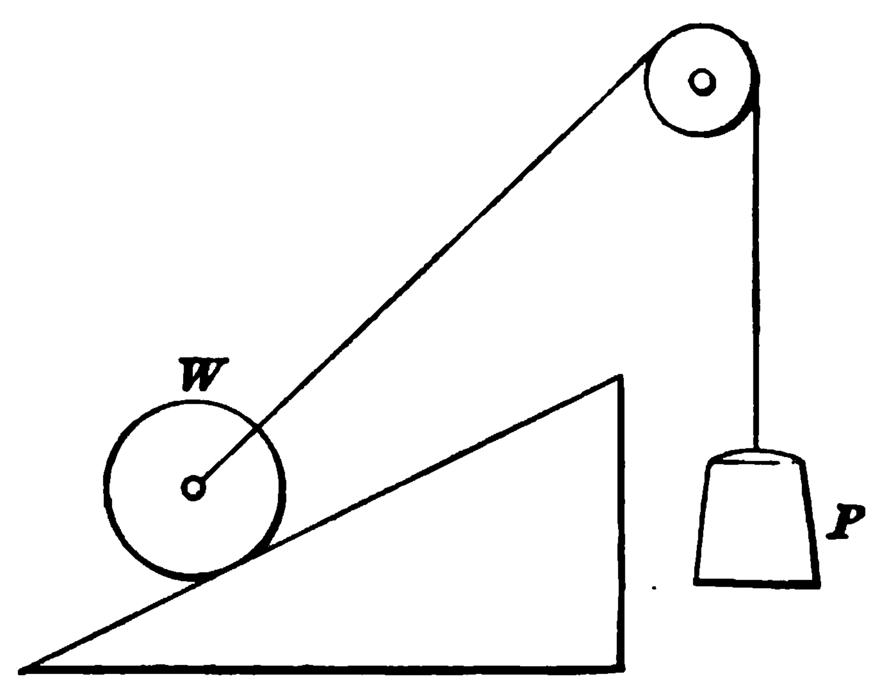


FIG. 623.

In Fig. 622, the power is supposed to act parallel to the base, for any position of W; therefore, while W is moving from the level $a\ c$ to b, or through the height $c\ b$ of the inclined plane, P will move through a distance equal to the length of the base $a\ c$. Hence, when the power acts parallel to the base, $W \times$ *height of the inclined plane* $= P \times$ *length of base.*

If the length of the base is 40 feet, and the height of the inclined plane is 20 feet, $W \times 20 = P \times 40$, and 1 pound at P will balance 2 pounds at W.

For Fig. 623 no rule can be given. The ratio of the power to the weight must be determined by trigonometry for every position of W.

1886. The **wedge** is a movable inclined plane, and is used for moving a great weight a short distance. A common method of moving a heavy body is shown in Fig. 624.

Simultaneous blows of equal force are struck on the heads of the wedges, thus raising the weight *W*. The laws for wedges are the same as for Case 2 of the inclined plane.

FIG. 624.

THE SCREW.

1887. A **screw** is a cylinder with a helical groove winding around its circumference. This helix is called the *thread* of the screw. The distance that a point on the helix is drawn back or advanced in one turn of the screw is called the *pitch* of the screw.

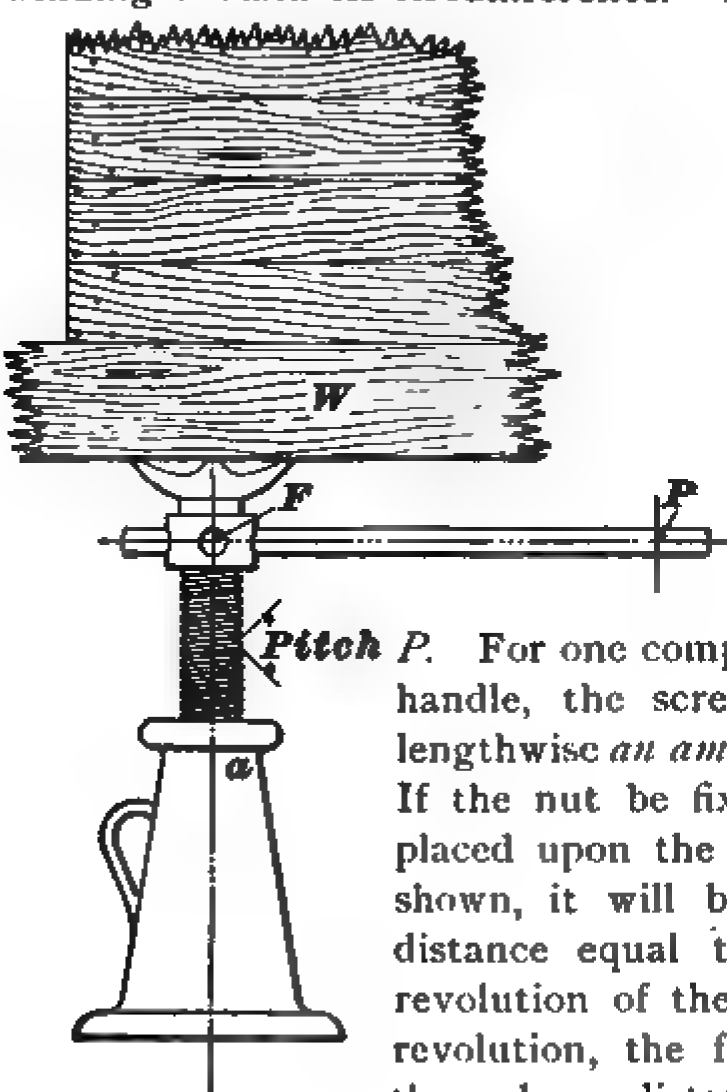


FIG. 625.

1888. The screw in Fig. 625 is turned in a *nut a*, by means of a force applied at the end of the handle *P*. For one complete revolution of the handle, the screw will be advanced lengthwise *an amount equal to the pitch.* If the nut be fixed, and a weight be placed upon the end of the screw, as shown, it will be raised vertically a distance equal to the pitch, by one revolution of the screw. During this revolution, the force at *P* will move through a distance equal to the

circumference, whose radius is PF. Hence, $W \times$ pitch of thread $= P \times$ circumference of P.

Let W = weight lifted;
P = force applied to handle;
p = pitch of screw;
R = radius of circle of force P.

Then,
$$W = \frac{6.2832\,PR}{p}. \qquad \textbf{(110.)}$$

$$P = \frac{p\,W}{6.2832\,R}. \qquad \textbf{(111.)}$$

Rule.—*Represent the required force or weight by x; multiply the force by the distance from the center of the screw to the point of the handle where the force is applied; multiply this product by 2 and by 3.1416, and place the result equal to the weight multiplied by the pitch. Divide the product of the known numbers by the number or product of the numbers by which x is multiplied, and the result will be the value of x.*

Single-threaded screws of less than 1-inch pitch are generally classified by the number of threads they have in 1 inch of their length. In such cases, *one inch divided by the number of threads equals the pitch;* thus, the pitch of a screw that has 8 threads per inch is $\frac{1}{8}$, one of 32 threads per inch is $\frac{1}{32}$, etc.

EXAMPLE.—It is desired to raise a weight by means of a screw having 5 threads per inch. The force applied is 40 pounds at a distance of 14 inches from the center of the screw; how great a weight can be raised?

SOLUTION.—The pitch is $\frac{1}{5}$ inch. Using formula **110**,

$$W = \frac{6.2832 \times 40 \times 14}{\frac{1}{5}} = 17{,}592.96 \text{ lb. Ans.}$$

1889. Velocity Ratio.—The ratio of the distance that the power moves to the distance which the weight moves on account of the movement of the power is called the **velocity ratio.**

Thus, if the power is moving 12 inches while the weight is moving 1 inch, the velocity ratio is 12 to 1, or 12; that is, P moves 12 times as fast as W.

If the velocity ratio is known, the weight which any machine can raise equals the *power multiplied by the velocity ratio.* If the velocity ratio is 8.7 to 1, or 8.7, $W = 8.7 \times P$, since $W \times 1 = P \times 8.7$.

NOTE.—In all of the preceding cases, including the last, friction has been neglected.

FRICTION.

1890. Friction is the resistance that a body meets from the surface on which it moves.

1891. The **ratio** between the *resistance* to the motion of a body due to friction and the *perpendicular* pressure between the surfaces is called the **coefficient of friction.**

If a weight W, as in Fig. 626, rests upon a horizontal plane, and has a cord fastened to it passing over a pulley a,

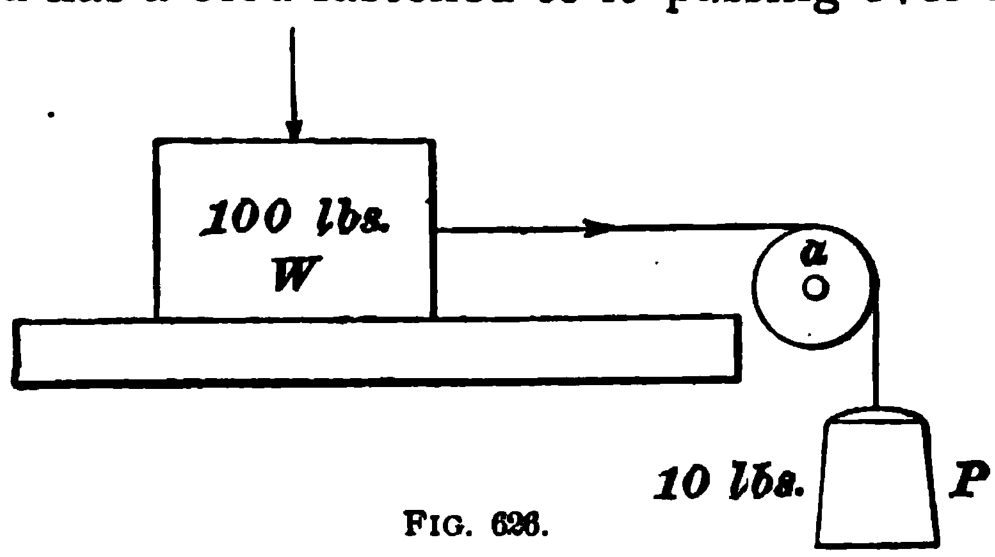


FIG. 626.

from which a weight P is just sufficient to start W, the ratio of P to W, or $\frac{P}{W}$, is the *coefficient of friction* between W and the surface it slides upon.

The weight W is the perpendicular pressure, and P is the force necessary to overcome the resistance to the motion of W due to friction.

If $W = 100$ pounds and $P = 10$ pounds, the coefficient of friction for this particular case would be $\frac{P}{W} = \frac{10}{100} = .1$.

1892. Laws of Friction:

I. *Friction is directly proportional to the perpendicular pressure between the two surfaces in contact.*

II. *Friction is independent of the extent of the surfaces in contact when the total perpendicular pressure remains the same.*

III. *Friction increases with the roughness of the surfaces.*

IV. *Friction is greater between surfaces of the same material than between those of different materials.*

V. *Friction is greatest at the beginning of motion.*

VI. *Friction is greater between soft bodies than between hard ones.*

VII. *Rolling friction is less than sliding friction.*

VIII. *Friction is diminished by polishing or lubricating the surfaces.*

1893. Law I shows why the friction is so much greater on journals after they begin to heat than before. The heat causes the journal to expand, thus increasing the pressure between the journal and its bearing, and, consequently, increasing the friction.

Law II states that no matter how small the surface may be which presses against another, if the perpendicular pressure is the same, the friction will be the same. Therefore, large surfaces are used where possible, not to reduce the friction, but to reduce the wear and diminish the liability of heating.

For instance, if the perpendicular pressure between a journal and its bearing is 10,000 pounds, and the coefficient of friction is .2, the amount of friction is $10,000 \times .2 = 2,000$ pounds. Suppose that one-half the area of the surface of the journal is 80 square inches, then the amount of friction for each square inch of bearing is $2,000 \div 80 = 25$ pounds.

If half the area of the surface had been 160 square inches, the friction would have been the same, that is, 2,000 pounds; but the friction per square inch would have been $2,000 \div 160 = 12\frac{1}{2}$ pounds, just one-half as much as before, and the wear and liability to heat would be one-half as great also.

TABLE 31.

COEFFICIENTS OF FRICTION.

Description of Surfaces in Contact.	Disposition of Fibers.	State of the Surfaces.	Coefficient of Friction
Oak on Oak	Parallel	Dry	.48
Oak on Oak	Parallel	Soaped	.16
Wrought Iron on Oak	Parallel	Dry	.62
Wrought Iron on Oak	Parallel	Soaped	.21
Cast Iron on Oak	Parallel	Dry	.49
Cast Iron on Oak	Parallel	Soaped	.19
Wrought Iron on Cast Iron	....	Slightly Unctuous	.18
Wrought Iron on Bronze	....	Slightly Unctuous	.18
Cast Iron on Cast Iron	....	Slightly Unctuous	.15

1894. The power which is required to raise a weight, or overcome an equal resistance in any machine, is thus always *greater than this weight or resistance divided by the velocity ratio of the machine.*

Thus, if there were no friction, a machine whose velocity ratio were 5 would, by an application of a force of 100 pounds, raise a weight of 500 pounds.

Now, suppose that the friction in the machine is equivalent to the application of a force of 10 pounds; then, it would take a force of 110 pounds to raise the weight of 500 pounds.

If, in the above illustration, friction were neglected, 110 pounds $\times$ 5 = 550 pounds, or the weight that 110 pounds would raise; but, owing to the frictional resistance, it only raised 500 pounds. Therefore, we have for the ratio between the two $\frac{500}{550} = .91$. That is,

$$500 : 550 :: .91 : 1.$$

1895. Efficiency.—This ratio between the weight actually raised and the power multiplied by the velocity ratio is called the **efficiency of the machine.**

For example, if the weight actually raised by a machine, say a screw, is 1,600 pounds, and the power multiplied by the velocity ratio is 2,400 pounds, the efficiency of this machine is $\frac{1,600}{2,400} = .66\frac{2}{3}$, or $66\frac{2}{3}\%$.

EXAMPLE.—In a machine having a combination of pulleys and gears, the velocity ratio of the whole is 9.75. A force of 250 pounds just lifts a weight of 1,626 pounds. What is the efficiency of the machine?

SOLUTION.—Efficiency $= \frac{1,626}{250 \times 9.75} = .6671$, or 66.71%. Ans.

1896. Since the total amount of friction varies with the load, it follows that the efficiency will also vary for different loads.

If the efficiency of a machine is known, the force actually required to raise a given load may be found by dividing the load by the product of the velocity ratio of the machine and the efficiency. Thus, if a certain machine has a velocity ratio of 10.6, and its efficiency is 60%, the force which must actually be applied to raise a load of 840 pounds is $840 \div 10.6 \times .60 = 840 \div 6.36 = 132.1$ pounds, nearly. If there had been no losses through friction, etc., the force required would have been $840 \div 10.6 = 79.25$ pounds, nearly.

If the efficiency is known, the weight which a certain force will raise may be found by multiplying together the force, velocity ratio, and the efficiency. Thus, if a certain machine has a velocity ratio of $6\frac{1}{2}$ and an efficiency of 78%, a force of 140 pounds will raise a weight of $140 \times 6\frac{1}{2} \times .78 = 709.8$ pounds.

When finding the force necessary to overcome the friction, the *perpendicular pressure* on the surface considered must always be taken. Thus, to find the greatest perpendicular pressure on the guides of a steam-engine due to the action of the piston-rod and connecting-rod on the cross-head, multiply the total piston pressure by the length of the crank, and divide by the length of the connecting-rod. This result,

multiplied by the proper coefficient of friction, will give the friction of the cross-head on the guides.

EXAMPLE.—An engine whose piston is 16 inches in diameter carries a steam pressure of 80 pounds per square inch. If the crank is 12 inches long and the connecting-rod is 66 inches long, what is the perpendicular pressure on the guides? The coefficient of friction for this case being 12%, what force will be required to overcome the friction?

SOLUTION.—Pressure on piston $= 16^2 \times .7854 \times 80 = 16{,}085$ lb. $\frac{16{,}085 \times 12}{66} = 2{,}924.55$ lb. = perpendicular pressure. Ans. $2{,}924.55 \times .12 = 350.95$ lb. = force required to overcome the friction. Ans.

EXAMPLES FOR PRACTICE.

1. How great a force must be applied to the free end of the rope of a block and tackle which has four movable pulleys, to raise a weight of 746 pounds? Ans. $93\frac{1}{4}$ lb.

2. An inclined plane is 30 feet long and 7 feet high; what force is required to roll a barrel of flour weighing 196 pounds up the plane, friction being neglected? Ans. $45.7\frac{1}{3}$ lb.

3. The distance from the axis of a screw to the point on the handle where the force is applied is twelve inches. The screw has 8 threads per inch. What force is necessary to raise a weight of 1,248 pounds? Ans. 2.07 lb., nearly.

4. In example 3, what should be the length of the handle to raise a weight of 5,400 pounds by the application of a force of 20 pounds? Ans. 5.371 inches, nearly.

5. What is the velocity ratio (*a*) in example 3? (*b*) in example 4?
Ans. { (*a*) 603, nearly. (*b*) 270.

6. An engine-piston is 24 inches in diameter. If the steam pressure is 93 pounds per square inch; the length of the connecting-rod, 8 feet 4 inches; the length of crank 20 inches, and coefficient of friction 14%, (*a*) what is the perpendicular pressure on the guides? (*b*) the force required to overcome the friction?
Ans. { (*a*) 8,414.46 lb. (*b*) 1,178 lb.

CENTRIFUGAL FORCE.

1897. If a body be fastened to a string and whirled so as to give it a circular motion, there will be a pull on the string which will be greater or less according as the velocity increases or decreases. The cause of this pull on the string will now be explained.

Suppose that the body is revolved horizontally, so that the action of gravity upon it will always be the same. According to the first law of motion, a body put in motion tends to move in a straight line unless acted upon by some other force, causing a change in the direction. When the body moves in a circle, the force that causes it to move in a circle instead of a straight line is exactly equal to the tension of the string. If the string were cut, the pulling force that draws it away from the straight line would be removed, and the body would then "fly off at a tangent;" that is, it would move in a straight line tangent to the circle, as shown in Fig. 627.

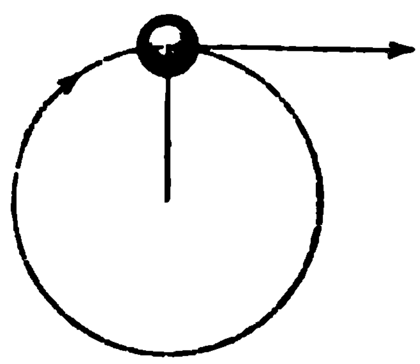


FIG. 627.

1898. Since, according to the third law of motion, every action has an equal and opposite reaction, we call that force which acts as an equal and opposite force to the pull of the string the **centrifugal force,** and it acts *away* from the center of motion.

The other force, or the pull of the string towards the center, is called the **centripetal force,** and it acts *towards* the center of motion. It is evident that these two forces, acting in opposite directions, tend to pull the string apart, and, if the velocity be increased sufficiently, the string will break. It is also evident that no body can revolve without generating centrifugal force.

The value of the centrifugal force, expressed in pounds, of any revolving body is calculated by the following rule:

Rule.—*The centrifugal force is equal to the continued product of .00034, the weight of the body in pounds, the radius in feet (taken as the distance between the center of gravity of the body and the center about which it revolves), and the square of the number of revolutions per minute.*

Let F = centrifugal force in pounds;
W = weight of revolving body in pounds;

R = radius in feet of circle described by center of gravity of revolving body;

N = revolutions per minute of revolving body.

Then, $F = .00034\ W R N^2$. **(112.)**

In calculating the centrifugal force of fly-wheels, it is the usual practice to consider the rim of the wheel only, and not take the arms and hub of the wheel into account. In this case, R would be taken as the *distance between the center of the rim and the center of the shaft.*

EXAMPLE.—A crank-pin weighing 65 pounds revolves in a circle whose radius is 21 inches. The number of revolutions is 180. What is the centrifugal force set up by the pin?

SOLUTION.— 21 in. = $1\frac{3}{4}$ ft. Using formula **112**,

$$F = .00034 \times 65 \times 1\tfrac{3}{4} \times 180^2 = 1{,}253.07 \text{ lb. Ans.}$$

SPECIFIC GRAVITY.

1899. The **specific gravity of a body** is the ratio between its weight and the weight of a like volume of water.

Since gases are so much lighter than water, it is usual to take the specific gravity of a gas as the ratio between the weight of a certain volume of the gas and the weight of the same volume of air.

EXAMPLE.—A cubic foot of cast iron weighs 450 pounds; what is its specific gravity, a cubic foot of water weighing 62.5 pounds?

SOLUTION.— $\frac{450}{62.5} = 7.2$. Ans.

1900. The specific gravities of different bodies are given in the tables of Specific Gravities; hence, if it is desired to know the weight of a body that can not be conveniently weighed, *calculate its cubical contents, and multiply the specific gravity of the body by the weight of a like volume of water, remembering that a cubic foot of water weighs 62.5 pounds.*

EXAMPLE.—How much will 3,214 cubic inches of cast iron weigh? Take its specific gravity as 7.21.

SOLUTION.—Since 1 cubic foot of water weighs 62.5 pounds, 3,214 cubic inches weigh $\frac{3,214}{1,728} \times 62.5 = 116.25$ pounds.

$$116.25 \times 7.21 = 838.16 \text{ pounds. Ans.}$$

EXAMPLE.—What is the weight of a cubic inch of cast iron?

SOLUTION.— $\frac{62.5}{1,728} \times 7.21 = .2608$ pound. Ans.

NOTE.—One cubic foot of pure distilled water at a temperature of 39.2° Fahrenheit weighs 62.42 pounds, but the value usually taken in making calculations is 62½ pounds.

EXAMPLE.—What is the weight in pounds of 7 cubic feet of oxygen?

SOLUTION.—One cubic foot of air weighs .08073 lb. (see table of Specific Gravities), and the specific gravity of oxygen is 1.1056 compared with air; hence, $.08073 \times 1.1056 \times 7 = .62479$ pound, nearly. Ans.

EXAMPLES FOR PRACTICE.

1. The balls of a steam-engine governor each weigh 5 pounds. If they revolve in a circle whose diameter is 14 inches at the rate of 80 revolutions per minute, what is the centrifugal force of each ball?
Ans. 6.347 lb., nearly.

2. If a cubic foot of a certain alloy weighs 678 pounds, what is its specific gravity? Ans. 10.848.

3. What is the weight of (*a*) 12.4 cubic inches of lead? (*b*) of steel? (*c*) of aluminum?

$$\text{Ans.} \begin{cases} (a)\ 5.0964 \text{ lb.} \\ (b)\ 3.5216 \text{ lb.} \\ (c)\ 1.116 \text{ lb.} \end{cases}$$

4. The specific gravity of an alloy of lead and zinc is 8.26; what is the weight of a cubic foot? Ans. 516.25 lb.

WORK AND ENERGY.

1901. Work *is the overcoming of resistance continually occurring along the path of motion.* Mere motion is not work, but if a body in motion constantly overcomes a resistance, it does work.

1902. *The* **measure of work** *is one pound raised vertically one foot*, and is called **one foot-pound.** All work is measured by this standard. A horse going up hill does an amount of work equal to his own weight, plus the weight of the wagon and contents, plus the frictional resistances reduced to an equivalent weight, multiplied by the vertical

height of the hill. Thus, if the horse weighs 1,200 pounds, the wagon and contents 1,200 pounds, and the frictional resistances equal 400 pounds, then, if the vertical height of the hill is 100 feet, the work done is equal to $(1,200 + 1,200 + 400) \times 100 = 280,000$ foot-pounds.

Rule.—*To find the work, multiply the force (or resistance) by the distance through which it acts. If the work consists in raising a weight, it is equal to the product of the weight multiplied by the vertical height of the lift.*

The *total* amount of work is independent of time, whether it takes one minute or one year in which to do it, but in order to compare the work done by different machines with a common standard, time must be considered. If one machine does a certain amount of work in 10 minutes, and another machine does exactly the same amount of work in 5 minutes, the second machine can do twice as much work as the first in the same time.

1903. The common standard to which all work is reduced is the *horsepower*, which is abbreviated H. P. *One horsepower is equal to 33,000 foot-pounds per minute; in other words, it is the work done in raising 33,000 pounds vertically one foot in one minute, or in raising 1 pound vertically 33,000 feet in one minute, or any combination that will, when multiplied together, give 33,000 foot-pounds in one minute.*

Thus, the work done in raising 110 pounds vertically 5 feet in one second is a horsepower, for, since in one minute there are 60 seconds, $110 \times 5 \times 60 = 33,000$ foot-pounds in one minute.

EXAMPLE.—If the coefficient of friction is .3, how many horsepower will be required to draw a load of 10,000 pounds on a level surface, a distance of one mile in one hour?

SOLUTION.— $10,000 \times .3 = 3,000$ pounds = the force necessary to overcome the resistance (resistance of the air is neglected). One mile = 5,280 feet; one hour = 60 minutes. Therefore, $\frac{5,280}{60} = 88$ feet per minute.

Work done = force multiplied by the space $= 3,000 \times 88 = 264,000$ foot-pounds per minute.

Horsepower $= \frac{264,000}{33,000} = 8$. Ans.

1904. Energy is a term used to express *the ability of an agent to do work.* Work can not be done without motion, and the work that a moving body is capable of doing in being brought to rest is called the **kinetic energy** of the body.

Kinetic energy means the actual visible energy of a body in motion. The work which a moving body is capable of doing in being brought to rest is exactly the same as the kinetic energy developed by it in falling in a vacuum through a height sufficient to give it the same velocity.

Rule.—*The kinetic energy of a moving body in foot-pounds equals its weight in pounds multiplied by the square of its velocity in feet per second, and divided by 64.32.*

Let W = weight of body in pounds;
v = velocity in feet per second;
K = kinetic energy in foot-pounds.

Then, $$K = \frac{Wv^2}{64.32}. \qquad \textbf{(113.)}$$

If a weight is raised to a certain height, a certain amount of work is done equal to the product of the weight and the vertical height. If a weight is suspended at a certain height and allowed to fall, it will do the same amount of work in foot-pounds that was required to raise the weight to the height through which it fell.

EXAMPLE.—If a body weighing 25 pounds falls from a height of 100 feet, how much work can it do?

SOLUTION.—Work $= Wh = 25 \times 100 = 2{,}500$ foot-pounds. Ans.

It requires the same amount of work or energy to stop a body in motion within a certain time as it does to give it that velocity in the same time.

EXAMPLE.—A body weighing 50 pounds has a velocity of 100 feet per second; what is its kinetic energy?

SOLUTION.—Applying formula **113**,

$$K = \frac{Wv^2}{64.32} = \frac{50 \times 100^2}{64.32} = 7{,}773.63 \text{ foot-pounds. Ans.}$$

EXAMPLE.—In the last example, how many horsepower will be required to give the body this amount of kinetic energy in 3 seconds?

SOLUTION.— 1 H. P. = 33,000 pounds raised 1 foot in 1 minute.

If 7,773.63 foot-pounds of work are done in 3 seconds, in 1 second there would be done $\frac{7,773.63}{3} = 2,591.21$ foot-pounds of work. One horsepower = 33,000 ft.-lb. per min. = 33,000 ÷ 60 = 550 ft.-lb. per sec.

The number of horsepower developed will be $\frac{2,591.21}{550} = 4.71$ H. P. Ans.

1905. Potential energy *is latent energy; it is the energy which a body at rest is capable of giving out under certain conditions.*

If a stone is suspended by a string from a high tower, it has potential energy. If the string is cut, the stone will fall to the ground, and during its fall its potential energy will change into kinetic energy, so that at the instant it strikes the ground its potential energy is wholly changed into kinetic energy.

At a point equal to one-half the height of the fall, the potential and kinetic energies are equal. At the end of the first quarter, the potential energy was $\frac{3}{4}$, and the kinetic energy $\frac{1}{4}$; at the end of the third quarter, the potential energy was $\frac{1}{4}$, and the kinetic energy $\frac{3}{4}$.

A pound of coal has a certain amount of potential energy. When the coal is burned, the potential energy is liberated and changed into kinetic energy in the form of heat. The kinetic energy of the heat changes water into steam, which thus has a certain amount of potential energy. The steam acting on the piston of an engine causes it to move through a certain space, thus overcoming a resistance, changing the potential energy of the steam into kinetic energy, and thus doing work.

Potential energy, *then, is the energy stored within a body, which may be liberated and produce motion, thus generating kinetic energy, and enabling work to be done.*

1906. The principle of **conservation of energy** teaches that energy, like matter, can never be destroyed. If a clock is put in motion, the potential energy of the spring is changed into kinetic energy of motion, which turns the wheels, thus producing friction.

The friction produces heat, which dissipates into the surrounding air, but still the energy is not destroyed—it merely exists in another form. The potential energy in coal was received from the sun in the form of heat ages ago, and has lain dormant for millions of years.

BELTS.

1907. A **belt** is a flexible connecting-band which drives a pulley by its frictional resistance to slipping at the surface of the pulley. Belts are most commonly made of leather or rubber, and united in long lengths by *cementing*, *riveting*, or *lacing*.

Belts are made *single* and *double*. A **single belt** is one composed of a single thickness of leather; a **double belt** is one composed of two thicknesses of leather cemented and riveted together the whole length of the belt.

1908. To Find the Length of a Belt.—In practice, the necessary length for a belt to pass around pulleys that are already in their position on a shaft is usually obtained by passing a tape-line around the pulleys; the stretch of the tape-line is allowed as that necessary for the belt. The lengths of open-running belts for pulleys not in position can be obtained as follows:

Rule.—*The length of a belt for open-running pulleys equals $3\frac{1}{4}$ times one-half the sum of the diameters of the pulleys plus 2 times the distance between the centers of the shafts.*

Let D = diameter of one pulley;
D_1 = diameter of other pulley;
L = distance between the centers of the shafts;
B = length of the belt.

Then, $$B = 3\tfrac{1}{4}\left(\frac{D + D_1}{2}\right) + 2L. \qquad \textbf{(114.)}$$

EXAMPLE.—The distance between the centers of two shafts is 9 feet 7 inches; the diameter of the large pulley is 36 inches, and the diameter of the small one is 14 inches; what is the necessary length of the belt?

SOLUTION.—Substituting in formula **114**, we have, since 9 feet 7 inches = 115 inches,

$$B = 3\tfrac{1}{4}\left(\frac{36 + 14}{2}\right) + 2 \times 115 = 311\tfrac{1}{4} \text{ in., or } 25 \text{ ft. } 11\tfrac{1}{4} \text{ in. Ans.}$$

1909. To find the width of a single leather belt that will transmit any given horsepower when equal pulleys are used:

Rule.—*The width of the belt in inches equals 800 times the horsepower to be transmitted divided by the speed of the belt in feet per minute.*

Let W = width of single belt in inches;
H = horsepower to be transmitted;
S = speed of belt in feet per minute.

Then, $$W = \frac{800\,H}{S}. \qquad \textbf{(115.)}$$

EXAMPLE.—What width of single leather belt is required to transmit 20 horsepower when equal pulleys are used and the speed is 1,600 feet per minute?

SOLUTION.—Substituting in formula **115**,

$$W = \frac{800 \times 20}{1,600} = 10 \text{ inches. Ans.}$$

1910. To find the number of horsepower that a single leather belt will transmit, its width and speed being given:

Rule.—*The number of horsepower equals the product of the width in inches and the speed in feet per minute divided by 800.*

Or, $$H = \frac{W S}{800}. \qquad \textbf{(116.)}$$

EXAMPLE.—If a 16-inch single leather belt is to be run at a speed of 700 feet per minute, what horsepower will it transmit?

SOLUTION.—Substituting in formula **116**, we have

$$H = \frac{16 \times 700}{800} = 14 \text{ horsepower. Ans.}$$

When the pulleys are of different diameter, the arc of contact must be considered. To find the number of degrees in the arc of contact, *multiply the length of belt in contact on the smaller pulley by 360, and divide the product by the*

circumference of the pulley, calculating the result to the nearest whole number. The quotient is the arc of contact.

Having found the arc of contact, *subtract it from 180° and multiply the result by 3. Add this last result to 800; the number thus obtained should be used instead of 800 in formulas* **115** *and* **116.**

EXAMPLE.—What should be the width of a single leather belt to transmit 25.24 horsepower at a speed of 1,500 feet per minute, the diameter of the smaller pulley being 24", and the belt having 30" of its length in contact with it?

SOLUTION.—Arc of contact $= \frac{30 \times 360}{24 \times 3.1416} = 143°$. $(180 - 143) \times 3 = 111$. $800 + 111 = 911$. Using formula **115**, and 911 instead of 800,

$$W = \frac{911 \times 25.24}{1,500} = 15.33'', \text{ say } 15\tfrac{1}{4}''. \quad \text{Ans.}$$

1911. To find the width of a double belt that will transmit the same horsepower as a given single belt, let W_1 represent the width of the double belt; then,

Rule.—*Multiply the width of a single belt that will transmit the same horsepower by $\frac{2}{3}$.*

Or, $$W_1 = \tfrac{2}{3} W. \qquad \textbf{(117.)}$$

EXAMPLE.—If a single leather belt is 15" in width and transmits 21.818 horsepower, what must be the width of a double belt to transmit the same horsepower?

SOLUTION.—Applying formula **117**,

$$W_1 = 15 \times \tfrac{2}{3} = 10 \text{ in.} = \text{width of double belt.} \quad \text{Ans.}$$

1912. Lacing Belts.—Many good methods of fastening the ends of belts are employed, but lacing is generally used, as it is flexible like the belt, and runs noiselessly over the pulleys.

When punching a belt for lacing, use an oval punch, the long diameter of the hole to be parallel with the side of the belt.

In a 3-inch belt, there should be four holes in each end, two in each row. In a 6-inch belt, seven holes, four in the row nearest the end. A 10-inch belt should have nine holes, five in the row nearest the end. The edges of the holes

should not be nearer than $\frac{3}{4}$ of an inch from the sides, and $\frac{7}{8}$ of an inch from the ends of the belt. The second row should be at least $1\frac{3}{4}$ inches from the end.

Always begin to lace from the center of the belt, and take care to get the ends exactly in line. The lacing should not be crossed on the side of the belt that runs next to the pulley. Always run the hair side of the belt next to the pulley.

EXAMPLES FOR PRACTICE.

1. How many foot-pounds of work are required to overcome for 7 minutes the friction of the cross-head of an engine which has a stroke of 4 feet and makes 160 strokes per minute, if the coefficient of friction is 8% and the average perpendicular pressure is 12,460 pounds?
Ans. 4,465,664 ft.-lb.

2. In the above example, what horsepower is required?
Ans. 19.332 H. P.

3. A cannon-ball weighing 500 pounds is fired with a velocity of 1,600 feet per second; what is its kinetic energy?
Ans. 19,900,497.5 ft.-lb.

4. An open belt drives two pulleys which are respectively 42 inches and 20 inches in diameter and 23 feet apart between their centers; what should be the length of the belt? Ans. $652\frac{1}{4}$ in., or 54 ft. $4\frac{1}{4}$ in.

5. What width of single leather belting, which has 2 feet 9 inches contact on the small pulley, is required to transmit 10 horsepower at a speed of 1,500 feet per min.? Give width to nearest half inch. Diameter of small pulley, 26 inches. Ans. 6 in.

6. What should be the width of the main belt of a steam-engine to transmit 120 horsepower? The engine runs at 80 R. P. M., the band wheel is 8 feet in diameter, the belt is double and has a contact of 6 feet on the smaller pulley, which is 5 feet in diameter. Take the speed of the belt the same as that of a point on the circumference of the band-wheel. Ans. $36\frac{1}{4}$ in.

7. A 26-inch double belt runs at a speed of 2,830 feet per min. and has a contact of 5 feet on the smaller pulley; what horsepower is it transmitting? Diameter of small pulley is 48 inches.
Ans. 121.15 H. P.

MECHANICS.

(PART 2.)

THE COMPOSITION OF FORCES.

1913. When two forces act upon a body at the same time but at different angles, their final result may be obtained as follows:

In Fig. 628, let A be the common *point of application* of the two forces, and let $A\ B$ and $A\ C$ represent the *magnitude* and *direction* of the forces. According to the second law of motion, the final effect of the movement due to these two forces would be the same, whether they acted singly or together. Suppose that the line $A\ B$ represents the distance that the force $A\ B$ would cause the body to move; similarly, that $A\ C$ represents the distance which the force $A\ C$ would cause the body to move when both forces were acting separately.

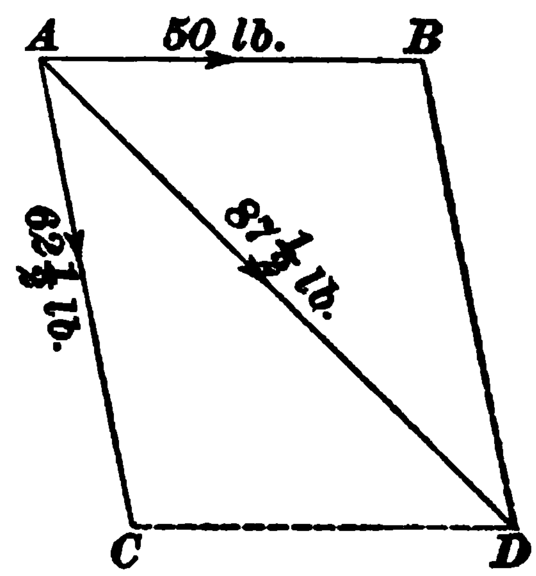


FIG. 628.

The force $A\ B$, acting alone, would carry the body to B; if the force $A\ C$ were now to act upon the body, it would carry it along the line $B\ D$, parallel to $A\ C$, to a point D, at a distance from B equal to $A\ C$. Join C and D; then, $C\ D$ is parallel to $A\ B$, and $A\ B\ D\ C$ is a parallelogram. Draw the diagonal $A\ D$. According to the second law of motion, the body will stop at D, whether the forces act separately or together, but if they act together, the path of the body will be along $A\ D$, the diagonal of the parallelogram. Moreover, the length of the line $A\ D$ represents the *magnitude* of a force, which, acting at A in the *direction* $A\ D$, would

cause the body to move from A to D; in other words, $A\ D$, measured to the same scale as $A\ B$ and $A\ C$, represents in *magnitude* and *direction* the combined effect of the two forces $A\ B$ and $A\ C$.

This line $A\ D$ is called the **resultant.** Suppose that the scale used was 50 pounds to the inch; then, if $A\ B = 50$ pounds, and $A\ C = 62\frac{1}{2}$ pounds, the length of $A\ B$ would be $\frac{50}{50} = 1$ inch, and the length of $A\ C$ would be $\frac{62.5}{50} = 1\frac{1}{4}$ inches. If $A\ D$, or the *resultant*, measures $1\frac{3}{4}$ inches, its *magnitude* would be $1\frac{3}{4} \times 50 = 87\frac{1}{2}$ pounds.

Therefore, a force of $87\frac{1}{2}$ pounds acting upon a body at A in the direction $A\ D$ will produce the same result as the combined effects of a force of 50 pounds acting in the direction $A\ B$, and a force of $62\frac{1}{2}$ pounds acting in the direction $A\ C$.

1914. The above method of finding the resulting action of two forces acting upon a body at a common point is correct, whatever may be their direction and magnitudes. Hence, to find the **resultant** of two forces when their common point of application, their direction, and magnitudes are known:

Rule.—*Assume a point, and draw two lines parallel to the directions of the lines of action of the two forces. With any convenient scale, measure off from the point of intersection (common point of application) distances corresponding to the magnitudes of the respective forces, and complete the parallelogram. From the common point of application, draw the diagonal of the parallelogram; this diagonal will be the resultant, and its direction will be away from the point of application. Its magnitude should be measured with the same scale that was used to measure the two forces.*

This method is called the **graphical method of the parallelogram of forces.**

1915. The principle of the parallelogram of forces is clearly shown in Fig. 629. $A\ B\ D\ C$ is a wooden frame, jointed to allow motion at its four corners. The length

$A B$ equals $C D$; that of $A C$ equals $B D$, and the corresponding adjacent sides are in the ratio of 2 to 3. Cords pass over the pulleys M and N, carrying weights W and w, of 90 and 60 pounds. The ratio between the weights equals the ratio of the corresponding adjacent sides. A weight V of 120 pounds is hung from the corner A.

When the frame comes to rest, the sides $A B$ and $A C$ lie

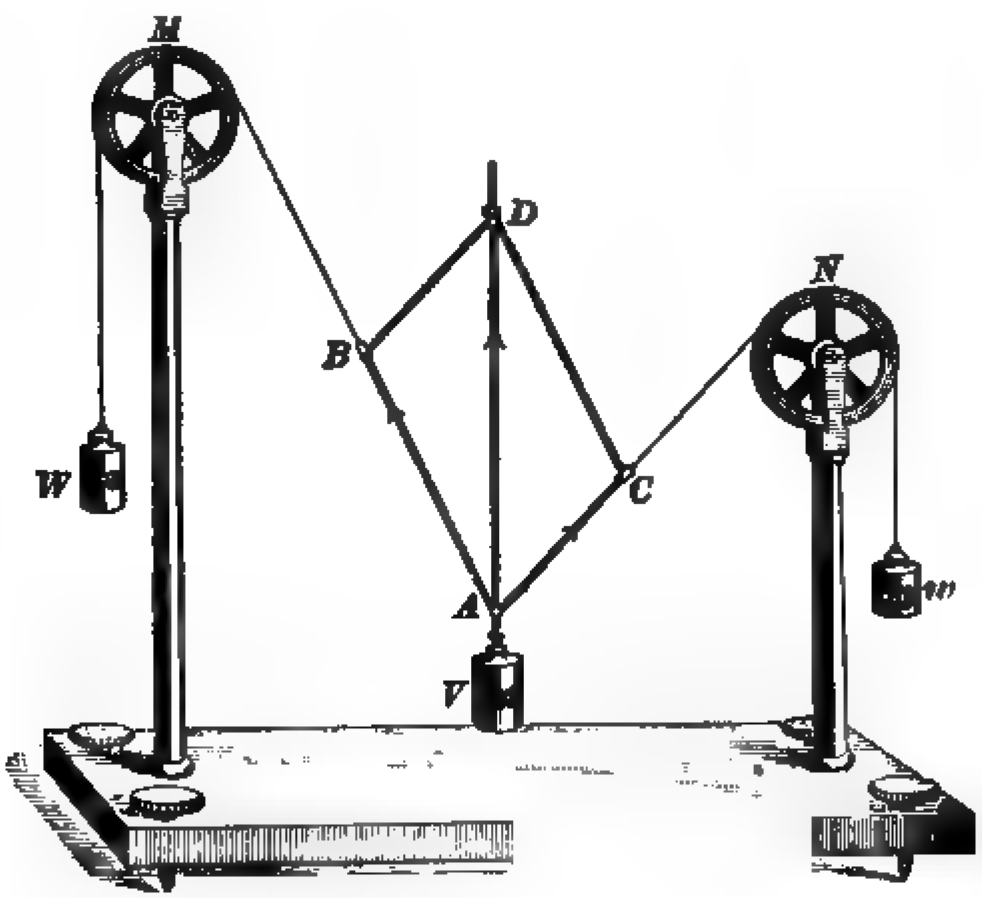


FIG. 620.

in the direction of the cords. These sides $A B$ and $A C$ are accurate graphic representations of the two forces acting upon the point A. It will be found that the diagonal $A D$ is vertical, and twice as long as $A C$; hence, since $A C$ represents a force of 60 pounds, $A D$ will represent a force of 2×60, or 120 pounds.

Thus, we see that the line $A D$ represents the *resultant* of the two forces $A B$ and $A C$; in other words, it represents the resultant of the two weights W and w. This resultant is equal and opposite to the vertical force, which is due to the weight of V.

Satisfactory results of this kind will be secured when we have the proportion

$$A B : A C = W : w.$$

EXAMPLE.—If two forces act upon a body at a common point, both acting away from the body, and the angle between them is 80°, what is the value of the resultant, the magnitude of the two forces being 60 pounds and 90 pounds respectively?

SOLUTION.—Draw two indefinite lines having an angle of 80° between them. With any convenient scale, say 10 pounds to the inch, measure off $A B = 60 \div 10 = 6$ inches, and $A C = 90 \div 10 = 9$ inches.

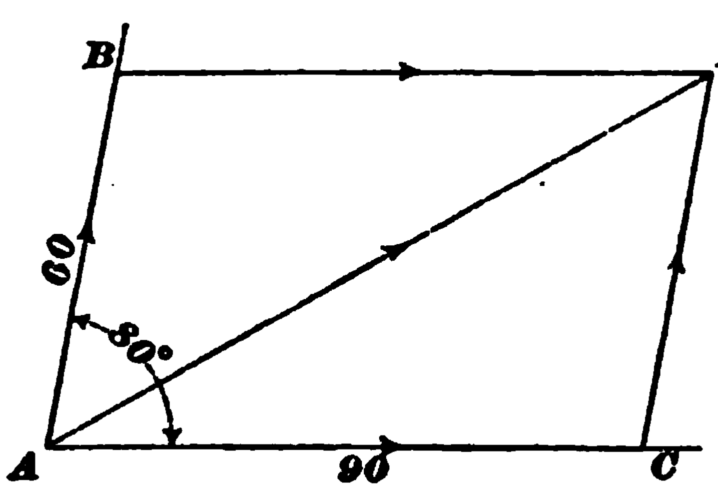


FIG. 630.

Through B, draw $B D$ parallel to $A C$, and through C, draw $C D$ parallel to $A B$, intersecting at D. Then, draw $A D$, and $A D$ will be the *resultant;* its *direction* is towards the point D, as shown by the arrow.

Measuring $A D$ we find that its length = 11.7 inches. Hence, $11.7 \times 10 = 117$ pounds. Ans.

CAUTION.—In solving problems by the graphical method, *use as large a scale as possible.* More accurate results are then obtained.

1916. The above example might also have been solved by the method called the **triangle of forces,** which is as follows:

In Fig. 630, suppose that the two forces acted separately, first from A to B, and then from B to D, in the direction of the arrows.

Draw $A D$; then $A D$ is the *resultant* of the forces $A B$ and $A C$, since $B D = A C$; but $A D$ is a side of the triangle $A B D$. It will also be noticed that the direction of $A D$ is *opposed* to that of $A B$ and $B D$; hence, to find the **resultant** of two forces acting upon a body at a common point, by the method of triangle of forces:

Rule.—*Draw the lines of action of the two forces as if each force acted separately, the lengths of the lines being proportional to the magnitude of the forces. Join the extremities of the two lines by a straight line, and it will*

be the resultant; its direction will be opposite to that of the two forces.

NOTE.—When we speak of the resultant being opposed in direction to the other forces around the polygon, we mean that, starting from the point where we began to draw the polygon, and tracing each line in succession, the pencil will have the same general direction around the polygon as if passing around a circle, from left to right, or from right to left, but that the closing line or resultant must have an *opposite direction*; that is, *the two arrow-heads must point towards the point of intersection of the resultant and the last side.*

1917. EXAMPLE.—Suppose the center of a headwheel is elevated 100 feet above the center of a hoisting-drum, as shown in Fig. 631. The rope from the headwheel to the hoisting-drum makes an angle of 30° with a vertical line, and the weight of the carriage and the load to be hoisted is 5 tons. (1) What force will there be on the shaft of the headwheel? (2) In what direction will the resultant force act, or what would be the direction in which the headwheel would be thrown if its shaft should break?

SOLUTION.—In Fig. 631, *A B C* represents the rope and its direction, with one end fastened to load *C*. The other end is passed over headwheel *B*, and wound around drum *A*. Now, as the rope is held in position by drum *A*, the tension at any point is equal to load *C*. Consequently, there is a force of 5 tons acting in the direction from *B* to *A*, as indicated by the arrow, and a like force acting in the direction from *B* to *C*, as indicated by the arrow. *B C* is assumed to be vertical. If we produce the lines *A B* and *C B* to *d*, *d* is the point of application; thus, we have the point of application, magnitude, and direction of the acting forces. Now, if we use a scale 1 inch = 1 ton, and lay off from *d*, the point of application, five inches or divisions on each component, as *d* to *1'*, *1'* to *2'*, *2'* to *3'*, *3'* to *4'*, *4'* to *5'*, and *d* to *1*, *1* to *2*, *2* to *3*, *3* to *4*, *4* to *5*, each inch or division represents one ton, and, consequently, the five inches

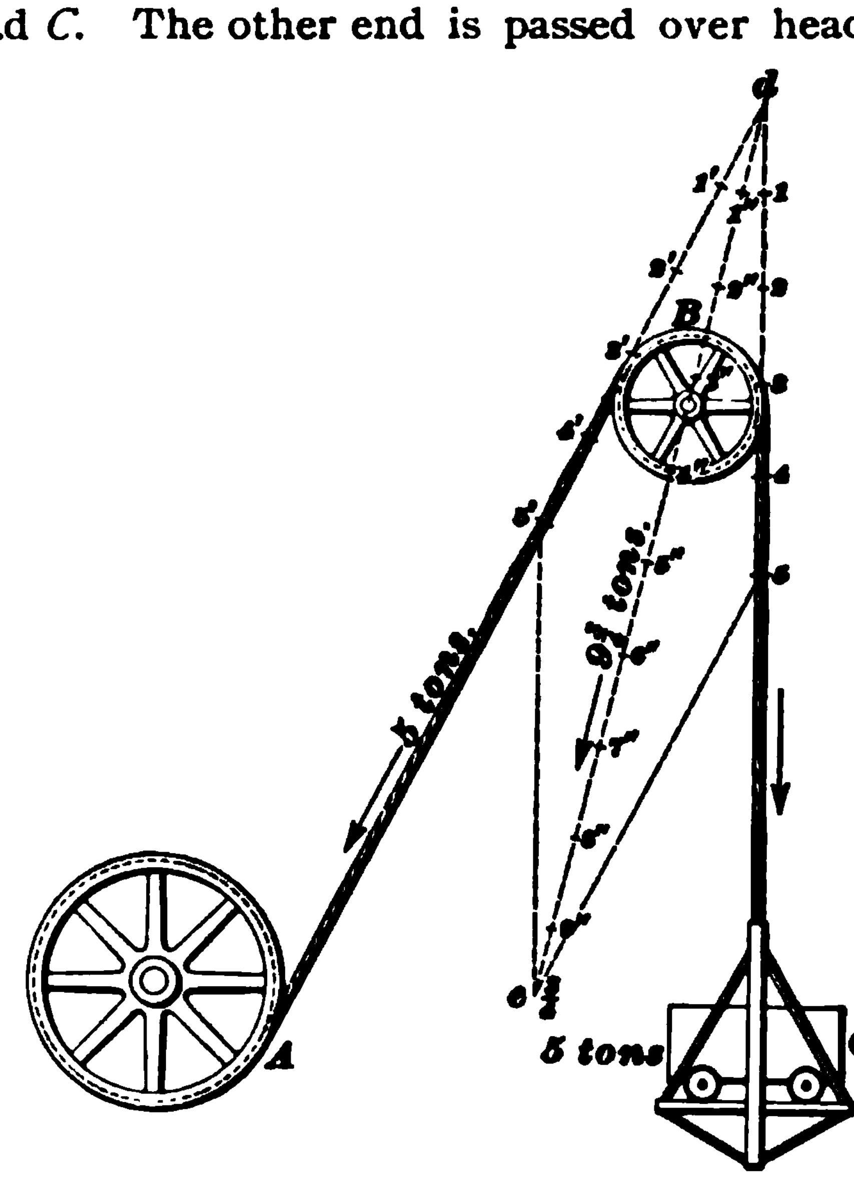


FIG. 631.

or divisions represent five tons, or the total force of each component. Then, by completing the parallelogram $d\,5'\,e\,5$, by drawing line $5'\,e$ parallel to $C\,B\,d$, and line $5\,e$ parallel to $A\,B\,d$, we have only to find how many times the resultant $d\,e$ contains the distance $d\,1$. If the resultant contains $d\,1$ seven times, then there is a force of 7 tons on the shaft B, acting in the direction $d\,e$, or if it contains $d\,1$ ten times, then there is a force of 10 tons on the shaft B, and so on. Consequently, there is one ton for each division we get on the line $d\,e$. Fig. 631 shows 9¾ such divisions; consequently, there are 9¾ tons on the shaft B, acting in the direction $d\,e$. The above discussion supposes the parts to be at rest.

1918. When three or more forces act upon a body at a given point, their *resultant* may be found by the following rule:

Rule.—*Find the resultant of any two forces; treat this resultant as a single force, and combine it with a third force to find a second resultant. Combine this second resultant with a fourth force, to find a third resultant, etc. After all the forces have been thus combined, the last resultant will be the resultant of all the forces, both in magnitude and direction.*

EXAMPLE.—Find the resultant of all the forces acting on the point O in Fig. 632, the length of the lines being proportional to the magnitude of the forces.

SOLUTION.—Draw $O\,E$ parallel and equal to $A\,O$, and $E\,F$ parallel and equal to $B\,O$; then, $O\,F$ is the resultant of these two forces, and its

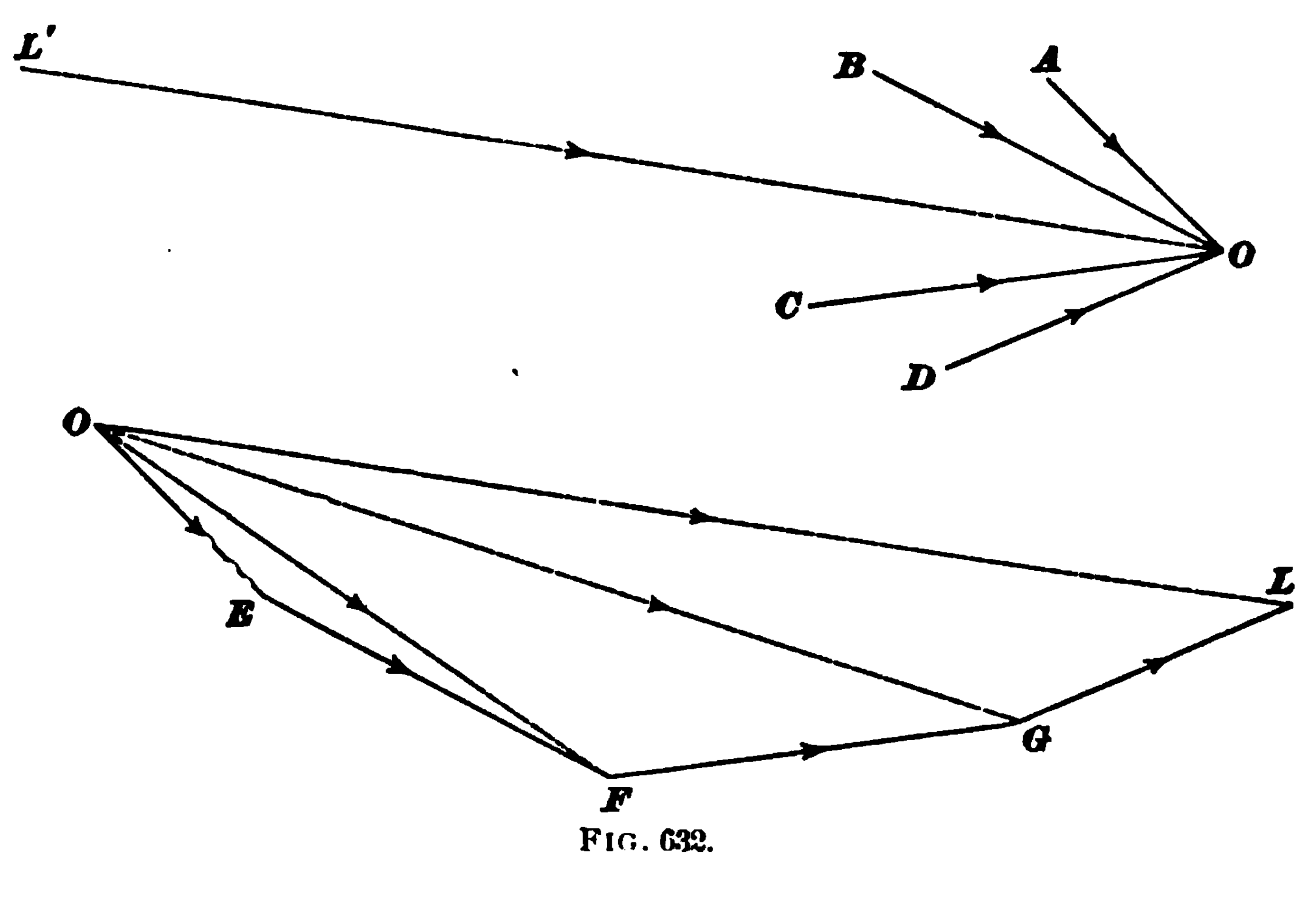


FIG. 632.

direction is from O to F, opposed to $O\,E$ and $E\,F$. Treat $O\,F$ as if $O\,E$ and $E\,F$ did not exist, and draw FG parallel and equal to $C\,O$; $O\,G$ will be the resultant of $O\,F$ and FG; but $O\,F$ is the resultant of $O\,E$ and $E\,F$; hence, $O\,G$ is the resultant of $O\,E$, $E\,F$, and FG, and likewise of $A\,O$, $B\,O$, and $C\,O$. The line FG parallel to $C\,O$ could not be drawn from the point O to the right of $O\,E$, for in that case it would be opposed in direction to $O\,F$; but FG must have the same direction as $O\,F$, in order that the resultant may be opposed to both $O\,F$ and FG.

For the same reason, draw $G\,L$ parallel and equal to $D\,O$. Join O and L, and $O\,L$ will be the ***resultant*** of all the forces $A\,O$, $B\,O$, $C\,O$, and $D\,O$ (both in magnitude and direction), acting at the point O. If $L'O$ were drawn parallel and equal to $O\,L$, and having the same direction, it would represent the effect produced on the body by the combined action of the forces $A\,O$, $B\,O$, $C\,O$, and $D\,O$.

In Fig. 632, it will be noticed that $O\,E$, $E\,F$, $F\,G$, $G\,L$, and $L\,O$ are sides of a polygon $O\,E\,F\,G\,L$, in which $O\,L$, the resultant, is the closing side, and that its direction is opposed to that of all the other sides. This fact is made use of in what is called the **method of the polygon of forces.**

1919. To find the resultant of several forces acting upon a body at the same point:

Rule.—*Through a convenient point on the drawing, draw a line parallel to one of the forces, and having the same direction and magnitude. At the end of this line, draw another line parallel to a second force, and having the same direction and magnitude as this second force; at the end of the second line, draw a line parallel and equal in magnitude and direction to a third force. Thus continue until lines have been drawn parallel and equal in magnitude and direction to all of the forces.*

The straight line joining the free ends of the first and last lines will be the closing sides of the polygon; mark it opposite in direction to that of the other forces around the polygon, and it will be the resultant of all the forces.

Example.—If five forces act upon a body at angles of 60°, 120°, 180°, 240°, and 270°, towards the same point, and their respective magnitudes

are 60, 40, 30, 25, and 20 pounds, find the magnitude and direction of their resultant by the method of polygon of forces.*

SOLUTION.—From a common point O, Fig. 633, draw the lines of action of the forces, making the given angles with a horizontal line through O, and mark them as acting towards O, by means of arrowheads, as shown. Now, choose some convenient scale, such that the

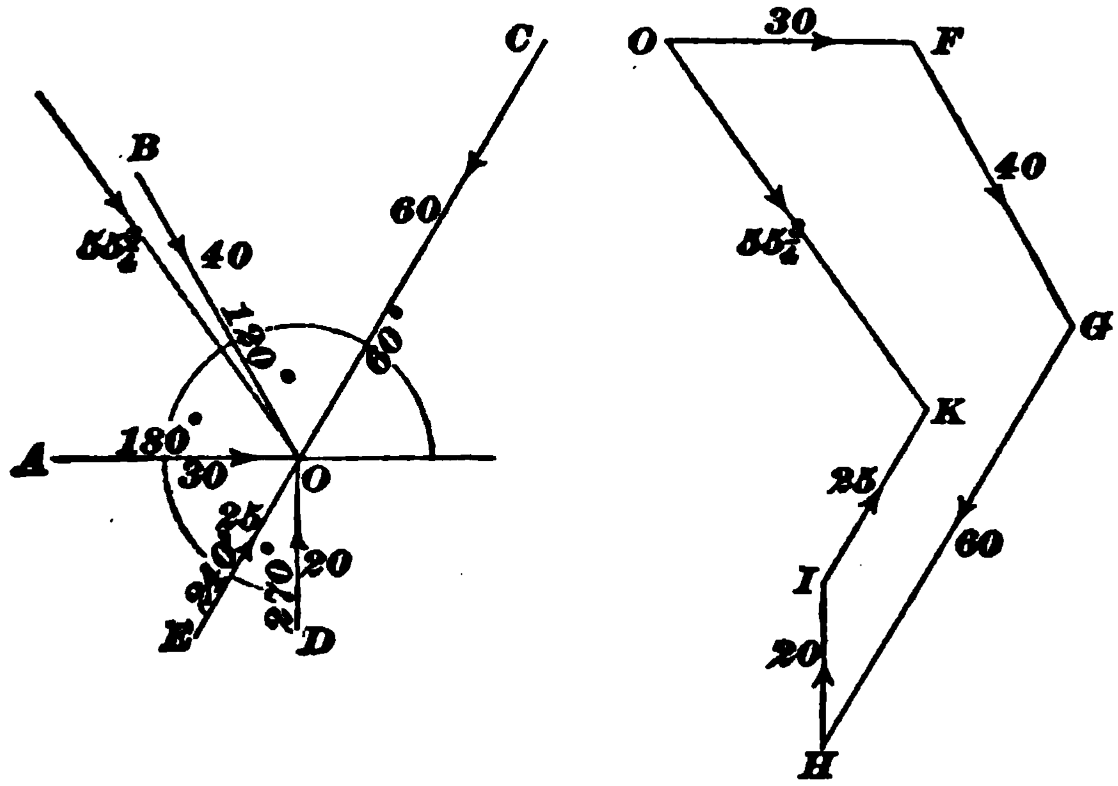


FIG. 633.

whole figure may be drawn in a space of the required size on the drawing. Choose any one of the forces, as $A\,O$, and draw $O\,F$ parallel to it, and equal in length to 30 pounds on the scale. It must also act in the same direction as $A\,O$. At F, draw $F\,G$ parallel to $B\,O$, and equal to 40 pounds. In a similar manner, draw $G\,H$, $H\,I$, and $I\,K$ parallel to $C\,O$, $D\,O$, and $E\,O$, and equal to 60, 20, and 25 pounds, respectively. Join O and K by $O\,K$, and $O\,K$ will be the resultant of the combined action of the five forces; its direction is opposite to that of the other forces around the polygon $O\,F\,G\,H\,I\,K$, and its magnitude = 55¾ pounds. Ans.

1920. If the resultant $O\,K$, in Fig. 633, were to act alone upon the body in the direction shown by the arrowhead with a force of 55¾ pounds, it would produce exactly the same effect upon a body as the combined action of the five forces.

If $O\,F$, $F\,G$, $G\,H$, $H\,I$, and $I\,K$ represent the distances and directions that the forces would move the body, if acting

* NOTE.—As stated in Trigonometry, all angles are measured from a horizontal line in a direction opposite to the movement of the hands of a watch (from around the circle to the left), from 1° or less, up to 360°.

separately, $O K$ is the direction and distance of movement of the body when all the forces act together.

From what has been said before, it is seen that any number of forces acting on the body at the same point, or having their lines of action pass through the same point, can be replaced by a *single force* (resultant) whose line of action shall pass through that point.

1921. Heretofore it has been assumed that the forces acted upon a single point on the *surface* of the body, but it

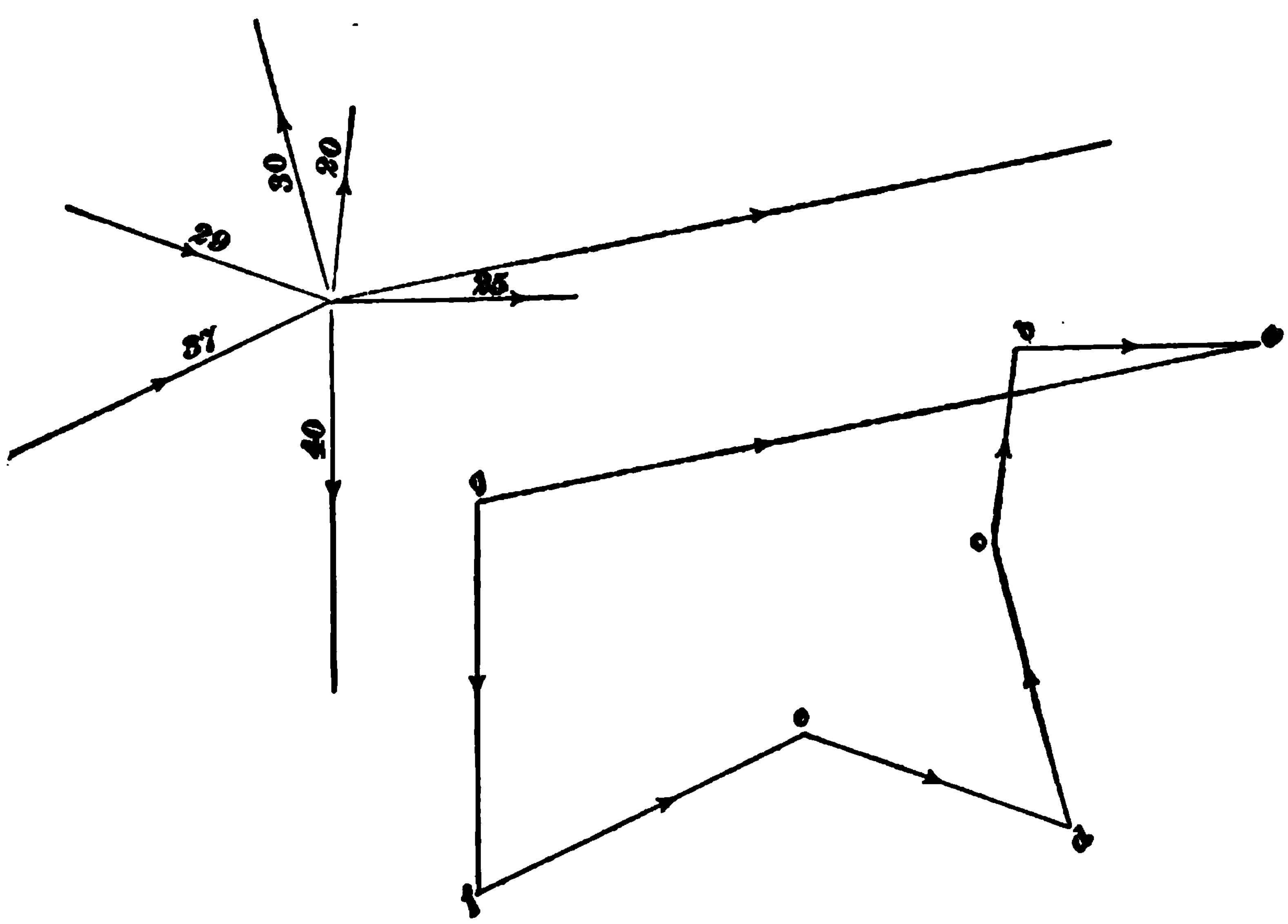


FIG. 634.

will make no difference where they act, so long as the lines of action of all the forces intersect at a *single point* either within or without the body, only so that the resultant can be drawn through the *point of intersection*. If two forces act upon a body in the same straight line and in the same direction, their *resultant* is the *sum of the two forces*; but if they act in opposite directions, their *resultant* is the *difference of the two forces*, and its direction is the same as that of the

greater force. If they are equal and opposite, the ***resultant*** is *zero*, or one force just balances the other.

EXAMPLE.—Find the resultant of the forces whose lines of action pass through a single point, as shown in Fig. 634.

SOLUTION.—Take any convenient point *g*, and draw a line *g f*, parallel to one of the forces, say the one marked 40, making it equal in length to 40 pounds on the scale, and indicate its direction by the arrow-head. Take some other force—the one marked 37 will be convenient; the line *f e* represents this force. From the point *e*, draw a line parallel to some other force, say the one marked 29, and make it equal in magnitude and direction to it. So continue with the other forces, taking care that the general direction around the polygon is not changed. The last force drawn in the figure is *a b*, representing the force marked 25. Join the points *a* and *g*; then, *a g* is the resultant of all the forces shown in the figure. Its direction is from *g* to *a*, opposed to the general direction of the others around the polygon. It does not matter in what order the different forces are taken, the resultant will be the same in magnitude and direction, if the work is done correctly.

THE RESOLUTION OF FORCES.

1922. Since two forces can be combined to form a single resultant force, we may also treat a single force as if it was the resultant of two forces, whose action upon a body will be the same as that of a single force. Thus, in Fig. 635, the force *O A* may be resolved into two forces, *O B* and *B A*, whose directions are opposed to *O A*.

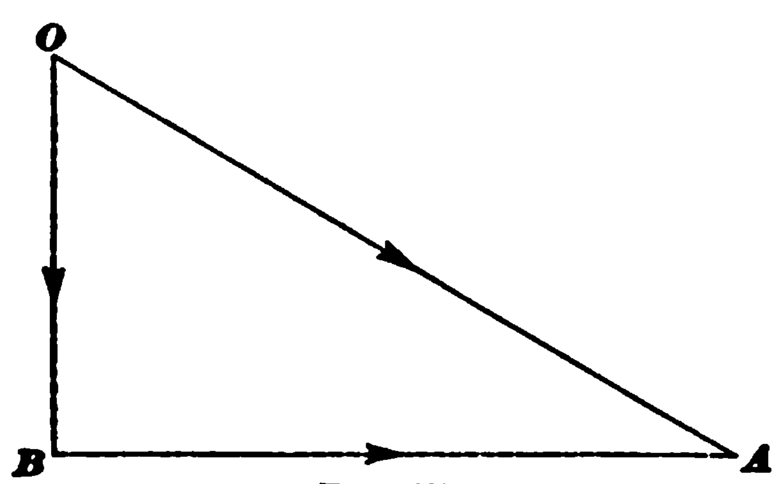


FIG. 635.

If the force *O A* acts upon a body, moving or at rest upon a horizontal plane, and the resolved force *O B* is vertical, and *B A* horizontal, *O B*, measured to the same scale as *O A*, is the magnitude of that part of *O A* which pushes the body *downwards*, while *B A* is the magnitude of that part of the force *O A* which is exerted in pushing the body in a *horizontal* direction. *O B* and *B A* are called the **components** of the force *O A*, and when these components

are vertical and horizontal, as in the present case, they are called the *vertical component* and the *horizontal component* of the force $O\ A$.

1923. It frequently happens that the position, magnitude, and direction of a certain force is known, and that it is desired to know the effect of the force in some direction other than that in which it acts. Thus, in Fig. 635, suppose that $O\ A$ represents, to some scale, the magnitude, direction, and line of action of a force acting upon a body at A, and that it is desired to know what effect $O\ A$ produces in the direction $B\ A$. Now, $B\ A$, instead of being horizontal, as in the cut, may have any direction. To find the value of the component of $O\ A$ which acts in the direction $B\ A$, we employ the following rule:

Rule.—*From one extremity of the line representing the given force, draw a line parallel to the direction in which it is desired that the component shall act; from the other extremity of the given force, draw a line perpendicular to the component first drawn, and intersecting it. The length of the component, measured from the point of intersection to the intersection of the component with the given force, will be the magnitude of the effect produced by the given force in the required direction.*

Thus, suppose $O\ A$, Fig. 635, represents a force acting upon a body resting upon a horizontal plane, and it is desired to know what *vertical pressure* $O\ A$ produces on the body. Here the desired direction is vertical; hence, from one extremity, as O, draw $O\ B$ parallel to the desired direction (vertical in this case), and, from the other extremity, draw $A\ B$ perpendicular to $O\ B$, and intersecting $O\ B$ at B. Then $O\ B$, when measured to the same scale as $O\ A$, will be the value of the vertical pressure produced by $O\ A$.

EXAMPLE.—If a body weighing 200 pounds rests upon an inclined plane whose angle of inclination to the horizontal is 18°, what force does it exert perpendicular to the plane, and what force does it exert parallel to the plane, tending to slide downwards?

SOLUTION.—Let $A\ B\ C$, Fig. 636, be the plane, the angle A being

equal to 18°, and let W be the weight. Draw a vertical line $FD =$ 200 pounds, to represent the magnitude of the weight. Through F, draw FE parallel to AB, and through D draw DE perpendicular to EF, the two lines intersecting at E. FD is now resolved into two components, one, FE, tending to pull the weight downwards, and the other, ED, acting as a perpendicular pressure on the plane.

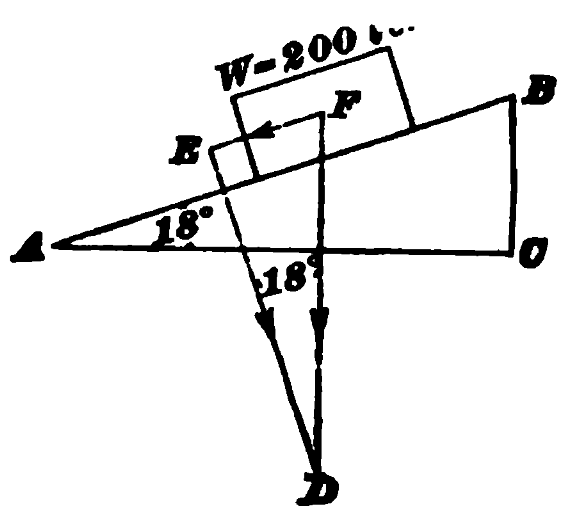


FIG. 636.

Since FD is perpendicular to AC and ED is perpendicular to AB, the angle $D =$ angle $A = 18°$.

Hence, $FE = 200 \times \sin 18° = 200 \times .30902 = 61.804$ pounds, and $ED = 200 \times \cos 18° = 200 \times .95106 = 190.212$ pounds.

Force parallel to the plane = 61.804 pounds.
Force perpendicular to the plane = 190.212 pounds. } Ans.

1924. EXAMPLE.—In Fig. 637, a body W is shown resting on an inclined plane AB, whose dimensions are marked on the cut; the weight P acts to pull the body up the plane by means of the rope r and pulley p. It is required to find what the weight of P must be in order to start W up the plane. Suppose W weighs 120 pounds, and that friction is neglected. It is also required to find the perpendicular pressure which W exerts against the plane.

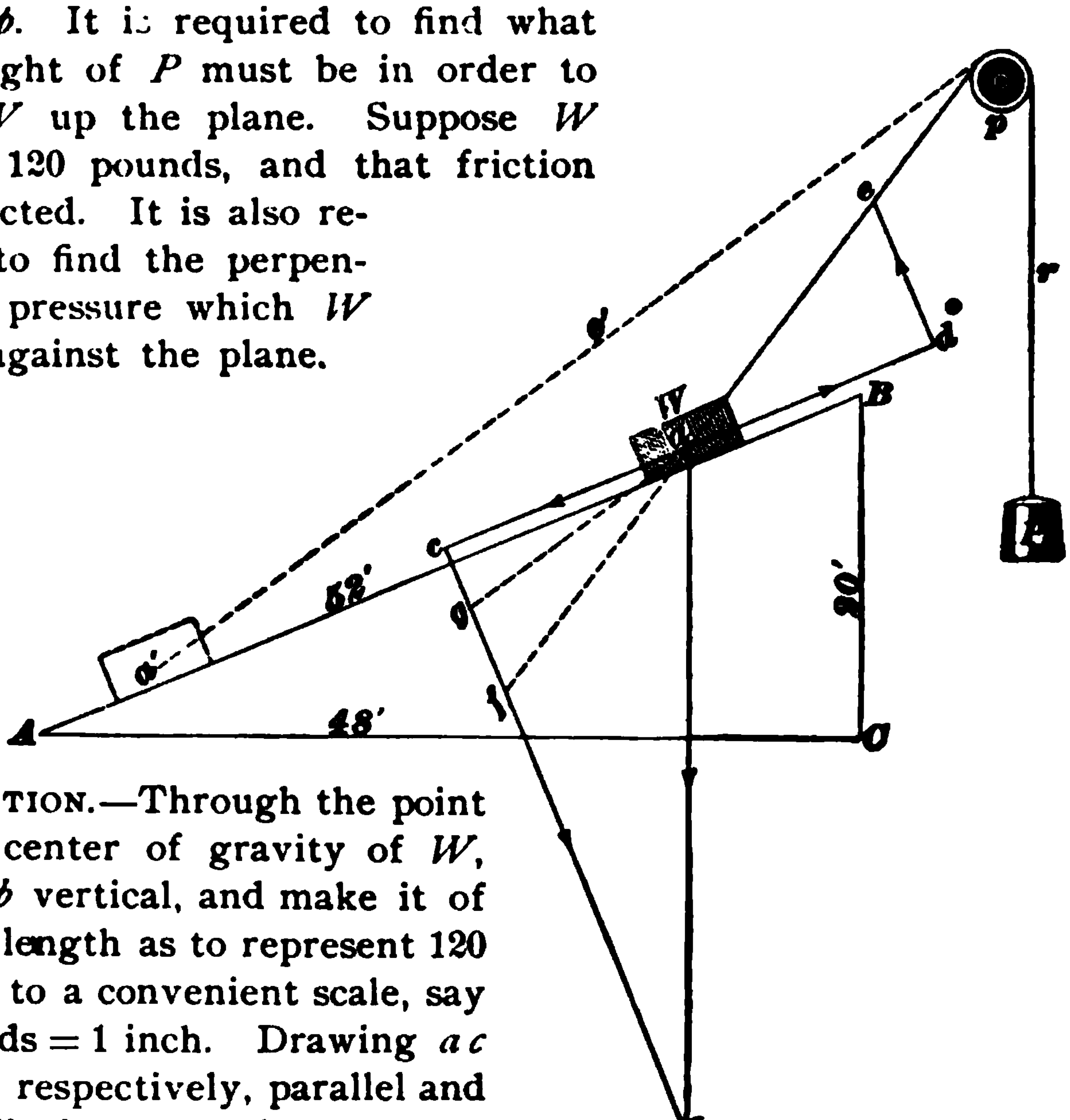


FIG. 637.

SOLUTION.—Through the point a, the center of gravity of W, draw ab vertical, and make it of such a length as to represent 120 pounds to a convenient scale, say 60 pounds = 1 inch. Drawing ac and cb, respectively, parallel and perpendicular to the plane, ac represents the magnitude of the force

which must be exerted parallel to AB in order to put the body in equilibrium, i. e., to balance the force which gravity exerts in pulling the body down the plane. If the rope r were parallel to AB, ac would represent the weight of P; but, since r makes an angle with the plane, P will not be equal to ac. To find what the weight of P must be, draw ad parallel to ac, but indicate it as acting in the opposite direction, or from a to d instead of from a to c. Now treat ad as though it were a *component* of the force acting in the rope; i. e., draw de perpendicular to ad, instead of perpendicular to ae. The reason for this is that if de were drawn perpendicular to ae, it could be resolved into components, one of which would be parallel to ad, a result which we wish to avoid; in other words, we want de perpendicular to the plane. The line ae, measured to the same scale as ab, will give the value of P. Measuring it, its length is .89 inch; hence, $P = .89 \times 60 = 53.4$ pounds. Ans.

To determine the perpendicular pressure against the plane, it will be noticed that ab equals the pressure due to gravity. Since cb and de are both perpendicular to AB, they are parallel, and since de acts in the opposite direction to cb, the actual pressure against the plane is given by the difference between cb and de. Making cf equal to de, fb represents the perpendicular pressure against the plane when the force P $(= ae)$ acts as shown. The length of fb is 1.39 inches; hence, the perpendicular pressure is $1.39 \times 60 = 83.4$ pounds. Ans.

Since ca and ad are parallel and equal, and cf and de are also parallel and equal, it follows that af and ae must also be parallel and equal. Consequently, the force P might have been found by drawing af parallel to the direction in which the pull on the rope acts, and bf perpendicular to the plane AB. Thus, suppose that the weight occupies the position shown by the dotted lines. Then, drawing ag parallel to $a'e'$, ag represents the weight of P, and gb represents the perpendicular pressure of the body W against the plane. Measuring ag, its length is .79 inch; hence, $P = .79 \times 60 = 47.4$ pounds. Measuring gb, its length is 1.65 inches; hence, the perpendicular pressure $= 1.65 \times 60 = 99$ pounds.

1925. The results obtained by the graphic method can be obtained by trigonometry when the inclination of the plane and the angle the rope makes with the plane for any position of the weight W_1 are given.

Thus, $ac = ab \times \sin abc = 120 \times \frac{20}{52} = 46.1538$ pounds.

Assuming the weight w to be in such a position that the rope r makes an angle of $30° \ 12'$ with the inclined plane, and

since in the triangle $a\,d\,e$ the side $a\,d$ equals the side $c\,a$ in the triangle $a\,b\,c$, we have

$$a\,e = \frac{a\,d}{\cos e\,a\,d} = \frac{46.1538}{.86427} = 53.4 \text{ pounds.} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

1. The current in a river which is ½ mile wide has a velocity of 3¾ miles an hour. (*a*) What will be the actual distance that a boat will pass over in crossing the river, if the boat is rowed at the rate of 5 miles an hour? (*b*) How far down the river will the boat have been carried when it reaches the other side? (*c*) What time will the boat require to cross the river?

Ans. (*a*) ⅝ mi. (*b*) ⅜ mi. (*c*) 6 min.

2. What force acting parallel to a plane whose inclination is 30° will be required to support a trip of cars whose total weight is 25 tons?

Ans. 12½ tons.

3. If a driver takes a side-hitch on a trip of cars standing on the turnout, with a mule that pulls with a force of say 400 pounds in a direction making an angle of 45° with the track, what force will tend to move the trip along the track? Ans. 282.85 lb.

4. Referring to Fig. 637, what would the angle $e\,a\,d$ become, if $P = 65.271$ pounds? Ans. 45°.

5. * The two ends of a rope 7 feet long are attached to the under side of a beam at points 5 feet apart; if a weight of one hundred pounds is firmly attached to the rope at a point 4 feet from one end, what will be the tension in each segment of the rope?

Ans. 60 lb. tension in long segment. 80 lb. tension in short segment.

6. What weight can be supported on a plane by a horizontal force of 1,521 pounds, if the ratio of the height to the base is ¾?

Ans. 2,028 lb.

7. What force is required (neglecting friction) to roll a barrel of oil weighing 300 pounds into a wagon 3 feet high by means of a plank 14 feet long resting against the wagon? Ans. 64$\frac{2}{7}$ lb.

* HINT.—To work this example by graphics, represent the weight by a vertical line drawn to scale; from one end of the line draw an indefinite line parallel to one of the segments of the rope, and from the other end of the line draw another indefinite line parallel to the other segment of the rope. These lines will intersect, and the distances from the point of intersection to the extremities of the vertical line will represent the tensions in the segments of the rope.

STRENGTH OF MATERIALS.

STRESSES AND STRAINS.

1926. When a force is applied to a body, it changes either its form or volume. A force, when considered with reference to the internal changes it tends to produce in any solid, is called a **stress.**

Thus, if a weight of 2 tons be held in suspension by a rod, the stress in the rod will be 2 tons. This stress is accompanied by a lengthening of the rod, which increases until the internal stress or resistance is in equilibrium with the external weight.

Stresses may be classified as follows:

Tensile, or pulling stress.

Compressive, or pushing stress.

Transverse, or bending stress.

Shearing, or cutting stress.

Torsional, or twisting stress.

1927. A **unit stress** is the amount of stress on a unit of area, and may be expressed either in pounds per square inch or in tons per square foot; or it is the load per square inch or square foot on any body.

Thus, if 10 tons are suspended by a wrought-iron bar which has an area of 5 square inches, the unit stress is 2 tons per square inch, because $\frac{10}{5} = 2$ tons.

1928. Strain is the deformation or change of shape of a body resulting from stress.

For example, if a rod 100 feet long is pulled in the direction of its length, and if it is lengthened 1 foot, it has a strain of $\frac{1}{100}$ of its length, or 1 per cent.

1929. Elasticity is the power which bodies have of returning to their original form after the external force on the body is withdrawn, providing the stress has not exceeded the elastic limit.

Consequently, we see from this that all material is

lengthened or shortened when subjected to either tensile or compressive stress, and the change of the length is directly proportional to the stress within the elastic limit.

For stresses within the elastic limits, materials are perfectly elastic, and return to their original length on removal of the stresses; but when their elastic limits are exceeded, the changes of their lengths are no longer reguiar, and a permanent **set** takes place; the destruction of the material has then begun.

1930. The **measure of elasticity** of any material is the change of length under stress within the elastic limit.

1931. The **elastic limit** is that unit stress under which the permanent set becomes visible.

The elasticity of wrought iron and that of steel are practically equal; that is, each material will change an equal amount of length under the same stress within the elastic limits.

The elastic limit of steel is higher than that of wrought iron; consequently, the former will lengthen or shorten more than the latter before its elasticity is injured.

TENSILE STRENGTH OF MATERIALS.

1932. The tensile strength of any material is the resistance offered by its fibers to being pulled apart.

The tensile strength of any material is proportional to the area of its cross-section.

Consequently, when it is required to find the safe tensile strength of any material, we have only to find the area at the minimum cross-section of the body, and multiply it by its strength per square inch, as given in Table 32 under the heading "Working Stress."

NOTE.—The minimum cross-section referred to in the above paragraph is that section of the material which is pierced with holes; such as bolt or rivet holes in iron, or knots in wood, if there are any.

1933. In Table 32 are given the average breaking and the working tensile stress of some materials.

The table shows that the tensile breaking strength

of cast iron is 16,000 pounds per square inch of cross-section, and that the working strength is from 1,500 to 3,500 pounds per square inch of cross-section.

TABLE 32.

Materials.	Breaking Stress in Pounds per Square Inch.	Working Stress in Pounds per Square Inch.
Timber................	10,000	600 to 1,200
Cast Iron..............	16,000	1,500 to 3,500
Wrought Iron..........	50,000	5,000 to 12,000
Steel..................	70,000	6,000 to 13,000

In machinery, such as steam-engines, where the parts are subjected to shocks, or are alternately compressed and extended, it is not safe to strain cast iron with more than 1,500 pounds per square inch of section, wrought iron with more than 5,000 pounds per square inch of section, or steel with more than 6,000 pounds per square inch of section.

But in structures in which the strains are constantly in one direction, as is the case with steam-boilers, wrought iron may be strained with from 6,000 to 8,000 pounds per square inch of section, or steel with from 8,000 to 10,000 pounds per square inch of section.

Consequently, strict attention must be given as to what working stress must be allowed for the materials of different structures.

NOTE.—For structures on which the load is applied suddenly, use the smaller working stresses given in the table, and for those on which the load is applied gradually, use the larger working stresses.

RULES AND FORMULAS FOR TENSILE STRENGTH.

1934. In the following formulas,

Let W = safe load in pounds;

A = area of minimum cross-section;

S = working stress in pounds per square inch, as given in Table 32.

Rule.—*The load in pounds on any bar subjected to a tensile strain is equal to the minimum sectional area of the bar, multiplied by the working stress in pounds per square inch, as given in Table 32.*

That is, $W = A\,S.$ **(118.)**

EXAMPLE.—A bar of good wrought iron which is 3 inches square is to be subjected to a steady tensile stress; what is the maximum load that it should carry ?

SOLUTION.—From what has been said above in regard to the materials and to the nature of the load, it will be safe in this case to use a working stress of 12,000 pounds per square inch.

Applying formula **118,** we have

$$W = 3 \times 3 \times 12{,}000 = 108{,}000 \text{ pounds. Ans.}$$

1935. Rule.—*The minimum sectional area of any bar subjected to a tensile stress is equal to the load in pounds, divided by the working stress in pounds per square inch, as given in Table 32.*

That is, $A = \dfrac{W}{S}.$ **(119.)**

EXAMPLE.—What should be the area of a wrought-iron bar to carry a steady load of 66,000 pounds, if it is to resist a tensile stress of 12,000 pounds per square inch ?

SOLUTION.—Applying formula **119,**

$$A = \frac{66{,}000}{12{,}000} = 5.5 \text{ sq. in. Ans.}$$

1936. Rule.—*The working stress in pounds per square inch is equal to the load in pounds divided by the minimum sectional area of the bar.*

That is, $S = \dfrac{W}{A}.$ **(120.)**

EXAMPLE.—A bar of wrought iron 3 inches square, subjected to tensile stress, carries a load of 86,400 pounds; what is the stress per square inch ?

SOLUTION.—Applying formula **120,**

$$S = \frac{86{,}400}{3 \times 3} = 9{,}600 \text{ lb. per sq. in. Ans.}$$

STRENGTH OF CHAINS.

1937. Chains made of the same size iron vary in strength, owing to the different kinds of links from which they are made.

It is a good practice to anneal old chains which have become brittle by overstraining. This renders them less liable to snap from sudden jerks. It reduces their tensile strength, but increases their toughness and ductility, which are sometimes more important qualities.

1938. In the following formulas,

Let W = safe load in pounds sustained by chain;
D = diameter in inches of the iron from which the links are made.

Rule.—*The safe load in pounds of a stud-link wrought-iron chain is equal to 18,000, multiplied by the square of the diameter of the iron from which the links are made.*

That is, $W = 18{,}000\ D^2$. **(121.)**

Example.—What is the maximum load that should be carried by a stud-link wrought-iron chain, if its links are made from ¾-inch round iron?

Solution.—Applying formula **121**, we have

$W = 18{,}000 \times \frac{3}{4} \times \frac{3}{4} = 10{,}125$ pounds. Ans.

1939. Rule.—*The safe load in pounds of a close-link wrought-iron chain is equal to 12,000 multiplied by the square of the diameter of the iron from which the links are made.*

That is, $W = 12{,}000\ D^2$. **(122.)**

Example.—What is the maximum load that should be carried by a close-link wrought-iron chain, if its links are made from ¾-inch round iron?

Solution.—Applying formula **122**, we have

$W = 12{,}000 \times \frac{3}{4} \times \frac{3}{4} = 6{,}750$ pounds. Ans.

STRENGTH OF HEMP ROPES.

1940. The strength of hemp ropes does not depend so much upon the quality of the material and the cross-section of the rope as upon the method of manufacture and the amount of twisting.

The ropes in common use are three-strand shroud-laid rope, and hawser or cable-laid rope.

The strongest ropes are three-strand shroud-laid made without tar. Ropes made with tar are less flexible, and are reduced in strength about 25 per cent., but have better wearing qualities.

1941. In the following formulas,

Let W = maximum working load in pounds;

C = circumference of rope in inches.

Rule.—*The maximum working load in pounds that should be allowed on any hemp rope is equal to the square of the circumference of the rope multiplied by 100.*

That is, $W = 100\ C^2$. **(123.)**

Example.—What is the maximum load in pounds that should be carried by a hemp rope which has a circumference of 8 inches?

Solution.—Substituting the value of C in formula **123**,

$$W = 100 \times 8^2 = 6{,}400 \text{ lb. Ans.}$$

1942. Rule.—*The circumference of any hemp rope is equal to the square root of the maximum working load in pounds which it is capable of carrying, multiplied by .1.*

That is, $C = .1\sqrt{W}$. **(124.)**

Example.—A maximum working load of 1,000 pounds is to be carried by a hemp rope; what should be the circumference of the rope?

Solution.—Applying formula **124**,

$$C = .1\sqrt{1{,}000} = 3.16 \text{ inches. Ans.}$$

When measuring ropes, the circumference is sought instead of the diameter, because the ropes are not round and the circumference is not 3.1416 times the diameter. For three strands, the circumference is about 2.86 d; for seven strands, 3 d.

STRENGTH OF WIRE ROPES.

1943. Wire rope is made of iron and steel wire. It is stronger than hemp rope, and, to carry the same load, is of smaller diameter.

In substituting steel for iron rope, the object in view

should be to gain an increase of wear from the rope, rather than to reduce the size.

A steel rope to be serviceable should be of the best obtainable quality, because ropes made from low grades of steel are inferior to good iron ropes.

1944. In the following formulas,

Let W = maximum working load in pounds;

C = circumference of rope in inches.

Rule.—*The maximum working load in pounds that should be allowed on any wire rope is equal to the square of the circumference of the rope in inches, multiplied by 600.*

That is, $$W = 600\ C^2. \qquad \textbf{(125.)}$$

EXAMPLE.—What is the maximum load in pounds that should be carried by an iron wire rope whose circumference is $4\frac{1}{2}$ inches?

SOLUTION.—Applying formula **125**,

$$W = 600 \times 4.5^2 = 12,150 \text{ lb.} \quad \text{Ans.}$$

1945. Rule.—*The circumference of any iron wire rope in inches is equal to the square root of the maximum working load in pounds multiplied by .0408.*

That is, $$C = .0408\sqrt{W}. \qquad \textbf{(126.)}$$

EXAMPLE.—A maximum working load of 12,150 pounds is to be carried by an iron wire rope; what should be the minimum circumference of the rope?

SOLUTION.—Applying formula **126**,

$$C = .0408\sqrt{12,150} = 4\tfrac{1}{2} \text{ inches.} \quad \text{Ans.}$$

1946. Rule.—*The above rules and formulas are also applicable when computing the safe strength of steel wire rope by substituting the constant 1,000 for the constant 600, and .0316 for .0408.*

EXAMPLE.—What is the maximum load in pounds that should be carried by a steel wire rope, the circumference of which is $4\frac{1}{2}$ inches?

SOLUTION.—Applying formula **125** as modified by the rule,

$$W = 1,000 \times 4.5^2 = 20,250 \text{ lb.} \quad \text{Ans.}$$

EXAMPLE.—A maximum working load of 10,485 pounds is to be

carried by a steel wire rope; what should be the minimum circumference of the rope?

SOLUTION.—Applying formula **126** as modified by the rule,

$$C = .0316\sqrt{10{,}485} = 3.24 \text{ inches. Ans.}$$

EXAMPLES FOR PRACTICE.

1. What should be the diameter of a steel piston-rod of a steam-engine to resist tension, if the piston is 19 inches in diameter and the pressure is 85 pounds per sq. in.? Ans. 2¼ in., nearly.

2. What safe load will a cast-iron bar of rectangular cross-section 7½ inches by 3½ inches support if subjected to shocks? The bar is in tension. Ans. 39,375 lb.

3. What is the stress per square inch on a piece of timber 8 inches square, which is subjected to a steady pull of 60,000 pounds? Ans. 937.5 lb. per sq. in.

4. What should be the safe load for a close-link wrought-iron chain whose links are made from ⅞-inch iron? Ans. 9,187.5 lb.

5. What safe load may a hemp rope carry whose circumference is 4 inches? Ans. 1,600 lb.

6. What should be the allowable working load for a steel wire rope whose circumference is 3¾ inches? Ans. 14,062.5 lb.

7. What should be the circumference of an iron wire rope to support a load of 20,000 pounds? Ans. 5⅝ in., nearly.

CRUSHING STRENGTH OF MATERIALS.

1947. The crushing strength of any material is the resistance offered by its fibers to being pushed together.

If a bar is long compared with its cross dimensions, any slight disturbance from uniformity will cause it to bend sideways under the compressive force, and we have, then, not only compression, but compression compounded with bending.

To obtain only compression, the length of a rod should not be more than five times greater than its least diameter, or its least thickness when it is a rectangular rod.

Experimental tests on pillars have shown that their strengths are approximately inversely proportional to the squares of their lengths. That is, if there are two pillars of the same material, having the same cross-section, but

one is twice as long as the other, the long one will sustain only about one-quarter the load of the short one.

1948. Attention should be given to the ends of pillars, as their shape has great influence upon their strength. In Fig. 638 are shown three pillars with different shaped ends.

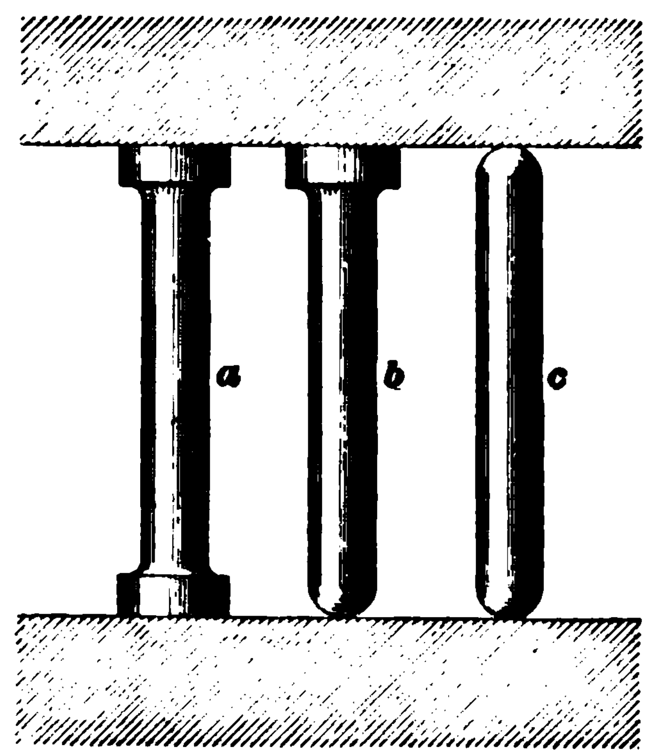


Fig. 638.

It has been proved by the aid of higher mathematics that, theoretically, a pillar having flat or fixed ends, as shown at *a*, is four times as strong as one that has round or movable ends, as shown at *c*, and one and seven-ninths times as strong as one having one flat and one round end, as shown at *b*; *b* is thus two and one-fourth times as strong as *c*. It has also been found that if three pillars, *a*, *b*, *c*, which have the same cross-section, are to carry the same load and be of equal strength, their lengths must be as the numbers 2, $1\frac{1}{2}$, and 1, respectively.

In practice, however, the ends of the pillars *b* and *c* are not generally made as shown by the figure, but have holes at their ends into which pins are fitted which are fastened to some other piece; as, for example, a connecting-rod of an engine. In such cases, it has been found that *a* is two times as strong as *c*, and that *b* is one and one-half times as strong as *c*. That is, in actual practice, a column fixed as at *c* is really one-half as strong as one fixed as at *a*, instead of being only one-quarter as strong, as given above.

Green or wet timber has only one-half the strength of dry and seasoned timber; consequently, its crushing strength is only one-half of that given in the table below.

1949. In Table 33 is given the mean crushing strength of some short specimens of materials in tons (of 2,000 pounds) per square inch.

TABLE 33.

Materials.	Crushing Strength in Tons per Square Inch.
Cast Iron	40
Wrought Iron	18
Mild Steel	26
Cast Copper	5
Cast Brass	4.5
Timber (Dry)	3.5
Brick	1
Stone	3

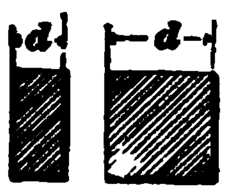
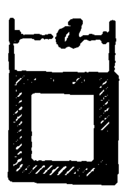
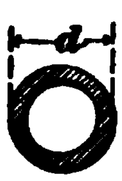
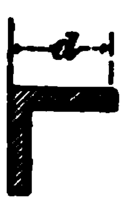
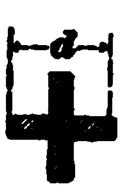
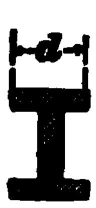
STRENGTH OF PILLARS.

1950. The following formula is applicable to pillars which are commonly used in practice, the lengths of which are about from 10 to 40 times their least diameter, or, if rectangular, their least thickness as indicated by d:

Let C = crushing strength in tons per square inch of a short specimen of the material as given in Table 33;
S = sectional area in square inches;
L = length in inches;
d = least thickness of rectangular pillar, or diameter of round pillar in inches;
W = breaking load in tons;
A = the area of the two flanges;
B = the area of the web;
a = a constant for the particular form of cross-section and material of which the pillar is made; its value is given in Tables 34 to 36 for such cross-sections as are given in the first column of those tables, and for such material as is mentioned at the top of the tables.

TABLE 34.

WROUGHT-IRON PILLARS.

Cross-section of Pillar.	When Both Ends of the Pillar are Flat or Fixed.	When One End of the Pillar is Flat or Fixed, and the Other Round or Movable.	When Both Ends of the Pillar are Round or Movable.
 Round.	2,250	1,500	1,125
 Square or Rectangle.	3,000	2,000	1,500
 Thin Square Tube.	6,000	4,000	3,000
 Thin Round Tube.	4,500	3,000	2,250
 Angle with Equal Sides.	1,500	1,000	750
 Cross with Equal Arms.	1,500	1,000	750
 I Beam.	$3{,}000 \times \frac{A}{A+B}$	$2{,}000 \times \frac{A}{A+B}$	$1{,}500 \times \frac{A}{A+B}$

1951. Rule.—*The breaking load of a pillar in tons is equal to the crushing strength of a short specimen of the material as given in Table 33, multiplied by the sectional area of the pillar in square inches, and the product divided by*

TABLE 35.

CAST-IRON PILLARS.

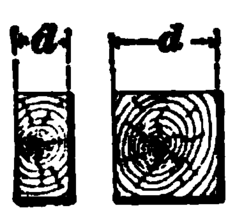
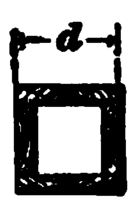
Cross-section of Pillar.	When Both Ends of the Pillar are Flat or Fixed.	When One End of the Pillar is Flat or Fixed, and the Other Round or Movable.	When Both Ends of the Pillar are Round or Movable.
Round.	281.25	187.5	140.625
Square or Rectangle.	375.00	250.0	187.500
Thin Square Tube.	750.00	500.0	375.000
Thin Round Tube.	562.50	375.0	281.250
Angle with Equal Sides.	187.50	125.0	93.750
Cross with Equal Arms.	187.50	125.0	93.750
I Beam.	$375 \times \frac{A}{A+B}$	$250 \times \frac{A}{A+B}$	$125 \times \frac{A}{A+B}$

the result obtained by dividing the square of the length of the pillar in inches by the square of the diameter, or least thickness if rectangular, multiplied by the value of a, plus 1.

That is, $$W = \frac{C\,S}{\frac{L^2}{a\,d^2} + 1}. \qquad \textbf{(127.)}$$

TABLE 36.

WOODEN PILLARS.

Cross-section of Pillar.	When Both Ends of the Pillar are Flat or Fixed.	When One End of the Pillar is Flat or Fixed, and the Other Round or Movable.	When Both Ends of the Pillar are Round or Movable.
 Round.	187.5	125.00	93.75
 Square or Rectangle.	250.0	166.66	125.00
 Hollow Square Made of Boards.	500.0	333.33	250.00

The result obtained by the formula must be divided by 6 to get the safe working load.

NOTE.—If the length of the pillar is given in feet, be sure to reduce it to inches before substituting in the formula.

EXAMPLE.—A wooden pillar 6 inches square and 144 inches long is fixed at both ends; what load will it sustain with safety?

SOLUTION.—Using formula **127**, we have

$$W = \frac{3.5 \times 6 \times 6}{\frac{144 \times 144}{250 \times 6 \times 6} + 1} = 38.14 \text{ tons, nearly.}$$

Which, divided by 6, gives $\frac{38.14}{6} = 6.357$ tons, or the load it is capable of sustaining with safety. Ans.

EXAMPLE.—A wrought-iron pillar 4 inches in diameter and 60 inches long is fixed at one end and movable at the other; what load will it sustain with safety?

SOLUTION.—Using formula **127**,

$$W = \frac{18 \times 4 \times 4 \times .7854}{\frac{60 \times 60}{1,500 \times 4 \times 4} + 1} = 196.69 \text{ tons.}$$

Which, divided by 6, gives $\frac{196.69}{6} = 32.78$ tons, nearly, or the load it is capable of sustaining with safety. Ans.

EXAMPLE.—A cast-iron pillar is 20 feet long and its cross-section is a cross with equal arms which are 1 inch thick and 10 inches long. (See dimension d, Table 35.) The two ends of the pillar are movable. What load will the column safely sustain?

SOLUTION.—Area of cross-section is equal to $(10 \times 1) + 2(4.5 \times 1) =$ 19 square inches; 20 feet are equal to 240 inches.

Now, applying formula **127**,

$$W = \frac{40 \times 19}{\frac{240 \times 240}{93.75 \times 10 \times 10} + 1} = 106.38 \text{ tons.}$$

Which, divided by 6, gives $\frac{106.38}{6} = 17.73$ tons, the load it is capable of sustaining with safety. Ans.

When using formula **127**, first obtain the value of C from Table 33. Next, calculate the area of the cross-section of the pillar. Then, find the value of a from one of the tables. Finally, be sure that the length of the pillar has been reduced to inches before substituting in the formula.

EXAMPLES FOR PRACTICE.

1. What load may be safely carried by a hollow cylindrical cast-iron pillar 20 ft. long, inside diameter 8", and outside diameter 10"? Both ends of the pillar are fixed. Ans. 93.13 tons.

2. A rectangular wooden column is 14 ft. long, and has one end rounded; if the cross-section is 12" × 8", what load will be required to break it? Ans. 92.15 tons.

3. A solid wrought-iron column, which has both ends movable, is 3" in diameter and 8 ft. long; what load will it safely support? Ans. 11.1 tons.

TRANSVERSE STRENGTH OF MATERIALS.

1952. The transverse strength of any material is the resistance offered by its fibers to being broken by bending. As, for example, when a beam, bar, rod, etc., which is supported at its ends, is broken by a force applied between its supports.

The transverse strength of any beam, bar, rod, etc., is proportional to the product of the square of its depth multiplied by its width; consequently, it is more economical to increase the depth than the width.

TABLE 37.

CONSTANTS FOR TRANSVERSE STRENGTH.

Materials.	Safe Transverse Strength in Pounds.	Materials.	Safe Transverse Strength in Pounds.
METALS.		WOODS.	
Cast Iron........	100	Birch............	35
Wrought Iron....	150	Elm	25
Structural Steel..	160	Ash	45
Copper..........	50	Beech............	30
Brass...........	55	Hickory	50
		Maple	60
		Oak (American)..	45
		Pine (Pitch)	40
		Pine (White).....	30

1953. A **cantilever** is a beam, bar, rod, etc., fixed at one end and subjected to a transverse stress, as shown in Fig. 639. It has a tendency to overthrow the wall or structure to which it is attached.

The strength of a cantilever varies inversely as the distance of the load from the section acted upon; and the stress upon any section varies directly as the distance of the load from that section.

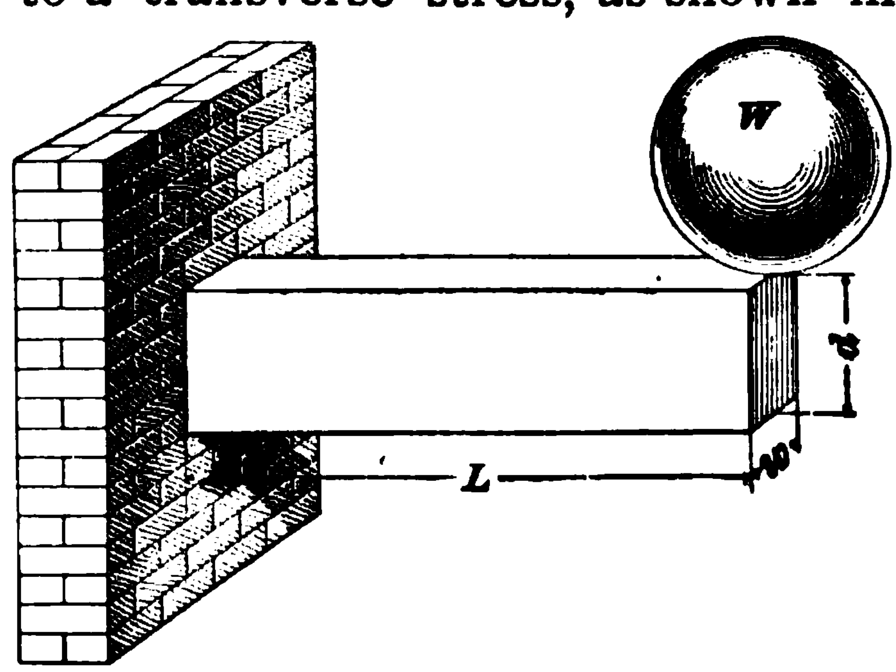


FIG. 639.

The strength of a beam, bar, rod, etc., which has both its ends supported, but not fixed, and which is loaded in the middle between its supports, is four times greater than when it is fixed at one end only.

A cantilever uniformly loaded will sustain twice as great a load as one on which the load is applied at one end; and a beam resting on two supports uniformly loaded will sustain twice as great a load as one on which the load is applied in the middle, between its supports.

In Table 37 is given the safe transverse strength of bars of different kinds of material, 1 inch square and 1 foot long, with the load suspended from one end.

RULES AND FORMULAS FOR THE TRANSVERSE STRENGTH OF BEAMS.

1954. In the following formulas,

Let d = the depth of beam in inches;
d_1 = diameter of cylindrical beam in inches;
w = the width of the beam in inches;
L = the length of the beam in feet between its supports;
S = the safe transverse strength as given in the above table;
W = the safe load in pounds.

For a rectangular or square cantilever to which the load is applied at one end, as shown in Fig. 639:

Rule.—*To find the maximum safe load in pounds that should be allowed at the end of any rectangular or square cantilever, multiply the square of the depth in inches by the width in inches and by the safe transverse strength of the material as given in Table 37; divide this product by the length in feet.*

That is, $$W = \frac{d^2 w S}{L}. \qquad \textbf{(128.)}$$

EXAMPLE.—What is the maximum safe load that can be placed at one end of a cast-iron bar which projects 4 feet, the depth being 6 inches and the width 3 inches?

SOLUTION.—Applying formula **128**, we have

$$W = \frac{6 \times 6 \times 3 \times 100}{4} = 2{,}700 \text{ pounds. Ans.}$$

1955. For a cylindrical cantilever to which the load is applied at one end, as shown in Fig. 640:

Rule.—*To find the maximum safe load in pounds that should be allowed at the end of any cylindrical cantilever, multiply the cube of its diameter in inches by .6 of the safe transverse strength of the material as given in Table 87, and divide the product by the length in feet.*

FIG. 640.

That is, $$W = \frac{d_1^3 \times .6\,S}{L}. \qquad \textbf{(129.)}$$

EXAMPLE.—What is the maximum load that can be placed with safety at one end of a cast-iron bar 4 inches in diameter that projects 8 feet?

SOLUTION.—Applying formula **129**, we have

$$W = \frac{4 \times 4 \times 4 \times .6 \times 100}{8} = 1{,}280 \text{ pounds. Ans.}$$

1956. When the load is uniformly distributed on a cantilever of any cross-section, as shown in Fig. 641, it will sustain a load twice as great as when the load is applied at one end. For example, if the cantilevers in the two examples above were to carry a uniformly distributed load, they would sustain $2{,}700 \times 2 = 5{,}400$ pounds and $1{,}280 \times 2 = 2{,}560$ pounds, respectively.

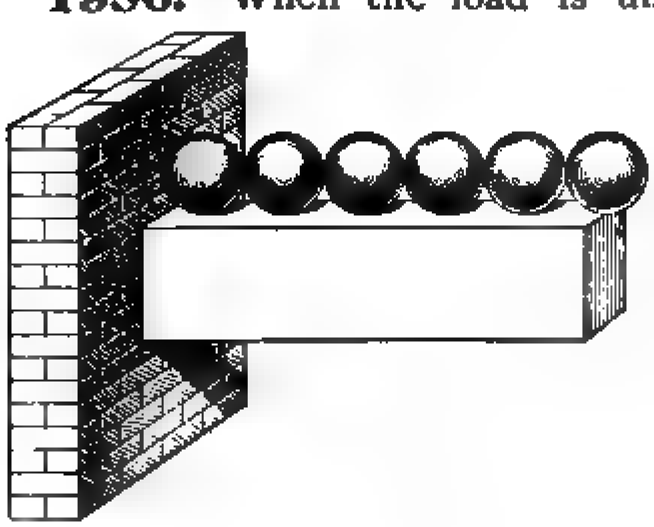


FIG. 641

1957. For a rectangular or square beam the ends of which merely rest upon supports, and loaded in the middle, as shown in Fig. 642:

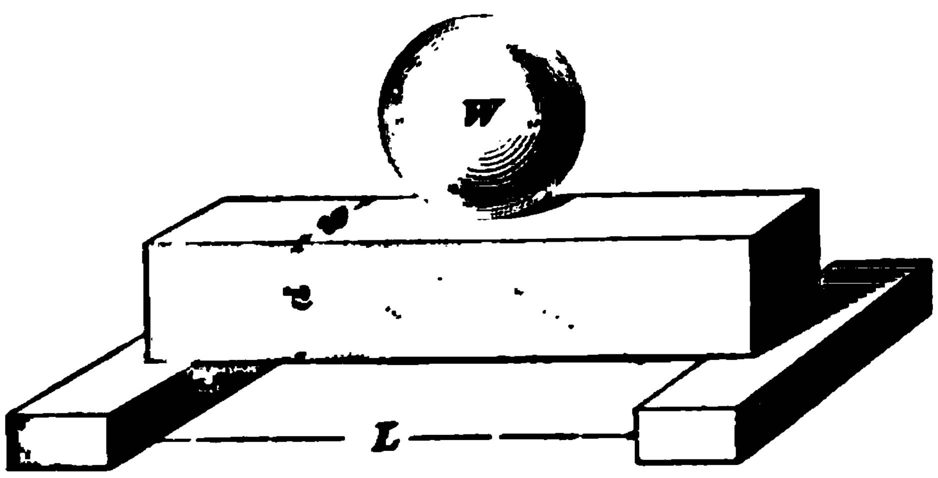


FIG. 642.

Rule.—*To find the maximum safe load in pounds that any rectangular or square beam is capable of sustaining at the middle when its ends merely rest upon supports, multiply four times the square of its depth in inches, by its width in inches, and by the safe strength of the material as given in Table 37; divide this product by the distance between its supports in feet;*

or, $$W = \frac{4\,d^2\,w\,S}{L}. \qquad \textbf{(130.)}$$

EXAMPLE.—What maximum safe load is a bar of cast iron capable of sustaining in the middle between its supports on which its ends merely rest, if its depth is 6 inches, its width 3 inches, and the distance between the supports is 4 feet?

SOLUTION.—Applying formula **130**,

$$W = \frac{4 \times 6^2 \times 3 \times 100}{4} = 10{,}800 \text{ lb.} \quad \text{Ans.}$$

1958. For a cylindrical beam supported at its ends and loaded in the middle, as shown in Fig. 643:

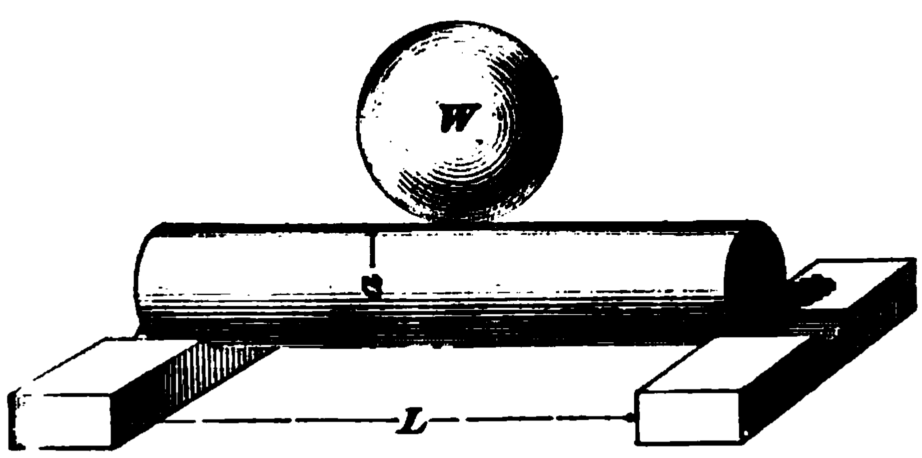


FIG. 643.

Rule.—*To find the maximum safe load in pounds that any cylindrical beam is capable of sustaining at the middle when its ends merely rest upon supports, multiply four times the cube of its diameter by .6 of the safe strength of the material as given in Table 37; divide this product by the distance between its supports in feet;*

or, $$W = \frac{4\,d_1^3 \times .6\,S}{L}. \qquad \textbf{(131.)}$$

Example.—What maximum safe load is a bar of cast iron capable of sustaining in the middle between its supports, on which its ends merely rest, if it is 4 inches in diameter, and if the distance between its supports is 3 feet?

Solution.—Applying formula **131**,

$$W = \frac{4 \times 4^3 \times .6 \times 100}{3} = 5{,}120 \text{ lb. Ans.}$$

1959. When the load is uniformly distributed on a beam of any cross-section, as shown in Fig. 644, it will sustain a load twice as great as when the load is applied in the middle between the supports.

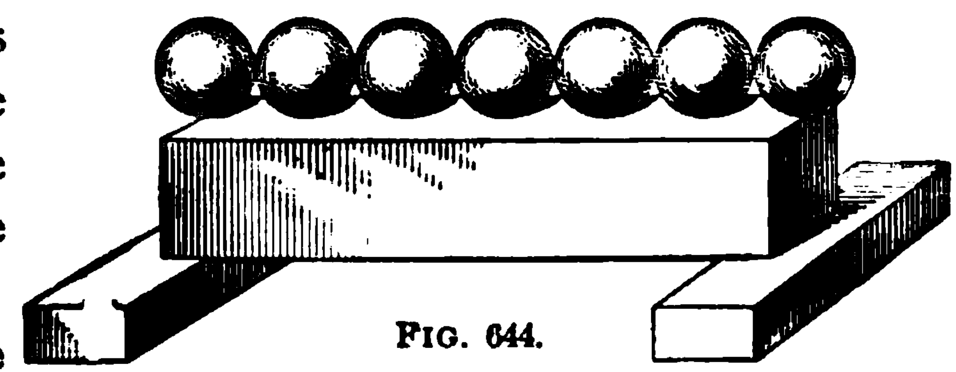


Fig. 644.

For example, if the beams in the last two examples were to carry a uniformly distributed load, they would sustain 10,800 × 2 = 21,600 pounds, and 5,120 × 2 = 10,240 pounds, respectively.

SHEARING OR CUTTING STRENGTH OF MATERIALS.

1960. The shearing strength of any material is the resistance offered by its fibers to being cut in two.

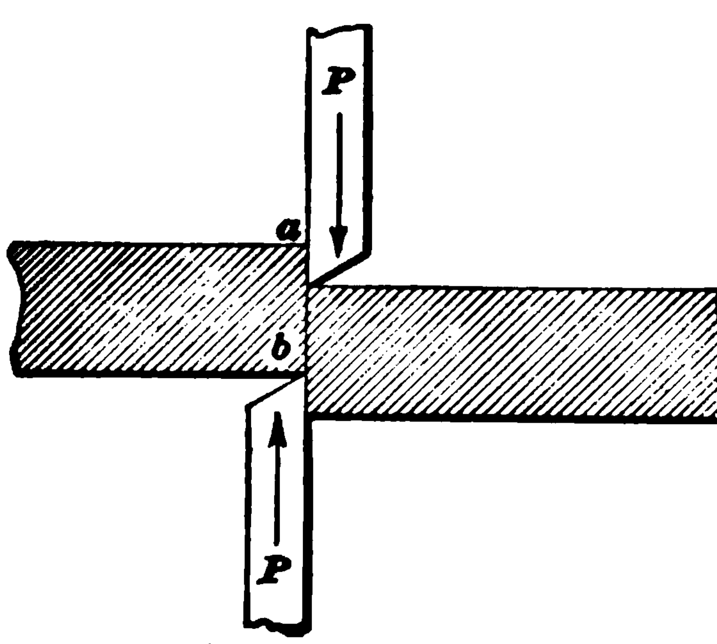


Fig. 645.

Thus the pressure of the cutting edges of an ordinary shearing machine, Fig. 645, causes a shearing stress in the plane *a b*. The unit shearing force may be found by dividing the force P by the area of the plane *a b*.

1961. Fig. 646 shows a piece in double shear; here the central piece *c d*

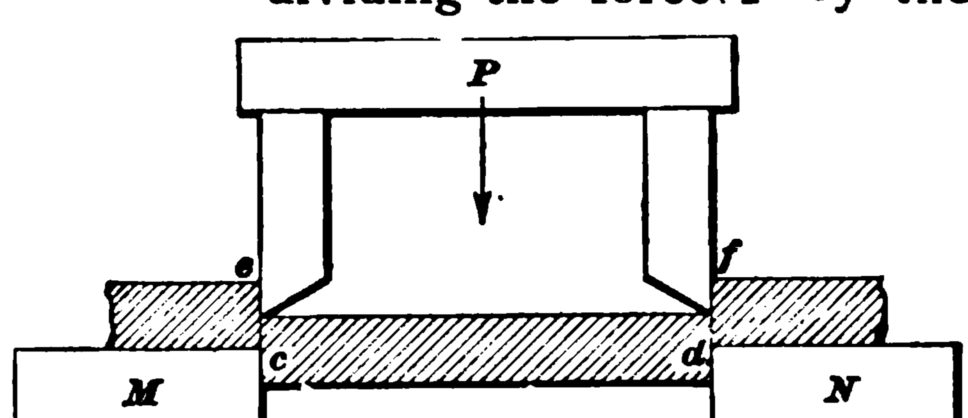


Fig. 646.

is forced out while the ends remain on their supports M and N.

The shearing strength of any body is directly proportional to its area.

1962. In Table 38 are given the greatest and the safe shearing strengths per square inch of different kinds of materials:

TABLE 38.

Materials.	Greatest Shearing Stress in Pounds per Square Inch.	Safe Shearing Stress in Pounds per Square Inch.
Cast Iron.........	18,000	1,500 to 3,000
Wrought Iron.....	40,000	4,000 to 10,000
Steel	60,000	5,000 to 12,000

1963. In the following formula,

Let a = area of cross-section in inches;

S = safe shearing stress, as given in Table 38;

W = safe load in pounds.

Rule.—*The safe load that any body which is subjected to a shearing stress is capable of sustaining is equal to the area of its cross-section in inches, multiplied by its safe shearing stress, as given in Table 38.*

That is, $W = a\,S$. **(132.)**

Example.—If the beam in Fig. 646 is made of wrought iron 4 inches in depth and 2 inches in width, what steady shearing stress is it capable of sustaining with safety?

Solution.—Applying formula **132**, $W = 4 \times 2 \times 10{,}000 = 80{,}000$ lb. This result must be multiplied by 2, since the beam is sheared in two places, along the lines $e\,c$ and $f\,d$. Hence, the stress which the beam will safely sustain is $80{,}000 \times 2 = 160{,}000$ lb. Ans.

Example.—What force is required to punch a hole $\frac{3}{8}''$ in diameter through a steel plate $\frac{1}{4}''$ thick?

Solution.—It is evident that punching is but shearing in a circle instead of a straight line. The area punched (sheared) is equal to the

thickness of the plate multiplied by the circumference of a circle having the same diameter as the punched hole. For, if the plate were cut through one of the diameters of the punched hole and the two semicircles were straightened out, the punched surface would be a rectangle, which would have a length equal to the circumference of a circle whose diameter was equal to that of the hole, and a breadth equal to the thickness of the plate. In this case, the area $= \frac{5}{8} \times 3.1416 \times \frac{1}{2} = .98175$ sq. in. Table 38 gives the ultimate shearing strength of steel as 60,000 lb. per sq. in. Hence, the total force required is $.98175 \times 60,000 = 58,905$ lb. Ans.

EXAMPLES FOR PRACTICE.

1. What is the greatest load that can be safely carried by a steel rectangular cantilever at its extreme end, if the bar is 2" wide, 3" deep, and 2 ft. 6" long? Ans. 1,152 lb.

2. What is the greatest uniform load that can be safely carried by a white pine girder 6" wide, 8" deep, 16 ft. long, and supported at its ends? Ans. 5,760 lb.

3. A cast-iron bar $1\frac{3}{4}$" in diameter and 5 ft. 3" long is supported at its ends; what load will it safely sustain in the middle? Ans. 245 lb.

4. What force is required to punch a $1\frac{1}{4}$" hole through a wrought-iron plate $\frac{7}{16}$" thick? Ans. 68,723 lb.

5. What force is required to cut off the end of a cast-iron bar whose diameter is $2\frac{1}{2}$"? Ans. 88,357 lb.

LINE SHAFTING.

1964. A line of shafting is one continuous run, or length, composed of lengths of shafts joined together by couplings.

The **main line** of shafting is that which receives the power from the engine or motor, and distributes it to the other lines of shafting, or to the various machines to be driven.

Line shafting is supported by hangers, which are brackets provided with bearings bolted either to the walls, posts, ceilings, or floors of the building. Short lengths of shafting called **countershafts** are provided to effect changes of speed, and to enable the machinery to be stopped or started.

Shafting is usually made cylindrically true, either by a special rolling process known as **cold-rolled shafting,** or

else it is turned up in a machine called a lathe. In the latter case, it is called **bright shafting.** What is known as **black shafting** is simply bar iron rolled by the ordinary process, and turned where it receives the couplings, pulleys, bearings, etc.

Bright-turned shafting varies in diameter by $\frac{1}{4}$ of an inch to about $3\frac{1}{2}$ inches in diameter; above this diameter the shafting varies by $\frac{1}{2}$ inch. The actual diameter of a bright shaft is $\frac{1}{16}$ of an inch less than the actual commercial diameter, it being designated from the diameter of the ordinary round bar-iron from which it is turned. Thus, a length of what is called 3-inch bright shafting is only $2\frac{15}{16}$ inches in diameter.

Cold-rolled shafting is designated by its actual commercial diameter; thus, a length of what is called 3-inch shafting is 3 inches in diameter.

1965. In Table 39 is given the maximum distance between the bearings of some continuous shafts which are used for the transmission of power.

TABLE 39.

Diameter of Shaft in Inches.	Distance Between Bearings in Feet.	
	Wrought-Iron Shaft.	Steel Shaft.
2	11	11.5
3	13	13.75
4	15	15.75
5	17	18.25
6	19	20.00
7	21	22.25
8	23	24.00
9	25	26.00

The necessary diameters of the various lengths of shafts composing a line of shafting should be proportional to the

quantity of power delivered by each respective length. In this connection, the positions of the various pulleys depend upon the distance between the pulley and the bearing and upon the amount of power given off by the pulleys. Suppose, for example, that a piece of shafting delivers a certain amount of power; then, it is obvious that the shaft will deflect or bend less if the pulley transmitting that power be placed close to a hanger or bearing than if it be placed midway between the two hangers or bearings.

NOTE.—It is impossible to give any rule for the proper distance of bearings which could be used universally, as in some cases the requirements demand that the bearings be nearer together than in others.

1966. To compute the horsepower that can be transmitted by a shaft of any given diameter:

Let D = diameter of shaft;
R = revolutions per minute;
H = horsepower transmitted;
C = constant taken from the following table:

TABLE 40.
CONSTANTS FOR LINE SHAFTING.

Material of Shaft.	No Pulleys Between Bearings.	Pulleys Between Bearings.
Steel or Cold-Rolled Iron..	65	85
Wrought Iron............	70	95
Cast Iron................	90	120

Rule.—*The horsepower that a shaft will transmit is equal to the product of the cube of the diameter and the number of revolutions, divided by the value of C for the given material.*

That is, $$H = \frac{D^3 R}{C}. \quad \textbf{(133.)}$$

EXAMPLE.—What horsepower will a 3-inch wrought-iron shaft transmit, which makes 100 revolutions per minute, and has pulleys between bearings?

SOLUTION.—Applying formula **133**, we have
$$H = \frac{3 \times 3 \times 3 \times 100}{95} = 28.42 \text{ horsepower. Ans.}$$

1967. To compute the number of revolutions a shaft must make to transmit a given horsepower:

Rule.—*The number of revolutions necessary for a given horsepower is equal to the product of the value of the constant C for the given material and the number of horsepower, divided by the cube of the diameter.*

That is, $$R = \frac{CH}{D^3}. \qquad \textbf{(134.)}$$

EXAMPLE.—How many revolutions must a 2-inch steel shaft make per minute to transmit 16 horsepower? There are no pulleys between bearings.

SOLUTION.—Applying formula **134**, we have $\frac{65 \times 16}{2 \times 2 \times 2} = 130$ revolutions. Ans.

1968. To compute the diameter of a shaft that will transmit a given horsepower, the number of revolutions the shaft makes per minute being given:

Rule.—*The diameter of a shaft equals the cube root of the quotient obtained by dividing the product of the value of the constant C for the given material and the number of horsepower by the number of revolutions.*

That is, $$D = \sqrt[3]{\frac{CH}{R}}. \qquad \textbf{(135.)}$$

EXAMPLE.—What must be the diameter of a cast-iron shaft to transmit 22.5 horsepower? The shaft makes 100 revolutions per minute, and there are pulleys between bearings.

SOLUTION.—Applying formula **135**, we have

$$D = \sqrt[3]{\frac{120 \times 22.5}{100}} = 3 \text{ in. Ans.}$$

1969. As the speed of shafting is used as a multiplier in the calculations of the horsepower of shafts, a shaft having a given diameter will transmit more power in proportion as its speed is increased. Thus, a shaft which is capable of transmitting 10 horsepower when making 100 revolutions per minute will transmit 20 horsepower when making 200 revolutions per minute. We may, therefore,

say *the horsepowers transmitted by two shafts are directly proportional to the number of revolutions.*

EXAMPLES FOR PRACTICE.

1. What horsepower will a $2\frac{1}{4}''$ wrought-iron shaft transmit when running at 110 revolutions per minute, it being used for transmission only? Ans. 24.55 horsepower.

2. A $6''$ cast-iron shaft transmits 150 horsepower; how many revolutions per minute must it make, there being no pulleys between bearings? Ans. 62.5 R. P. M.

3. What should be the diameter of a wrought-iron shaft to transmit 100 horsepower at 150 revolutions per minute, there being pulleys between bearings? Ans. 6.82 in.

4. A certain line shaft is to transmit to a number of machines by means of pulleys between its bearings 65 horsepower when running at 150 revolutions per minute; what should be its diameter?

Ans. $3\frac{1}{4}$ in., nearly.

HYDRAULICS AND HYDRAULIC MACHINERY

(PART 1)

HYDROSTATICS

LAWS OF LIQUID PRESSURE

1. Hydrostatics treats of liquids at rest under the action of forces. Liquids are very nearly *incompressible*. A pressure of 15 pounds per square inch compresses water less than $\frac{1}{300000}$ of its volume.

2. Fig. 1 represents two cylindrical vessels of the same size. The inside of vessel a is fitted with a wooden block up to the line P; the vessel b is filled with water to a depth equal to the length of the wooden block in a. Both vessels are fitted with air-tight pistons P, whose areas are each 10 square inches.

Suppose, for convenience, that the weights of the pistons, block, and water be neglected, and that a force of 100 pounds be applied to both pistons. The pressure per square inch will be $\frac{100}{10} = 10$ pounds. This pressure will be transmitted to the

FIG. 1

bottom of the vessel *a* and will be 10 pounds per square inch; there will be no pressure on the sides. In the vessel *b*, the pressure on the bottom will be the same as in the other case, that is, 10 pounds per square inch, but, owing to the fact that the molecules of the water are perfectly free to move, this pressure is *transmitted in every direction with the same intensity;* that is to say, the pressure at any point, *c*, *d*, *e*, *f*, *g*, *h*, etc., due to the force of 100 pounds, is exactly the same and equals 10 pounds per square inch.

3. Pascal's Law.—*The pressure per unit of area exerted anywhere on a mass of liquid is transmitted undiminished in all directions and acts with the same intensity on all surfaces in a direction at* **right angles** *to those surfaces.*

This may be proved experimentally by means of the apparatus shown in Fig. 2. Let the area of the pistons *a*, *b*, *c*, *d*, *e*, and *f* be 20, 7, 1, 6, 8, and 4 square inches, respectively.

a

FIG. 2

If the pressure due to the weight of the water be neglected and a force of 5 pounds be applied at *c* (whose area is 1 square inch), a pressure of 5 pounds per square inch will be transmitted in all directions; in order that there shall be no movement, a force of $6 \times 5 = 30$ pounds must be applied at *d*, 40 pounds at *e*, 20 pounds at *f*, 100 pounds at *a*, and 35 pounds at *b*.

If a force of 99 pounds were applied to *a*, instead of 100 pounds, the piston *a* would rise and the other pistons, *b*, *c*, *d*, *e*, and *f* would move inwards; but if the force applied to *a* were 100 pounds, they would all be in equilibrium. Suppose 101 pounds to be applied at *a*; the pressure

per square inch $\frac{1010}{200} = 5.05$ pounds would be transmitted in all directions; then, since the pressure due to *c* is only 5 pounds per square inch, it is now evident that the piston *a* will move downwards and the pistons *b*, *c*, *d*, *e*, and *f* will be forced outwards.

The pressure due to the weight of a liquid may be downwards, upwards, or sideways.

4. Downward Pressure.—In Fig. 3 the pressure on the bottom of the vessel *a* is equal to the weight of the water it contains. If the areas of the bottoms of vessels *a* and *b* and the depth of the liquids contained in them are the same, the pressures on the bottoms of the vessels will be the same. Suppose the bottoms of the vessels are 6 inches square, that the part *c d*, in the vessel *b*, is 2 inches square, and that the vessels are filled with water. The weight of 1 cubic inch of water is $\frac{62.5}{1,728}$ $= .03617$ pound. The number of cubic inches in *a* will be $6 \times 6 \times 24 = 864$. The weight of the water will be $864 \times .03617 = 31.25$ pounds. Hence, the total pressure on the bottom of *a* will be 31.25 pounds, or .868 pound per square inch. The pressure in *b* due to the weight contained in the part *b c* is $6 \times 6 \times 10 \times .03617 = 13.02$ pounds. The weight of the part contained in *c d* is $2 \times 2 \times 14 \times .03617 = 2.0255$ pounds, and the weight per square inch of area in *c d* is $\frac{2.0255}{4} = .5064$ pound.

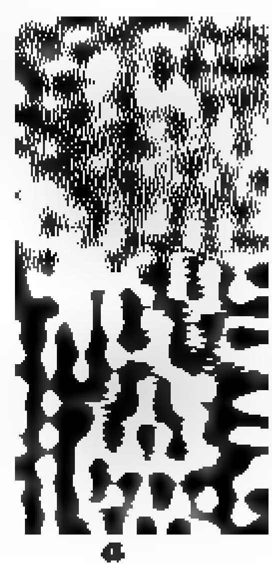


Fig. 3

According to Pascal's law, this weight (pressure) is transmitted equally in all directions; therefore, every square inch of the top of the large part of the vessel *b* will be subjected to a pressure of .5064 pound. The area of the part *b c* is

$6 \times 6 = 36$ square inches, and the total pressure due to the weight of the water in the small part will be $.5064 \times 36 = 18.23$ pounds. Hence, the total pressure on the bottom of *b* will be $13.02 + 18.23 = 31.25$ pounds, the same result as in the case of the vessel *a*.

If an additional pressure of 10 pounds per square inch were applied to the upper surface of both vessels, the total pressure on their bottoms would be $31.25 + (6 \times 6 \times 10) = 31.25 + 360 = 391.25$ pounds.

If this pressure were obtained by means of a weight placed on a piston, as shown in Figs. 1 and 2, the weight for the vessel *a* would be $6 \times 6 \times 10 = 360$ pounds and for the vessel *b*, $2 \times 2 \times 10 = 40$ pounds.

5. *The pressure on the bottom of a vessel containing a fluid is independent of the shape of the vessel and is equal to the weight of a prism of the fluid whose base is the same as the bottom of the vessel and whose altitude is the distance between the bottom and the upper surface of the fluid plus the pressure per unit of area upon the upper surface of the fluid multiplied by the area of the bottom of the vessel.*

FIG. 4

Suppose that the vessel *b*, Fig. 3, were inverted, as shown in Fig. 4, the pressure on the bottom would still be .868 pound per square inch, but it would require a weight of 3,490 pounds to be placed on a piston at the upper surface to make the pressure on the bottom 391.25 pounds, instead of a weight of 40 pounds, as in the other case.

EXAMPLE.—A vessel filled with salt water having a specific gravity of 1.03 has a circular bottom 13 inches in diameter. The top of the vessel is fitted with a piston 3 inches in diameter, on which is laid a weight of 75 pounds. What is the total pressure on the bottom if the depth of the water is 18 inches?

SOLUTION.—The weight of 1 cubic inch of the water is $\frac{62.5 \times 1.03}{1,728} = .037254$ pound. $13 \times 13 \times .7854 \times 18 \times .037254 = 89.01$ pounds, or the

pressure due to the weight of the water. $\frac{75}{3 \times 3 \times .7854} = 10.61$ pounds per square inch due to the weight on the piston. $18 \times 18 \times .7854 \times 10.61 = 1,408.29$ pounds. Total pressure $= 1,408.29 + 89.01 = 1,497.3$ lb. Ans.

6. Upward Pressure.—In Fig. 5 is represented a vessel of exactly the same size as that represented in Fig. 4. There is no upward pressure on the surface *c* due to the weight of the water in the large part *c d*, but there is an upward pressure on *c* due to the weight of the water in the small part *b c*. The pressure per square inch due to the weight of the water in *b c* was found to be .5064 pound; the area of the upper surface *c* of the large part *c d* is evidently $(6 \times 6) - (2 \times 2) = 36 - 4 = 32$ square inches, and the total upward pressure due to the weight of the water is $.5064 \times 32 = 16.2$ pounds.

FIG. 5

If an additional pressure of 10 pounds per square inch were applied to a piston fitting the top of the vessel, the total upward pressure on the surface *c* would be $16.2 + (32 \times 10) = 336.2$ pounds.

EXAMPLE.—A horizontal surface 6 inches by 4 inches is submerged in a vessel of water 26 inches below the upper surface. If the pressure on the water is 16 pounds per square inch, what is the total upward pressure on the horizontal surface?

SOLUTION.— $4 \times 6 \times 26 \times .03617 = 22.57$ pounds, the upward pressure due to the weight of the water. $6 \times 4 \times 16 = 384$ pounds, the upward pressure due to the outside pressure of 16 pounds per square inch. The total upward pressure $= 384 + 22.57 = 406.57$ lb. Ans.

7. Lateral Pressure.—Suppose that the top of the vessel shown in Fig. 6 is 10 inches square and that the projections at *a* and *b* are 1 inch $\times$ 1 inch and 10 inches long.

The pressure per square inch on the bottom of the vessel due to the weight of a liquid would be $1 \times 1 \times 18 \times$ the weight of a cubic inch of the liquid.

The pressure at a depth equal to the distance of the upper surface *b* would be $1 \times 1 \times 17 \times$ the weight of a cubic inch of the liquid.

Since both of these pressures are transmitted in every direction, they are also transmitted sideways, and the *pressure per unit of area on the projection b is a mean between the two* and equals 1 × 1 × 17½ × the weight of a cubic inch of the liquid.

To find the lateral pressure on the projection *a*, imagine that the dotted line *c* is the bottom of the vessel; then the conditions would be the same as in the preceding case, except that the depth is not so great.

FIG. 6

The lateral pressure on *a* is thus seen to be 1 × 1 × 11½ × the weight of a cubic inch of the liquid.

EXAMPLE.—A well 3 feet in diameter and 20 feet deep is filled with water; what is the pressure on a strip of the wall 1 inch wide, the center of which is 1 foot from the bottom? What is the pressure on the bottom? What is the upward pressure per square inch 2 feet 6 inches from the bottom?

SOLUTION.— 1 × 36 × 3.1416 = 113.1 square inches, the area of the strip. 113.1 × 19 × 12 × .03617 = 932.71 lb., the total pressure on the strip. Ans.

The pressure per square inch would be $\frac{932.71}{113.1}$ = 8.247 pounds, nearly.

36 × 36 × .7854 × 20 × 12 × .03617 = 8,836 lb., the pressure on the bottom. Ans.

20 − 2.5 = 17.5. 1 × 17.5 × 12 × .03617 = 7.596 pounds, the upward pressure per square inch 2 ft. 6 in. from the bottom. Ans.

8. A tall vessel *a* having a stop-cock *b* near its base and arranged to float on the water, as shown in Fig. 7, illustrates the effects of lateral pressure. When this vessel is filled with water, the lateral pressures at any two points of the surface of the vessel opposite to each other are equal. Being equal and acting in opposite directions, they balance each other,

and no motion can result; but if the stop-cock is opened, there will be no resistance to that pressure acting on the surface equal to the area of the opening, and it will cause the water to flow out, while its equal and opposite force will cause the vessel to move through the water in a direction opposite to that of the spouting water.

FIG. 7

9. The laws of liquid pressure given in the preceding articles may be embraced in the following formula:

$$P = a\,(d\,w + p), \qquad (1.)$$

where a = area of a submerged surface in square inches;
d = distance in inches of center of gravity of surface from surface of liquid;
w = weight of a cubic inch of the fluid in pounds;
p = pressure on surface of liquid in pounds per square inch;
P = total pressure on submerged surface in pounds.

10. Since the pressure on the bottom of a vessel due to the weight of the liquid is dependent only on the height of the liquid, and not on the shape of the vessel, it follows that if a vessel has a number of radiating tubes, as shown in Fig. 8, the water in each tube will be on the same level, no matter what may be the shape of the tubes. For, if the water were higher in one tube than in the others, the downward pressure on the bottom due to the height of the water in this tube would be greater than that due to the height of the water in the other tubes. Consequently, the upward pressure would also be greater, the equilibrium would be

destroyed, and the water would flow from this tube into the vessel and rise in the other tubes until it was at the same

FIG. 8

level in all, when it would be in equilibrium. This principle is expressed in the familiar saying *water seeks its level.*

EXAMPLE.—The water level in a city reservoir is 150 feet above the level of the street; what is the pressure of the water per square inch on the hydrant?

SOLUTION.— $1 \times 150 \times 12 \times .03617 = 65.106$ lb. per sq. in. Ans.

11. In Fig. 9 let the area of the piston a be 1 square inch, of b 40 square inches. According to Pascal's law, 1 pound placed on a will balance 40 pounds placed on b.

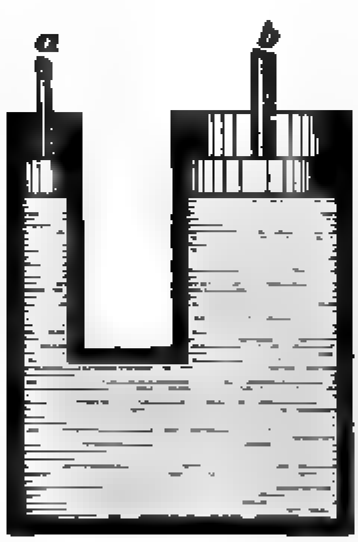


FIG. 9

Suppose that a moves downwards 10 inches; then 10 cubic inches of water will be forced into the tube b. This will be distributed over the entire area of the tube b in the form of a cylinder, whose cubical contents must be 10 cubic inches, whose base has an area of 40 square inches, and whose altitude must be $\frac{10}{40} = \frac{1}{4}$ of an inch; that is, a movement of 10 inches of the piston a will cause a movement of $\frac{1}{4}$ of an inch in the piston b.

12. Hydraulic Press.—The foregoing principles are made use of in the hydraulic press represented in Fig. 10. As the lever O is depressed, the piston a is forced down on the water in the cylinder A. The water is forced through

FIG. 10

the bent tube d into the cylinder in which the large piston C works and causes C to rise, thus lifting the platform K and compressing the bales. If the area of a be $\frac{1}{2}$ square inch and that of C be 50 square inches, it is evident, from the explanation of Fig. 9, that a force of 10 pounds on piston a will lift a load of $\frac{10}{.5} \times 50 = 1{,}000$ pounds on piston C. If, now, the length of the lever between the hand and the fulcrum is 10 times the length between fulcrum and piston a, a force of 10 pounds on the end of the lever will exert 100 pounds on a, and therefore 10,000 pounds on C.

13. Applications of this principle are seen in the hydraulic machines used for forcing locomotive driving wheels on their axles, etc., and for testing the strength of boiler shells.

EXAMPLE 1.—A suspended vertical cylinder is tested for the tightness of its heads by filling it with water. A pipe whose inside diameter is $\frac{1}{4}$ of an inch and whose length is 20 feet is screwed into a hole in the upper head and then filled with water; what is the pressure per square inch on each head if the cylinder is 40 inches in diameter and 60 inches long?

SOLUTION.—Area of heads $= 40^2 \times .7854 = 1{,}256.64$ square inches.

The pressure per square inch on the bottom head due to the weight of the water in the cylinder $= 1 \times 60 \times .03617 = 2.17$ pounds. $(\frac{1}{4})^2 \times .7854 = .04909$ square inch, the area of the pipe.

$.04909 \times 20 \times 12 \times .03617 = .426$ pound = the weight of water in pipe = the pressure on a surface area of .04909 square inch.

The pressure per square inch due to the water in the pipe is $\frac{1}{.04909} \times .426 = 8.68$ pounds per square inch on the upper head. Ans.

The pressure per square inch on the lower head is $8.68 + 2.17 = 10.85$ lb. Ans.

EXAMPLE 2.—In example 1, if the pipe be fitted with a piston weighing $\frac{1}{4}$ pound and a 5-pound weight be laid upon it, what will be the pressure on the upper head?

SOLUTION.—In addition to the pressure of .426 pound on the area of .04909 square inch, there is now an additional pressure on this area of $5 + \frac{1}{4} = 5.25$ pounds, and the total pressure on this area is $.426 + 5.25 = 5.676$ lb. Ans.

The pressure per square inch is $\frac{1}{.04909} \times 5.676 = 115.6$ pounds.

BUOYANT EFFECTS OF WATER

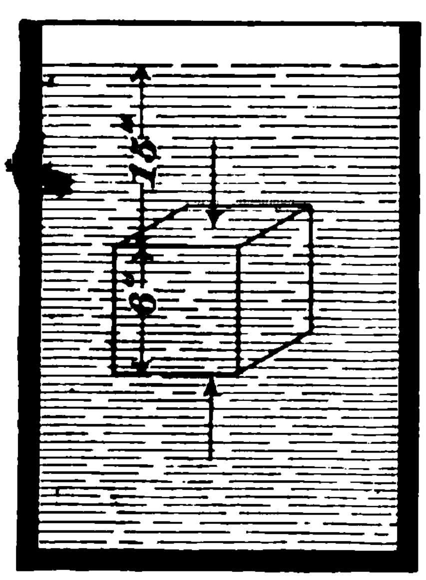


FIG. 11

14. In Fig. 11 is shown a 6-inch cube entirely submerged in water. The lateral pressures are equal and in opposite directions. The upward pressure $= 6 \times 6 \times 21 \times .03617$; the downward pressure $= 6 \times 6 \times 15 \times .03617$; and the difference $= 6 \times 6 \times 6 \times .03617$ = the volume of the cube in cubic inches $\times$ the weight of 1 cubic inch of water. That is, the upward pressure exceeds the downward pressure by

the weight of a volume of water equal to the volume of the body.

15. This excess of upward pressure acts against gravity; consequently, *if a body be immersed in a fluid, it will lose in weight an amount equal to the weight of the fluid it displaces.* This is called the **principle of Archimedes,** because it was first stated by him.

The principle may be experimentally demonstrated with beam scales, as shown in Fig. 12.

From one scale pan suspend a hollow cylinder of metal t and below that a solid cylinder a, of the same size as the hollow part of the upper cylinder. Put weights in the other scale pan until they exactly balance the two cylinders. If a be immersed in water, the scale pan containing the weights will descend, showing that a has lost some of its weight. Now fill t with water, and the volume of water that can be poured into t will equal that displaced by a. The scale pan that contains the weights will gradually rise until t is filled, when the scales balance again.

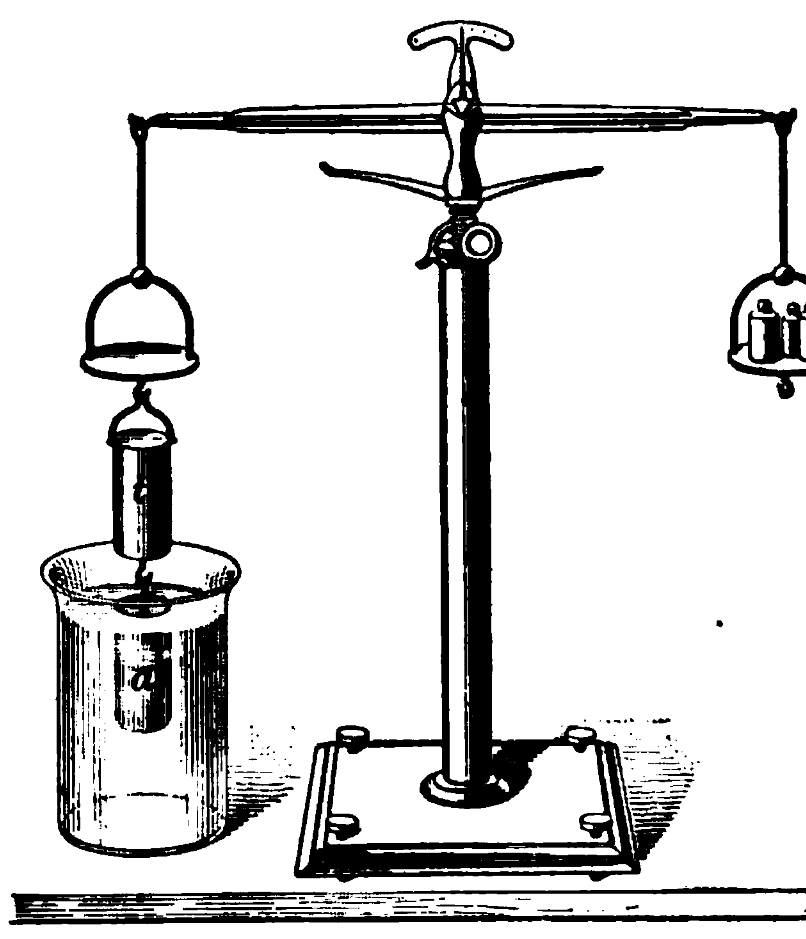


FIG. 12

If the immersed body is lighter than the liquid, the upward pressure will cause it to rise and extend partly out of the liquid, until the weight of the body and the weight of the liquid displaced are equal. If the immersed body is heavier than the liquid, the downward pressure plus the weight of the body will be greater than the upward pressure, and the body will fall downwards until it touches bottom or meets an obstruction. If the weights of equal volumes of the liquid and the body are equal, the body will remain stationary

and be in equilibrium in any position or depth beneath the surface of the liquid.

An interesting experiment in confirmation of the foregoing facts may be performed as follows: Place an egg in a glass jar filled with fresh water. The mean density of the egg being a little greater than that of water, it will fall to the bottom of the jar. Now dissolve salt in the water, stirring it so as to mix the fresh and salt water. The salt water will presently become denser than the egg and the egg will rise. Now, if fresh water is poured in until the egg and water have the same density, the egg will remain stationary in any position that it may be placed below the surface of the water.

HYDRAULICS

FLOW OF WATER THROUGH SHORT TUBES

16. Hydraulics treats of water in motion. The velocity of water is not the same at all points unless all cross-sections of the pipe or canal that it flows through are equal. That *velocity* which, being *multiplied by the area of the cross-section of the stream*, will equal the total quantity *discharged* is called the **mean velocity.**

Let Q = quantity that passes any section in 1 second;
A = area of the section;
v = mean velocity in feet per second.

Then,
$$Q = A\,v, \qquad (2.)$$
and
$$v = \frac{Q}{A}. \qquad (3.)$$

EXAMPLE 1.—The area of a certain cross-section of a stream is 27.9 square inches; the velocity of the water through this section is 51 feet per second. What is the quantity discharged in cubic feet?

SOLUTION.—Applying formula **2**, $Q = \frac{27.9}{144} \times 51 = 9.9$ cu. ft. per sec. Ans.

EXAMPLE 2.—In example 1, what would the velocity have been to discharge the same quantity had the area of the cross-section been 36 square inches?

SOLUTION.—Applying formula **3**, $v = \dfrac{9.9}{\frac{36}{144}} = \dfrac{9.9 \times 144}{36} = 39.6$ ft. per sec. Ans.

17. Velocity of Efflux.—If a small aperture be made in a vessel containing water, the velocity with which the water issues from the vessel is the same as if it had fallen from the level of the surface to the level of the aperture, all resistances being neglected. This velocity is called the **velocity of efflux.**

The vertical height of the level surface of the water above the center of the aperture is called the **head.** In Fig. 13, a is the head for the aperture A; b is the head for the aperture B; and c is the head for the aperture C.

FIG. 13

Let v = velocity of efflux in feet per second;
h = head in feet at the aperture considered.

Then, the theoretical velocity of efflux is expressed by the formula

$$v = \sqrt{2gh}. \qquad (4.)$$

Here $g = 32.16$; that is, *the velocity of efflux is the same as if the same weight of water had fallen through a height equal to its head.*

Were it not for the resistance of the air, friction, and the effect of the falling particles, the issuing water would spout to the level of the water in the vessel, that is, to a height equal to its head.

EXAMPLE 1.—A small orifice is made in a pipe 50 feet below the water level; what is the velocity of the issuing water?

SOLUTION.—Applying formula **4**, $v = \sqrt{2 \times 32.16 \times 50} = 56.7$ ft. per sec. Ans.

From the foregoing formula, as in the laws of falling bodies,

$$h = \frac{v^2}{2g}. \qquad (5.)$$

Hêre, h is called the *head due to the velocity v.* Consequently, if the velocity of efflux is known, the head can be found.

EXAMPLE 2.—An issuing jet of water has a velocity of 60 feet per second; what must be the head to give it this velocity?

SOLUTION.—Applying formula **5**, $h = \frac{60^2}{2 \times 32.16} = 55.97$ ft. Ans.

18. Suppose that a tall vessel is fitted with a piston and has an orifice near the bottom fitted with a stop-cock. If an additional pressure be applied to the piston, it is evident that the velocity of efflux will be increased.

Let p be the pressure per unit of area at the level of the water, due to the additional pressure on the piston. If the unit of area is 1 square inch, the height of a column of water that would cause a pressure equal to p would be

$$\frac{p}{.03617 \times 12} = \frac{p}{.434} \text{ feet.}$$

If the unit of area is in square feet, the height of a column of water would be $\frac{p}{62.5}$ feet. Denote this height corresponding to the additional pressure by h_1. The original head of the water in the vessel is h; hence, $h_1 + h =$ the total head, and the velocity of efflux, when the cock is opened, will be

$$v = \sqrt{2g(h_1 + h)}. \qquad (6.)$$

The total head $h_1 + h$ is called the **equivalent head,** and must in all cases be reduced to feet before substituting in the formula.

EXAMPLE.—The area of a piston fitting a vessel filled with water is 27.36 square inches. The total pressure on the piston is 80 pounds, the weight of the piston is 25 pounds, and the head of the water at the level of the orifice is 6 feet 10 inches; what is the velocity of efflux, assuming that there are no resistances?

SOLUTION.— $80 + 25 = 105$ pounds = the total pressure on the upper surface of the liquid. $\frac{105}{27.36} = 3.838$ pounds per square inch.

$\frac{3.838}{.03617} = 106.11$ = head in inches due to the pressure of 105 pounds.

$\frac{106.11}{12} = 8.84$ ft. $= h_1$. 6 feet 10 inches = 6.8333 feet $= h$.

Hence, applying formula **6**,

$$v = \sqrt{2g(8.84 + 6.8333)} = \sqrt{2 \times 32.16 \times 15.6733} = 31.75 \text{ ft. per sec.}$$

Ans.

19. When water issues from the side of a vessel, it is subjected to the same laws that govern projectiles. The range may be calculated in the same manner by taking the *velocity of efflux* as the *initial velocity* of the projectile.

The range may be calculated more conveniently by the formula

$$R = \sqrt{4\,h\,y}, \qquad (7.)$$

in which R is the range, h is the head or equivalent head at the level of the orifice, and y is the vertical height of the orifice above the point where the water strikes. In Fig. 14

FIG. 14

the upper surface of the water is free. For the orifice E, $h = B\,E$ and $y = E\,A$; for the orifice C, $h = B\,C$ and $y = C\,A$

The greatest range is obtained when $h = y$; that is, when the orifice is half way between the upper surface of the

water and the level of the place where the steam strikes. If two orifices are situated equally distant from the middle orifice giving the greatest range, as C and E, Fig. 14, the ranges of water issuing from them will be equal.

EXAMPLE.—The vertical height above the ground of the surface of the water in a vessel is 12 feet. If an orifice is situated 4 feet from the upper surface, what is the range? Where is the other point of equal range? What is the greatest range?

SOLUTION.—Applying formula **7**, $R = \sqrt{4 \times 4\,(12 - 4)} = 11.31$ feet, nearly; greatest range $= \sqrt{4 \times 6 \times 6} = 12$ feet. $6 - 4 = 2$; hence, the point of equal range is $6 + 2 = 8$ feet below the surface of the water. Ans.

PROOF.—Range $= \sqrt{4\,hy} = \sqrt{4 \times 8 \times 4} = 11.31$ feet, as before.

20. When the water flows through an orifice in the bottom of the vessel of large size compared with the area of the base, a different rule must be used from that given above. In Fig. 15 suppose that the area of the orifice in the bottom of the vessel is a and that the area of the bottom is A; then the velocity v is expressed by the formula

$$v = \sqrt{\frac{2\,g\,h}{1 - \frac{a^2}{A^2}}}. \qquad (8.)$$

FIG. 15

If the area of the orifice is not more than $\frac{1}{20}$ of the area of the cross-section of the vessel, use formula **4**. That is, *the velocity of efflux from a small orifice, not larger than $\frac{1}{20}$ of the cross-sectional area of the vessel, equals the square root of $2\,g$ times the head.*

EXAMPLE 1.—A vessel has a rectangular cross-section of 11 in. × 14 in.; the upper surface of the water is 14 feet above the bottom. If an orifice 4 inches square is made in the bottom of the vessel, what will be the velocity of efflux?

SOLUTION.—Area of the cross-section is $14 \times 11 = 154$ square inches. Area of orifice is $4 \times 4 = 16$ square inches. $\frac{16}{154} = \frac{1}{9.625}$. Since the area of the orifice is greater than $\frac{1}{20}$ the area of the bottom, apply formula **8**.

$$v = \sqrt{\frac{2gh}{1 - \frac{a^2}{A^2}}} = \sqrt{\frac{2 \times 32.16 \times 14}{1 - \frac{16^2}{154^2}}} = 30.17 \text{ ft. per sec. Ans.}$$

EXAMPLE 2.—If the orifice had been 2 inches square in example 1, what would have been the velocity of efflux? Also, if it had been 8 inches square?

SOLUTION.— $2 \times 2 = 4$ square inches, or the area of the orifice. $\frac{4}{154} = \frac{1}{38.5}$. Since the area of the orifice is less than $\frac{1}{25}$ the area of the vessel, apply formula **4.**

$$v = \sqrt{2gh} = \sqrt{2 \times 32.16 \times 14} = 30.008 \text{ ft. per sec. Ans.}$$

$8 \times 8 = 64$ square inches, or the area of the orifice in the second case; then, applying formula **8,**

$$v = \sqrt{\frac{2gh}{1 - \frac{a^2}{A^2}}} = \sqrt{\frac{2 \times 32.16 \times 14}{1 - \frac{64^2}{154^2}}} = 32.99 \text{ feet per second; prac-}$$

tically, 33 ft. per sec. Ans.

21. The Contracted Vein.—When water issues from an orifice in a thin plate (see Fig. 16) or from a square-edged

FIG. 16 FIG. 17

orifice (see Fig. 17), the stream is contracted a short distance from the orifice and expands again to the full size of the orifice. The point at which the contraction is greatest is at a distance from the orifice equal to the diameter of the orifice. In consequence of this contraction, the velocity of efflux is slightly reduced from the theoretical value and the quantity discharged is greatly

reduced. This contraction is called the **contracted vein,** a name given to it by Sir Isaac Newton.

For ordinary purposes, the actual velocity of efflux may be taken as 98% of the theoretical values calculated by the preceding rules.

The actual velocity of efflux from a small orifice is expressed by the formula

$$v = .98\sqrt{2gh}. \qquad \textbf{(9.)}$$

EXAMPLE.—What is the actual velocity of discharge from a small, square-edged orifice in the side of a vessel, if the head is 20 feet?

SOLUTION.—Applying formula **9,**

$$v = .98\sqrt{2gh} = .98\sqrt{2 \times 32.16 \times 20} = 35.15 \text{ ft. per sec. Ans.}$$

22. The diameter of the contracted vein at its smallest section is about .8 of the diameter of the orifice and its area is about $.8 \times .8 = .64$ of the area of the orifice. In Art. **16,** it was stated that the quantity discharged in cubic feet per second is equal to the area of the section multiplied by the mean velocity, or $Q = A\,v$. This is the theoretical value; *the actual value is the area of the contracted vein multiplied by the actual velocity of efflux*, or $Q = .64\,A \times .98\,v = .627\,A\,v$; that is, the actual discharge is about .627 of the theoretical discharge. This number .627 is called the **coefficient of efflux.**

The coefficient of efflux varies somewhat according to the head and the size and shape of the orifice; but for square-edged orifices or for orifices in thin plates, its average value may be taken as .615. Hence,

Rule.—*The actual quantity discharged is .615 times the theoretical amount,*

or $$Q = .615\,A\,v. \qquad \textbf{(10.)}$$

EXAMPLE.—The theoretical discharge from a certain vessel is 12.4 cubic feet per minute; what is the amount actually discharged per second?

SOLUTION.— $12.4 \times .615 = 7.626$ cubic feet per minute; $\frac{7.626}{60} = .1271$ cu. ft. per sec. Ans.

23. If the water discharges through a short tube whose length is from $1\frac{1}{2}$ to 3 times the diameter of the orifice (see Fig. 18), the discharge will be increased. From a large

FIG. 18

FIG. 19

number of experiments made by different persons, the coefficient of efflux for a short tube may be taken as .815; that is, the actual discharge may be taken as .815 times the theoretical discharge through an orifice of the same size. If the inside edges of the tube are well rounded and the tube is conical, as shown in Fig. 19, there will be no contraction, and the coefficient of discharge may be taken as .97; that is, the actual discharge through a tube of this form will be .97 times the theoretical discharge through an orifice whose area is the same as the area of the end of the tube.

24. If in a compound mouthpiece or tube, such as is shown in Fig. 20, the narrowest part *a b* be taken as the diameter of the orifice, the coefficient of discharge may be taken as 1.5526; that is, the actual discharge through a compound mouthpiece of this shape will be 1.5526 times the theoretical discharge through an orifice whose area is the same as the area of the smallest section of the mouthpiece.

FIG. 20

When the upper surface of the water remains at the same height above the orifice, there is said to be a *constant head.* The velocity of efflux varies for different points in the orifice; it is greater at the bottom of the orifice than at the top, since the head is greater at the bottom. A mean velocity may be obtained by *dividing the quantity of water discharged in cubic feet per second by the area of the orifice;* or

$$v_m = \frac{Q}{A}. \quad \text{(See formula } \mathbf{3.}\text{)}$$

25. Let Q = theoretical number of cubic feet discharged per second;
v_m = mean velocity through orifice;
A = area of orifice;
h = theoretical head necessary to give a mean velocity v_m;
Q_a = actual quantity discharged in cubic feet per second.

Then, for an orifice in a thin plate or a square-edged orifice (the hole itself may be of any shape—triangular, square, circular, etc.—but the edges must not be rounded), the actual quantity discharged is

$$Q_a = .615\,Q = .615\,A\,v_m = .615\,A\,\sqrt{2\,g\,h}. \qquad \mathbf{(11.)}$$

For a discharge through a short tube, as shown in Fig. 18,

$$Q_a = .815\,Q = .815\,A\,v_m = .815\,A\,\sqrt{2\,g\,h}. \qquad \mathbf{(12.)}$$

For a discharge through a mouthpiece, as shown in Fig. 19,

$$Q_a = .97\,Q = .97\,A\,v_m = .97\,A\,\sqrt{2\,g\,h}. \qquad \mathbf{(13.)}$$

For a discharge through the compound mouthpiece, as shown in Fig. 20, the area of the orifice being taken as the area of the smallest section,

$$Q_a = 1.5526\,Q = 1.5526\,A\,v_m = 1.5526\,A\,\sqrt{2\,g\,h}. \qquad \mathbf{(14.)}$$

In these four formulas it is assumed that the head remains constant.

FLOW OF WATER THROUGH PIPES

THE HYDRAULIC GRADE LINE

26. The **hydraulic grade line,** or **hydraulic gradient,** is the line drawn through a series of points to which water will rise in tubes attached to a pipe through which water flows. With a smooth pipe of uniform cross-section, without bends or other obstructions to flow, the hydraulic grade line is a straight line extending from the reservoir to the end of the pipe.

In Fig. 21 is shown a long horizontal pipe leading from a reservoir to a stop-valve *S*. When the valve is open so that water from the pipe discharges freely into the atmosphere,

FIG. 21

the hydraulic grade line is the line *a d f g*. The distance of the point *a* below the surface of the water in the reservoir represents the head absorbed in overcoming the resistances of entrance to the pipe and in producing the velocity with which the water flows. In the same way the difference in the height to which the water rises in any two tubes represents the head absorbed in overcoming the resistance to flow in the pipe between the points at which the tubes are joined.

27. The flow of water through the pipe *P* would be the same whether it were horizontal, as shown in the figure, or if it were laid along the grade line *a d f g*. The flow would also be the same if the reservoir were deepened and the pipe laid along the line *a' d' f'*. The pressures in the pipe, however, would vary greatly with the different positions. If it were laid along the line *a d f g*, there would be little or no pressure in any part of it, and if it were perforated at the top, little or no water would flow from the perforations. In the horizontal position, however, and still more in the position *a' a' f'*, there would be pressure at all points, the pressure for any point in the pipe being equivalent to the head represented by the vertical distance from that point to the hydraulic grade line; and if the pipe were perforated anywhere, water would issue from the perforations.

28. In laying a line of pipe to connect two points having different elevations, it is of the utmost importance to ascertain the position of the hydraulic grade line. Let *A* and *B*, Fig. 22, represent two reservoirs connected by a pipe line

Fig. 22

of uniform diameter through which the water flows by gravity from the upper to the lower level. The hydraulic grade line will be the straight line connecting the two reservoirs; in order to cover the most unfavorable conditions, it is usually drawn between the two ends of the pipe line, and not from surface to surface of the water in the two reservoirs, as the level of these surfaces may vary. The slope of the grade line will be represented by $\frac{H}{L}$. *In order*

that the discharge may take place under the full head, the pipe line must never rise above the grade line at any point.

Should the pipe rise above this grade line, as is shown at b, Fig. 23, the rate of slope is no longer $\frac{H}{L}$ through the entire pipe line, but it is broken into two others at the point b, one $\frac{h}{l}$ flatter and the other $\frac{h'}{l'}$ steeper than $\frac{H}{L}$. If the pipe were of the same diameter throughout, it would not discharge as much water as if it were kept entirely under the hydraulic grade line $a\,c$, because its flow would be governed by the flatter hydraulic grade line $a\,b$. From b to c the water would flow without completely filling the pipe. Sometimes, when a rocky ridge must be crossed, where it would be very difficult and expensive to keep the pipe low enough, two diameters are used; the larger one being laid

FIG. 23

between a and b and the smaller between b and c. By properly proportioning the diameters to the grades, according to the rules for the flow of water through pipes, the desired discharge can be economically secured in this way.

FLOW OF WATER THROUGH LONG PIPES

29. When comparing the length of pipe with head or pressure, the diameter of the pipe and the nature of its interior surface so entirely overshadow the mere height of fall, which is the only factor considered in the formula $v = \sqrt{2\,g\,h}$, that,

finally, the formula itself fades completely out of the problem. The only trustworthy knowledge of the velocity of flow and consequent volume of discharge through pipes of different diameters and under different circumstances rests wholly on direct experiment.

DARCY'S FORMULAS

30. The French engineer Darcy made a series of experiments with pipes of different diameters, from which he formulated certain algebraic expressions, which have remained standard. It was found by these experiments that the character of its interior surface affected to a remarkable degree the velocity of the water flowing through a pipe. The amount of water flowing with a given head through a clean, smooth pipe of given diameter and length was surprisingly diminished when another pipe, exactly similar, except having a rough and dirty interior surface, was substituted. The degree of reduction in this case was surprising because it had been supposed that the small projections caused by the roughness of the surface would at most only affect the flow by diminishing to that extent the inside diameter of the pipe. This would be the case if water were a perfect fluid, for then some of the particles of water would simply level up the irregularities of the surface and the other particles would flow freely over them. Water, however, is very far removed from a perfect fluid. It possesses the property of *viscosity* to a great degree, and the particles of which it is composed, instead of moving freely over one another, are held together by molecular attraction, and it requires considerable force to tear them apart. For this reason, the term "friction" is misapplied when used to express the resistance experienced by water in flowing over a rough surface. It is really a resistance to *shearing* that takes place.

31. It has been found within the extreme limits of roughness and smoothness that exist in practice that if a smooth pipe of given diameter discharges a certain quantity of water per second, a rough pipe, otherwise similar, will

require a diameter 15 per cent. greater to discharge the same amount in the same time. Thus, if the smooth pipe had a diameter of 36 inches, the rough pipe would require one of 41.40 inches to have an equal delivery. Did not this fact rest on actual experience, it would seem incredible that irregularities amounting to only a fraction of 1 per cent. of the diameter of a pipe could affect the flow to such an extent. It is explainable, however, the moment the great *viscosity* of water is realized.

These facts led Darcy to divide cast-iron water pipes into the two classes already mentioned, "smooth" and "rough," the formula for the flow through each being modified by an appropriate coefficient. The cleanest and best-conditioned pipes will not give a greater discharge than that assigned to them by the coefficient for smooth pipes, nor will the greatest amount of roughness, from the incrustations to which pipes are liable in practice, reduce the flow below that for rough pipes, although it frequently approaches it closely.

32. Fundamental Formula.—Darcy's formula for long pipes, by which is understood pipes of 1,000 diameters and over in length, is

$$\frac{D H}{C L V^2} = 1. \qquad \textbf{(15.)}$$

In this formula,

D = diameter of pipe in feet;
H = total head in feet;
L* = total length in feet;
V = velocity of efflux in feet per second;
C = an experimental coefficient.

From formula **15** we deduce

$$V = \sqrt{\frac{D H}{C L}}. \qquad \textbf{(16.)}$$

* Although L is, properly speaking, the actual length of the pipe, it differs in practice so little from its horizontal projection that the latter is taken as being, in general, a sufficiently close approximation.

Since the quantity Q in cubic feet per second is equal to the area A of the pipe in square feet multiplied by the velocity in feet per second, we have

$$Q = A\sqrt{\frac{D\,H}{C\,L}}. \qquad (17.)$$

Since $A = .7854\ D^2$,

$$Q = .7854\ D^2\sqrt{\frac{D\,H}{C\,L}}, \qquad (18.)$$

which may be written

$$Q = \sqrt{\frac{.617\ D^5\,H}{C\,L}}. \qquad (19.)$$

33. Coefficients.—The important matter now is to know the value of C. For this Darcy gives the following table, based on his experiments:

TABLE OF COEFFICIENTS

Diameters in Inches	Value of C for Rough Pipes	Value of C for Smooth Pipes
3	.00080	.00040
4	.00076	.00038
6	.00072	.00036
8	.00068	.00034
10	.00066	.00033
12	.00066	.00033
14	.00065	.00033
16	.00064	.00032
24	.00064	.00032
30	.00063	.00032
36	.00062	.00031
48	.00062	.00031

It will be observed that the coefficient for smooth pipes is in all cases half that of rough ones. As all pipes, no matter how clean and smooth they may be when first laid, become in course of time more or less incrusted, it is safer in practice to always use the coefficient for rough pipes when a permanent system is being laid down.

34. It will be noticed from the foregoing table that the coefficients for pipes from 8 to 48 inches in diameter do not greatly vary; moreover, from formula **17, 18,** or **19,** all other conditions being equal, the quantity discharged is affected by only the square root of the coefficient, so that slight differences in its value are insignificant in reference to the volume of water discharged. Formula **19** contains the factor .617, and if .000617 be taken as an approximate coefficient for pipes within limits of 8 and 48 inches, the formula becomes

$$Q = \sqrt{\frac{.617\, D^5 H}{.000617\, L}};$$

whence,

$$Q = \sqrt{\frac{1{,}000\, D^5 H}{L}}. \qquad \textbf{(20.)}$$

If, now, for $\frac{H}{L}$, or the total head divided by the total length of pipe, the *head per thousand*, or $\frac{h}{1{,}000}$, be substituted, the above formula becomes

$$Q = \sqrt{D^5 h}, \qquad \textbf{(21.)}$$

which may be generalized thus:

$$\frac{Q^2}{h\, D^5} = 1. \qquad \textbf{(22.)}$$

In this formula it must be borne in mind that h is the fall per thousand. When logarithms are used, formulas **20** and **21** are readily solved. Otherwise, they may be more conveniently written:

$$Q = D^2 \sqrt{D h}. \qquad \textbf{(23.)}$$

$$\frac{Q}{D^2 \sqrt{D h}} = 1. \qquad \textbf{(24.)}$$

For pipes of smaller diameter, from 3 to 6 inches, .000785 is assumed as a coefficient. Then, from formula **19**,

$$Q = \sqrt{\frac{.785 \times .785\, D^5\, H}{.000785\, L}};$$

whence,

$$\frac{Q^2}{h\, D^5} = .785; \qquad \textbf{(25.)}$$

also

$$Q = .89 \sqrt{D^5 h}. \qquad \textbf{(26.)}$$

That is to say, for these smaller diameters, the delivery will be, in round numbers, about 90 per cent. of that given by formula **21**.

35. Formulas for Smooth Pipes.—While in practice the formulas for rough pipes should always be used, it is sometimes useful to know the probable discharge through smooth ones. Since the coefficients for the latter are always one-half of those for the former, for smooth pipes, formulas **21** and **22** may be written,

$$Q = \sqrt{2\, D^5 h}. \qquad \textbf{(27.)}$$

$$\frac{Q^2}{h\, D^5} = 2. \qquad \textbf{(28.)}$$

Also, from formula **27**,

$$Q = 1.40 \sqrt{D^5 h}. \qquad \textbf{(29.)}$$

That is to say, *in general, the discharge through a smooth pipe is 1.40 times that through a rough pipe of the same diameter; and reciprocally, the discharge through a rough pipe is .70 times that through a smooth one of the same diameter.* These factors represent the practical limits between which the extremes of roughness and smoothness can affect the flow through long pipes.

36. Formulas for Velocity.—Formulas for velocity may be derived from those already established.

Since velocity is equal to quantity divided by area, there is obtained from formula **23**, for rough pipes of from 8 to 48 inches diameter,

$$V = \frac{D^2 \sqrt{D h}}{.7854\, D^2};$$

whence,

$$V = 1.27 \sqrt{D h}. \qquad \textbf{(30.)}$$

For rough pipes of smaller diameter,

$$V = 1.13 \sqrt{D h}. \qquad \textbf{(31.)}$$

For smooth pipes of large diameter,

$$V = 1.78 \sqrt{D h}. \qquad \textbf{(32.)}$$

For smooth pipes of small diameter,

$$V = 1.60 \sqrt{D h}. \qquad \textbf{(33.)}$$

The ratio of the velocities will be as the quantities; hence, the general rule in Art. **35** holds good for relative velocities also.

The terms "rough" and "smooth" here, as elsewhere, signify the extremes of both cases.

EXAMPLE 1.—A rough pipe 16 inches in diameter and 3,700 feet long connects two reservoirs, the difference of elevation between the two being 187 feet. With what velocity does the water flow through the pipe?

SOLUTION.—Substituting in formula **16**,

$$V = \sqrt{\frac{\frac{4}{3} \times 187}{.00064 \times 3{,}700}} = 10.26 \text{ ft. per sec.} \quad \text{Ans.}$$

EXAMPLE 2.—What is the velocity through the pipe in example 1 calculated by formula **30**.

SOLUTION.— $V = 1.27 \sqrt{\frac{4}{3} \times 50.5} = 10.42$ ft. per sec. Ans.

NOTE.—In approximate formulas, such as all those that apply to the flow of water through pipes necessarily are, the results obtained in examples 1 and 2 are equivalent to an agreement, and in practice one might happen to be as nearly right as the other. It is obvious that when the character of the pipe may vary as to interior surface so widely, a very close result can never be hoped for, and all that can be done is to keep within probable limits.

EXAMPLE 3.—A rough iron pipe 10 inches in diameter is laid with a fall of $7\frac{1}{2}$ feet per 1,000 feet. What is the discharge?

SOLUTION.—According to formula **21**, $Q = \sqrt{D^5 h}$. Substituting the values in the above example, $\left(\frac{10}{12}\right)^5$ and $7\frac{1}{2}$ for D and h, then the formula becomes, $Q = \sqrt{\frac{3,125}{7,776} \times 7\frac{1}{2}}$. Performing the indicated operations, $\sqrt{\frac{3,125 \times 7\frac{1}{2}}{7,776}} = \sqrt{3.01} = 1.736$, which is the discharge in cubic feet per second. Ans.

EXAMPLE 4.—It is desired to discharge 3 cubic feet per second from a pipe line having a fall of 5 feet per 1,000 feet. What diameter of rough cast-iron pipe will be required?

SOLUTION.—Insert the data given in formula **22.** Then, $D = \sqrt[5]{\frac{9}{5}} = \sqrt[5]{1.8} = 1.125$ feet, or $13\frac{1}{2}$ inches, diameter. Ans.

As cast-iron pipes are made to certain sizes, and there are no half inches, the nearest approach to the size would be a pipe 14 inches in diameter. It is usual for hydraulic engineers to provide themselves with tables, for the purpose of working such intricate examples as occur in this subject. Some, however, extract the roots by logarithms. Others provide themselves with tables containing fifth roots and their corresponding numbers. The extraction of the fifth root of numbers is very long and tedious, and the student is referred to the *Arithmetic* for the method of solving such examples.

EXAMPLE 5.—It is desired to discharge $\frac{1}{4}$ cubic foot per second from a 4-inch pipe. What head per 1,000 is necessary to accomplish this?

SOLUTION.—Substituting the data in formula **25**,

$$h = \frac{\frac{1}{4}}{.785 \times \frac{1}{243}} = 77.39 \text{ ft.} \quad \text{Ans.}$$

37. General Relations Between D, Q, L, H, and C, and D', Q', L', H', and C'.—From formula **15** we have for a given pipe line

$$\frac{D\,H}{C\,L\,V^2} = 1$$

For any other system

$$\frac{D' H'}{C' L' V'^2} = 1 \text{ and } \frac{D H C' L' V'^2}{D' H' C L V^2} = 1.$$

C and C' will generally be sufficiently near each other to be negligible; hence,

$$\frac{D H L' V'^2}{D' H' L V^2} = 1. \qquad \textbf{(34.)}$$

Also, from formula **19**, there results

$$\frac{Q^2 L}{D^5 H} = \frac{.617}{C} \text{ and } \frac{Q'^2 L'}{D'^5 H'} = \frac{.617}{C'}.$$

Then, letting $C = C'$,

$$\frac{Q^2 L D'^5 H'}{Q'^2 L' D^5 H} = 1. \qquad \textbf{(35.)}$$

EXAMPLE.—A pipe 16 inches in diameter, 3,700 feet long, with a total fall of 187 feet, has a velocity of 10.26 feet per second. Another pipe has exactly the same elements, except that its diameter is 18 inches. What is its velocity ?

SOLUTION.—Let the elements of the first pipe be D, H, L, and V, and those of the second, D', H', L', and V'. By the conditions given, $H = H'$ and $L = L'$. Then, in formula **34**, $\frac{D V'^2}{D' V^2} = 1$, and $V' = V\sqrt{\frac{D'}{D}}$.

Substituting the data,

$$V' = 10.26\sqrt{\frac{1.50}{1.33}} = 10.83 \text{ ft. per sec. Ans.}$$

38. From formula **35**,

$$Q' = \sqrt{\frac{Q^2 L D'^5 H'}{L' D^5 H}}.$$

If L and H equal respectively L' and H',

then $$\frac{Q'}{Q} = \sqrt{\frac{D'^5}{D^5}}. \qquad \textbf{(36.)}$$

That is, other elements being equal, the quantities discharged are as the square roots of the fifth powers of the diameters. This is a very important relation.

EXAMPLE 1.—A long pipe 24 inches in diameter gives a discharge of 2 cubic feet per second. What will be the discharge of a pipe under similar circumstances 30 inches in diameter?

SOLUTION.—By multiplying formula **36** by Q and factoring Q', which represents the quantity of water that will be discharged, is obtained in the form

$$Q' = Q\sqrt{\frac{D'^5}{D^5}}.$$

By substitution,

$$Q' = 2\sqrt{\frac{(2.5)^5}{(2)^5}} = 2\sqrt{\frac{(2.5)^5}{(2)^5}} = 2\sqrt{(1.25)^5} = 2\sqrt{3.052}$$

$= 1.747 \times 2 = 3.494$ cubic feet discharged per second. Ans.

EXAMPLE 2.—A 24-inch pipe discharges 2 cubic feet per second. What diameter pipe, with the same length and head, will be required in order to discharge 3 cubic feet per second?

SOLUTION.—The length of the pipe and the head being the same for both pipes, they may be neglected and the formula $D' = D\sqrt[5]{\frac{Q'^2}{Q^2}}$ may be obtained from formula **36** by taking D'^5 and D^5 from under the square-root sign and placing Q'^2 and Q^2 under the fifth-root sign.

Then, by the substitution of the given data, in this formula the diameter D', which represents the diameter of the pipe sought, may be found as follows:

$$D' = 2\sqrt[5]{\frac{9}{4}} = 2 \times 1.176$$

$= 2.352$ feet in diameter, or $12'' \times 2.352' = 28.224$ inches.

This would probably be taken in practice as 28 inches, for the next regular size of cast-iron pipe is 30 inches, which is much larger than needed. Ans.

The difference in area between 28-inch and 30-inch diameter pipes is 91 square inches, or nearly $\frac{2}{3}$ of a square foot. The rules given in this treatise are only approximate, and with pipes from 6 to 24 inches in diameter are liable to vary from 5 to 15 per cent. The formulas given are the textbook standards, but as stated are not absolutely correct.

EXAMPLE 3.—A 24-inch pipe, as in example 1, discharges 2 cubic feet per second. How many 8-inch pipes will be required to give the same discharge, the heads and lengths being the same?

SOLUTION.—Let x = the required pipes. Then the number required being inversely as the quantity discharged, invert formula **36** and obtain

$$x = \sqrt{\frac{D^5}{D'^5}}. \qquad \textbf{(37.)}$$

Inserting the data, $x = \sqrt{\frac{2^5}{(\frac{2}{3})^5}} = \sqrt{3^5}$.

$$x = \sqrt{3^4 \times 3} = 3\sqrt{27} = 15.588.$$

That is to say, 16 pipes would be required, each 8 inches in diameter. Ans.

The student is again reminded that all the preceding formulas apply to *long pipes* only; i. e., those of at least 1,000 diameters in length.

FLOW OF WATER THROUGH SHORT PIPES

39. All that precedes refers to the flow of water through long, rough pipes, where only the head necessary to maintain the flow against the interior resistance of the pipe has been taken into account. In such pipes, the additional head necessary to overcome resistance to entry into the pipe and that necessary to produce the velocity of flow are so insignificant in comparison with the

FIG. 24

so-called friction head that they are neglected as unnecessarily complicating the formulas.

In short pipes, however, the case is quite different, and the velocity and entrance heads must be taken into account. For this purpose take the resistance head at about one-half the velocity head.

Suppose, for example, a reservoir, Fig. 24, is tapped by a 24-inch pipe 20 feet long, the center of which is 20 feet below the surface of the water in the reservoir. What is the discharge, using formulas for rough pipe and ignoring the modifying action of the reducers shown in the figure ?

What is wanted here is the velocity of efflux, which can be obtained in the following manner:

The total head, 20 feet, is made up of the velocity head, the entrance head, and the frictional head. Call the velocity head x, and the entrance head will then be $\frac{x}{2}$. The frictional head call y. Then, $\frac{3\,x}{2} + y = 20$. The velocity head is that required by the law of falling bodies, $x = \frac{v^2}{2\,g}$. The velocity and entrance heads together are, therefore, $\frac{3\,v^2}{4\,g}$.

From formula **30**,

$$v = 1.27\sqrt{D \times \frac{1{,}000\,y}{L}},$$

where h is replaced by its value $\frac{1{,}000\,y}{L}$. Substituting the given data,

$$v = 1.27\sqrt{2 \times \frac{1{,}000\,y}{20}} \text{ and } y = \frac{v^2}{161}.$$

Therefore (neglecting small decimals), $v^2\left(\frac{3}{129} + \frac{1}{161}\right) = 20$. $v^2\left(\frac{1}{43} + \frac{1}{161}\right) = 20$. $v^2 = \frac{6{,}923}{204} \times 20 = 678.7$. $v = 26.05$. Area of 2-foot pipe $= 3.1416$.

Then the discharge is $Q = 26.05 \times 3.1416 = 81.84$ cubic feet per second.

40. The formula for finding the diameter of a short pipe to convey a given quantity of water with a given head is derived from the general formula as follows:

Solving the form of formula **30** given in the last article for y, $y = \frac{v^2 L}{1612.9 D}$; this substituted in the expression for the total head, $H = \frac{3 v^2}{4 g} + y$, gives $H = \frac{3 v^2}{4 g} + \frac{v^2 L}{1612.9 D}$

Substituting for v its value $\frac{Q}{.7854 D^2}$, and reducing,

$$H = \frac{Q^2}{26.45 D^4} + \frac{Q^2 L}{995 D^5},$$

from which

$$D = .251 \sqrt[5]{\frac{Q^2}{H}(37.6 D + L)}.$$

To use this formula, first assume a value for the D under the radical sign and solve, thus finding an approximate value for D. Then substitute this new value for the D under the radical and solve again, and if the new value of D agrees closely with the first approximation, the next larger commercial size may be taken as the required size of pipe. If, however, the second value differs greatly from the first approximation, it may be substituted for the D under the radical and a new value can thus be found. One or two approximations of this kind will usually give a value of D that will enable one to select the commercial size nearest to the theoretical diameter.

EXAMPLE.—What diameter of pipe must be used in order to draw 17.22 cubic feet of water per second from a reservoir if the total head is 20 feet and the pipe is 20 feet long?

SOLUTION.—Assuming for D a value of 16 inches = 1.33 feet and substituting it for the D under the radical in the last formula,

$$D = .251 \sqrt[5]{\frac{17.22^2}{20}(37.6 \times 1.33 + 20)} = 1.0067 \text{ feet, say 1 foot.}$$

Substituting this value under the radical,

$$D = 251\sqrt[5]{\frac{17.22^2}{20}(37.6 \times 1 + 20)} = .968 \text{ foot.}$$

Since this value is so near that of the first approximation, it is plain that the required diameter is 1 ft. Ans.

OTHER LOSSES OF HEAD

41. Besides the losses of head that have been considered, there are other minor ones, such as those occasioned by bends, changes of grade, or by passing from one diameter to another. In general, any change whatever in a pipe line produces some loss of head, but all such as occur in practice are so insignificant in comparison with the loss of head from interior surface resistance that no account is taken of them. In practice, changes of horizontal direction, when at all pronounced, are effected by special castings called *bends*,

FIG. 25

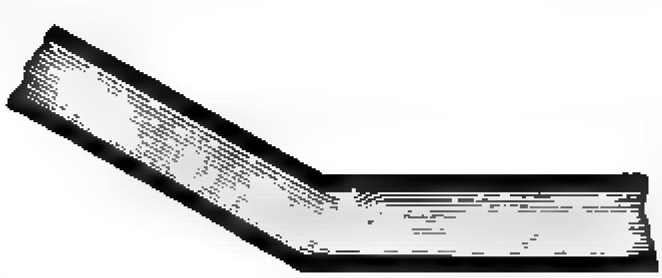


FIG. 26

FIG. 27

which effect the change with very little loss of head; and changes of diameter are made through other special castings, called *reducers*, tapering in form, so as to mold the stream of water into the proper shape for entering into the pipe of different diameter. Moreover, since water pipes are always cast to even sizes, when calculation calls for a fractional diameter, as it almost always does, the next larger size of even inches is taken, and this is generally more than enough

to cover all the small losses that can occur from the foregoing causes.

42. When bends and elbows are necessary, they should be as large as circumstances will permit, so as to change the direction gradually. Sudden changes in direction destroy the velocity very rapidly, and, consequently, reduce the discharge. A reduction or increase in the size of the pipe, owing to the screwing on of branch pipes smaller or larger than the main pipe, also reduces the velocity.

43. When bends are necessary, it is better to round them, as shown in Fig. 25, than to have a sharp bend, as shown in Fig. 26. A bend at right angles, as shown in Fig. 27, is very destructive to the velocity. A rounded elbow, as shown in Fig. 28, should be used, in which the radius should be made as large as possible.

FIG. 28

44. Wooden Pipes.—For moderate heads, wooden-stave pipes are commonly used. They are practicable for any desired head, but are only economical to the point where the pressure necessitates such close banding that the cost exceeds that of iron or steel pipe of the same strength. If kept full of water, the stave pipe will last indefinitely, provided the bands are protected from rust by a coating of asphaltum or mineral paint. The amount of iron in the bands for each foot of pipe is the same as that required for a foot of sheet-iron pipe of the same diameter calculated to withstand the same head, or pressure, with a considerable margin of safety. Fig. 29 illustrates wooden-stave pipe in which the bands are composed of round steel rods. One advantage of wooden-stave pipe is that it can be made to conform to the irregularities of the ground more easily than is the case with iron pipe.

45. Wooden Lining for Tunnels.—On some extensive ditch lines it was necessary to carry water through

Fig. 29

tunnels, and owing to the fact that the irregular rock lining of the tunnel interfered considerably with the flow of the

water, it was found best to line the tunnels with timber. This was done by building a wooden pipe inside the tunnel; the pipe being backed with cement, no bands were necessary. In fact, it was simply a wooden lining for the tunnel. Where such linings are used, the tunnels are sometimes driven below the water level of the ditch, so that they really become inverted siphons, and in case the water should be turned off in the ditch, the tunnel would always be filled, and hence there would be no tendency for the lining to dry out and warp.

46. Iron Pipes.—Wrought-iron or steel pipes are exclusively used for high heads. For low heads, either wood or iron may be employed, the choice between them being a matter of location and cost. Pipes are used as water conduits for replacing ditches or flumes, as supply pipes for passing water from the pressure box to the claim, and as distributing pipes taking water from the gates at the end of the supply-pipe line to deliver it to the Giants or nozzles. Pipes used for carrying water across depressions and placed so as to follow the natural surface of the ground are called *inverted siphons*. The thickness of the metal for pipes is determined by the pressure of the water and the diameter of the pipe. The pipe when put together soon becomes water-tight from the foreign matter in the water. This caulking may be hastened by throwing in a few bags of sawdust. Pipes thus rendered water-tight will resist a pressure as great as 200 pounds per square inch. In the Texas pipe line, Nevada County, California, there is an inverted siphon pipe 17 inches in diameter, 4,438.7 feet long, constructed of riveted sheet iron. The maximum head is 770 feet, which is equivalent to a pressure of 334 pounds per square inch on the pipe at its lowest point.

47. Joints.—Ordinarily, conduit pipes vary from 11 to 40 inches in diameter and are constructed of sheet iron or steel varying in thickness from No. 8 to No. 14 or 16 (Birmingham wire gauge). The sheets are riveted together into sections of from 30 to 36 inches in length, and these in

turn into lengths of from 20 to 30 feet or into convenient lengths for transportation. These longer pieces may be put together by a number of different devices Sometimes they are simply put together stovepipe fashion, neither rivets, wire, nor any other contrivance being necessary to hold the joint in place. Where there is great pressure, iron collars or lead joints are frequently used. Fig. 30 (*a*) shows this style of joint as it is frequently used; *f* is a wrought-iron collar about 5 inches in width and $\frac{1}{16}$ inch thicker than the pipe iron. The inside diameter of this collar is $\frac{3}{8}$ of an inch greater than the outside diameter of the pipe; *l* is a joint composed of lead, which is run in between the collar *f* and the pipe and then calked tight from both sides; *n* is a nipple about 6 inches in length, which is riveted in one of the sections by means of $\frac{3}{8}$-inch rivets. Sometimes, owing to expansion and contraction of the pipe, the lead in the joint has a tendency to work out, and to replace this lead or force it back into the joint, the clamp shown in Fig. 30 (*b*) has been devised.

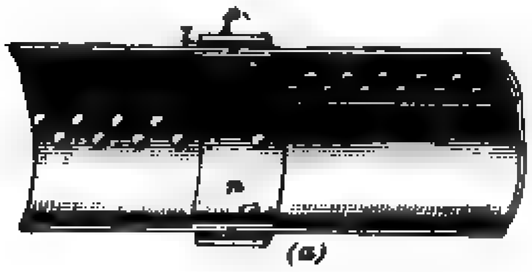


Fig. 30

At *a* is shown the clamp and its method of application for forcing back the lead that is worked out. The clamp is shown both in side view and in cross-section. At the lower part of Fig. 30 (*b*) will be seen another clamp *b*, which is driven over the joint to keep the lead in place after it has been forced in by means of the clamp *a*.

48. Sometimes wrought-iron pipes are provided with hooks, which are riveted near the ends of the pipe and are fastened together by winding wire about the hooks on the

adjacent lengths of pipe, thus counteracting the tendency that the pipes have to work apart, owing to expansion and contraction.

49. Elbows.—Sharp bends should always be avoided in pipe lines when possible, and all turns should be made by

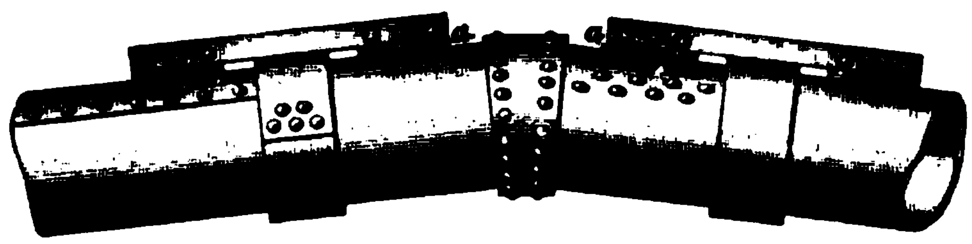


FIG. 31

gradually bending the pipe, if this can be accomplished. When short curves are necessary, elbows similar to that shown in Fig. 31 may be employed. In this case, *a*, *a* are the angle irons riveted on to the elbow and connected by straps to similar angle irons riveted on to the adjacent sections of pipe, as shown in the illustration. These angle irons and straps are necessary to prevent the pipe from pulling apart at this point, owing to expansion and contraction.

50. Air and Blow-Off Valves.—Blow-off valves are provided to allow the escape of air while the pipes are being filled; also to prevent the formation of a vacuum and the consequent collapse of the pipe, which might occur in case of a break. The simplest form is a loaded flap valve of leather on the inside of the pipe, arranged to cover a hole from 1 to 4 inches in diameter. A very simple automatic valve is shown in Fig. 32, which consists of a small chamber above the pipe, in which hangs an inverted bell or cylinder *a*, closed at

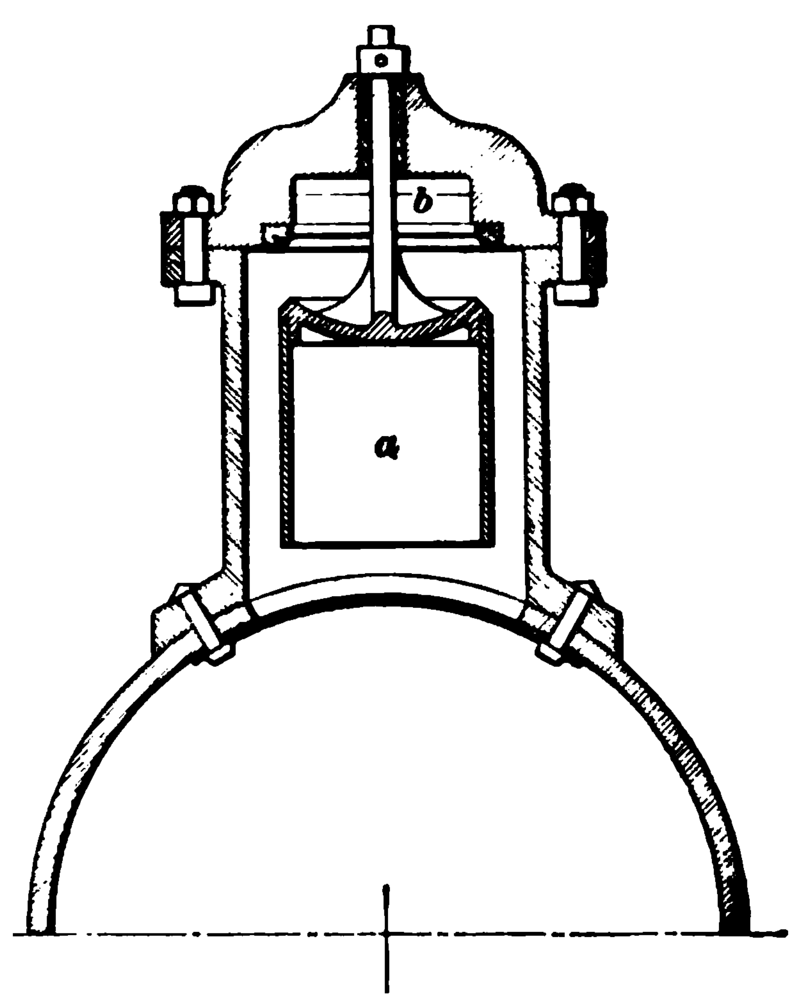


FIG. 32

the top. When simply air is escaping, this cylinder will remain in the position shown in the illustration, owing to its own weight, but as soon as water rises into the chamber, air will be trapped under the bell, causing it to float up and seat against the top of the chamber, thus closing the opening *b*, and hence preventing the escape of the water. Should the flow of water cease, the bell will immediately fall and air will enter through the opening *b*, thus protecting the pipe from collapse. Fig. 33 shows a form of blow-off, or drain, valve used at low points along the line for emptying the pipe. Fig. 34 shows a combination automatic blow-off and vacuum valve, which is employed at high points in the pipe line. The valve on the right, Fig. 34, is kept closed when the pipe is full and the valve immediately over the pipe open. The pressure in the horizontal tube will keep the central valve closed. In case any small amount of air does collect in the pipe, it can be easily discharged by opening the small valve at the right. Now, if a break should occur anywhere in the pipe line and a vacuum result at the upper point, the central valve would fall of its own weight, thus admitting air and

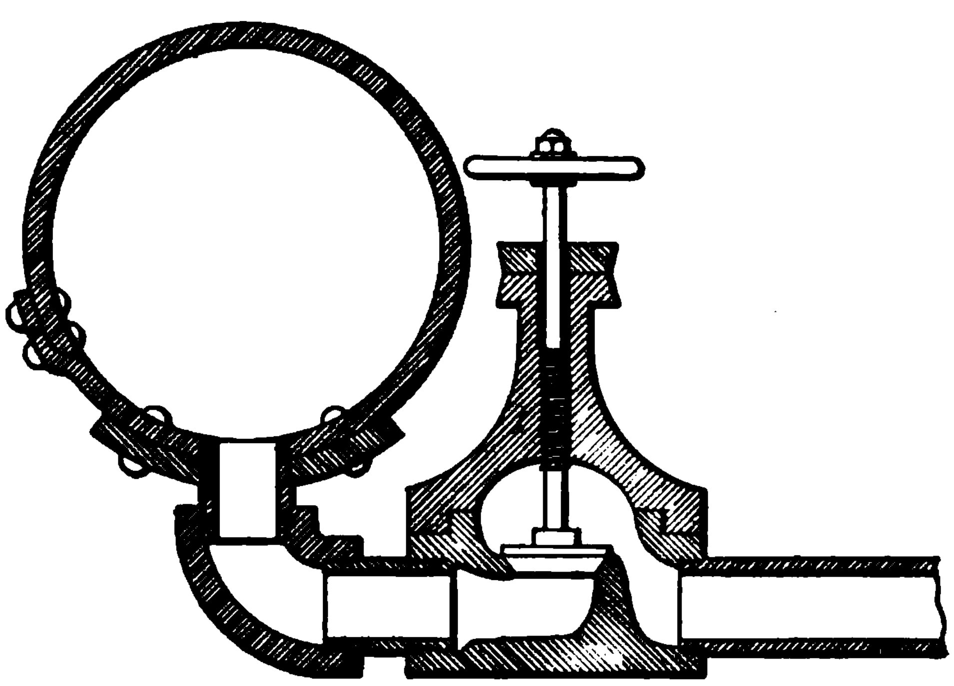


FIG. 33

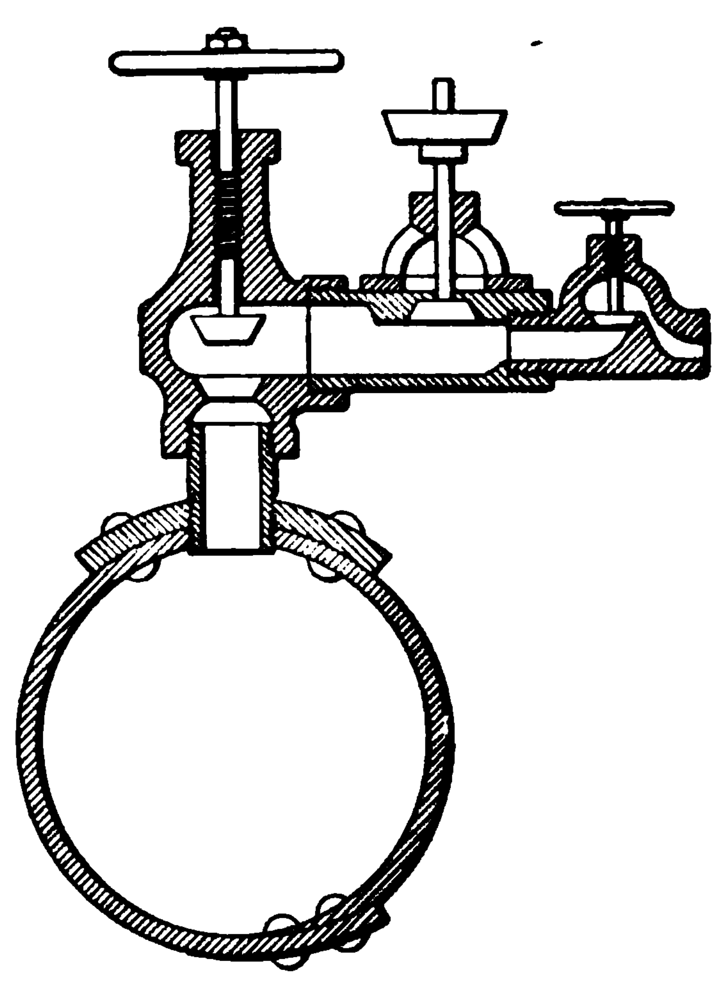


FIG. 34

preventing the collapse of the pipe. On refilling the pipe, this valve, being open, allows the air to escape, and when properly constructed will close on being reached by the water. This latter effect may be accomplished either by making the lower part of the valve so that it will trap some air and float up, or by shaping the upper disk properly, the escaping water will strike it and lift it high enough so that the current can catch and close it.

51. Laying Pipe Lines.—To preserve iron pipe, it should be laid in a trench and covered with earth to a depth of at least 1 foot. Wooden pipe should be painted on the outside with the same mixture that is used for covering the bands. Iron pipes should be covered inside and out with asphalt or coal tar. Such pipes well coated have been found in good condition after 15 years of continuous service.

The following mixtures have been found to give good results for this purpose: Crude asphalt, 28 per cent.; coal tar (free from oily matter), 72 per cent.; or refined asphalt, 16.5 per cent.; coal tar (free from oily matter), 83.5 per cent.

To prepare either of these, the asphalt is broken into small pieces and heated with the coal tar to a temperature of about 400° F. and well stirred. The pipe to be coated is dried and immersed in this mixture, where it should be allowed to remain until it acquires the temperature of the bath. When coated, it is removed and placed on trestles to drip and dry in the sun and air. For convenience in immersing, wrought-iron troughs of such a size that they will conveniently contain one section of pipe are provided.

52. Filling Pipes.—Pipes should be filled in such a manner as to prevent, as far as possible, the admission of air, which will be drawn in with the water in surprising quantities unless care is taken. The best plan is to put a gate in the pipe below the intake and thus regulate the flow and maintain a steady pressure. Where pipes that convey water to mills are supplied from flumes, a penstock, or

pressure box, as illustrated in Fig. 35, is necessary. A grating should be so placed in the flume as to catch all rubbish before the water passes into the penstock. The surplus water should be allowed to flow over a weir and escape, as shown in the illustration. The water in the pressure box should be sufficiently deep and quiet to prevent air from being carried into the pipe. To accomplish this, the box is frequently made of two parts, the water flowing from the flume into one and from it into the other through a grating or partition provided with small holes. As the water coming through ditches almost invariably carries more or less sand with it, and as this would be liable to cut and scour the inside of the pipe, it is important that it should settle before the water enters the pipe. This is accomplished by means of a sand box, which may be constructed in connection with the pressure box or at a point in the flume above the pressure box. The sand box is simply an enlargement in the flume, so arranged that the velocity of the current is reduced and the sand allowed to settle on the bottom of the box, where it accumulates and from where it is occasionally flushed out by means of a gate near the bottom of the box. Sometimes pressure boxes are made large and provided with a chamber below the intake pipe, it being intended that the sand shall accumulate in this chamber and be removed from there periodically.

FIG. 35

53. Supply, or Feed, Pipes.—Water is conveyed in iron pipes from the pressure box to the mill and discharged by means of iron gates. The supply pipe is usually funnel-shaped where it connects with the pressure box, and from there it is usually of a uniform diameter. Where pipes from 22 inches to 30 inches in diameter are used, metal

lighter than No. 14 B. G. is not advisable. The main supply pipe should descend to the mill in the most direct line possible. Sharp angles, rises, and depressions should be avoided. Air valves should be provided for the escape of the air when filling the pipe and to prevent collapse in case of a break. The pipes should be well braced and weighted at all turns or angles to prevent creeping due to expansion and contraction. In filling the supply pipe, water should be turned on gradually, for if

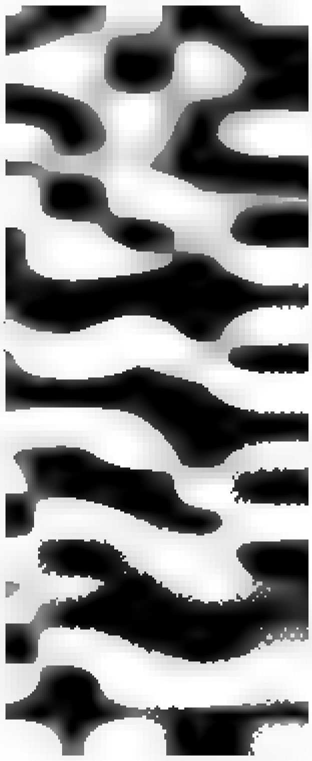


FIG 36

this is not the case, the moment the pipe becomes filled the sudden check in the flow of the water will result in a violent water hammer, which may strain the pipe badly or even burst it. Wherever it is necessary to join the supply pipe to a distributing pipe, the present practice is to fork the main pipe by means of a Y joint and to provide each branch with a gate valve similar to that shown in Fig. 36.

FLUMES

54. The use of flumes is to be avoided wherever possible in mountain regions, for long experience has demonstrated that they are liable to destruction by fire, wind, snow slides, or decay, and are expensive to maintain. There are instances where the formation of the country requires the use of flumes rather than ditches, for example, in cases where the water must be carried along the face of vertical cliffs. There are also certain kinds of rocks, independent of topography, where a ditch cannot be used as economically as a flume; for instance, when the ground is composed

either of very hard or of porous and broken material. Likewise, where the water is scarce and evaporation and absorption are great, either flumes or pipes may be advantageously used.

Flumes may be given a steeper grade than ditches, the fall frequently being as much as from 25 to 30 feet to the mile, and thus increase the velocity of the flow and permit a decrease in the cross-section of the flume.

55. Construction of Flumes.—Flumes are usually constructed of seasoned pine plank from $1\frac{1}{2}$ to 2 inches thick, from 12 to 24 inches wide, and from 12 to 16 feet long. The edge joints are battened on the inside with pine strips from 3 to 4 inches wide and $\frac{1}{2}$ inch thick. The structure is reenforced every 4 feet by a frame consisting of a sill, a cap, and two posts. A flume 4 feet wide by 3 feet deep requires $4'' \times 5''$ posts and caps and $4'' \times 6''$ sills. Posts are set into the sills with the gain $1\frac{1}{4}$ inches deep and are not mortised. The sills are allowed to extend from 12 to 20 inches beyond the posts, and diagonal braces are usually introduced, as illustrated in Fig. 37, which shows a cross-section of a flume and trestle. The posts are given sufficient length to allow a space of 3 or 4 inches between the top of the side planking and the cap. In carrying a flume along a hillside, it should be placed in close to the bank, so that snow or landslides will pass over them. For this purpose the ground is first graded, then stringers laid in place, and upon them the sills for the flume. The stringers prevent the sills coming into contact with the earth and thus protect them from rotting. Another advantage in having a flume close to the bank is that the snow usually stops up the space at the sides of the flume, thus

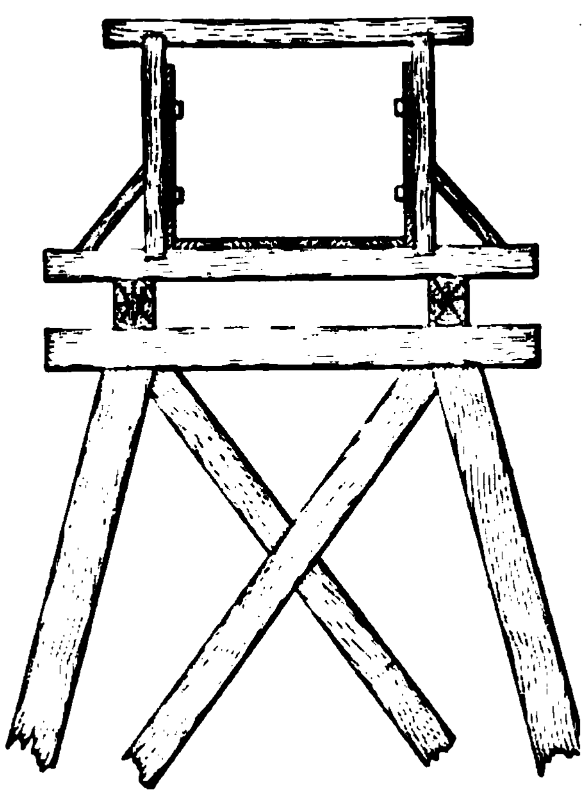


FIG. 37

preventing a circulation of cold air under it and its subsequent chilling effect.

56. Curves.—Curves should be laid with care to insure the maximum flow of water and to prevent splashing, otherwise excessive freezing is liable to occur in cold weather. For good curving, the side planks are sawed partly through in places so as to make them bend easily. Running water has a tendency to rise on the outer radius of a curve, and hence the flume must be blocked up on that side. This is usually accomplished by judging the elevation at first and changing it after the water is running by wedging up the flume until all splashing ceases.

57. Bed and Joints.—In constructing a flume, the sills when placed upon the stringers at proper intervals have bottom planks nailed to them, the end joints of the planks being carefully fitted. The side planks are nailed to the bottom plank and the posts, an occasional cap being placed on the posts to hold the flume in shape. Sixteenpenny and twentypenny nails are used for fastening the material together.

58. Connection With Ditch.—Where a flume connects with a ditch, the posts for a distance of several boxes are lengthened to permit the introduction of an additional plank on each side above the others. The end boxes of the flume are flared to permit a free entrance or discharge of the water. Where a flume passes through a bank of earth, an outer siding should be nailed on the outside of the post to protect the flume from rotting. The lumber should be prepared in exact sizes at the mill and should be delivered at the head of the flume, where water may be turned in to float the material down as the work progresses. Where trestles are used, the bents are usually placed from 8 to 12 feet apart. The life of a flume will not exceed 20 years, and as a rule is but little more than 10 years.

59. Waste Gates.—Waste gates should be placed every half mile to empty the flume whenever necessary.

In snow belts, flumes may be covered with sheds in exposed places to protect them from snowslides. If anchor ice freezes on the bottom of a flume, the water should be immediately turned out. If snow fills the flume when no water is running through it, it may be got rid of by turning on the water and flushing it out before it has time to pack.

60. Bracket Flumes.—When it becomes necessary to carry a flume along the face of a cliff at such an elevation that a trestle is practically out of the question, brackets may be used. Fig. 38 illustrates a bracket flume that was

FIG. 38

used in Butte County, California. The brackets are made of 30-pound **T** rails bent in the shape of an **L**; the longer arm (10 feet long), on which the bed of the flume rests, is placed horizontally, having the end next the cliff supported in a hole drilled in the rock. The short arm stands vertically and has in its upper end an eye into which is hooked one end of a ¾-inch round-iron rod connecting with a ring bolt soldered into a hole in the cliff above. The brackets were set 8 feet apart and were tested to stand a weight of 14½ tons. The flume is 4 feet wide and 3 feet deep, with a capacity of 3,000 miner's inch

61. Formula for the Flow of Water Through Wooden Flumes.—To ascertain the velocity of water

flowing through wooden flumes, the following formula is advanced:

$$v = \sqrt{\frac{100{,}000\, r^2 s}{6.6\, r + .46}}, \qquad \textbf{(38.)}$$

in which v = mean velocity of water in feet per second;

r = hydraulic radius = $\frac{a}{p}$;

and s = the slope $\frac{h}{l}$. (See Art. **4**, Part 2.)

Timber flumes are generally rectangular in shape and may be accurately proportioned to secure the best results. The most favorable rectangular cross-section is that in which the water has a depth equal to half the width.

EXAMPLE.—A timber flume 10 feet wide and running 5 feet deep has an inclination of 9 inches to the mile. What is its discharge per second in cubic feet?

SOLUTION.—Here, $r = \frac{50}{20} = 2.5$ and $s = \frac{.75}{5{,}280} = .000142$.

Then, $v = \sqrt{\frac{100{,}000 \times 6.25 \times .000142}{6.6 \times 2.5 + .46}} = 2.29$,

and $2.29 \times 50 = 114.5$ cu. ft. per sec. Ans.

Calculations for finding the dimensions of flumes to carry given quantities of water are difficult, since higher algebra and the solutions of equations of the sixth degree are involved. The student will not be required to work such examples. It has been thought advisable, however, to insert the method of calculating them, for the benefit of those sufficiently advanced in higher algebra. The following example will make the method of operation clear:

It is required to compute the dimensions of a wooden flume to convey 250 cubic feet of water per second with a grade of $8\frac{1}{2}$ feet per mile, the width of the flume to be twice the depth of the water flowing through it.

Let x = the depth of the water in the flume; then the width will be $2x$; the wetted perimeter, $4x$; the area of the

water cross-section, $2x^2$; and the hydraulic radius, $2x^2 \div 4x = \frac{1}{2}x$.

The slope is $8.5 \div 5{,}280 = .0016$; and, since the discharge is to be 250 cubic feet per second, the mean velocity v must be $250 \div 2x^2 = \frac{125}{x^2}$. Substituting the above terms in formula **38**,

$$\frac{125}{x^2} = \sqrt{\frac{100{,}000 \times \frac{x^2}{4} \times .0016}{6.6 \times \frac{x}{2} + .46}}.$$

Squaring,

$$\frac{15{,}625}{x^4} = \frac{100{,}000 \times \frac{x^2}{4} \times .0016}{6.6 \times \frac{x}{2} + .46},$$

from which $$x^6 - 1{,}289\,x = 179.7.$$

Assuming a depth of water of 5 feet for x and substituting this value in the left-hand member of the equation,

$$5^6 - 1{,}289 \times 5 = 15{,}625 - 6{,}445 = 9{,}280,$$

which is much greater than the second member of the equation and shows that our assumed value is too great.

Trying a value of $x = 4$, we have

$$4^6 - 1{,}289 \times 4 = 4{,}096 - 5{,}156 = -1{,}060,$$

which is less than the second member of the equation, but nearer to it than the value obtained when 5 was substituted.

Trying 4.2, we have

$$4.2^6 - 1{,}289 \times 4.2 = 5{,}489 - 5{,}413.8 = 75.2,$$

which is still less than the required quantity; but by trying 4.3, we get

$$4.3^6 - 1{,}289 \times 4.3 = 6{,}321.5 - 5{,}542.7 = 778.8,$$

which is too great. We, therefore, see that a depth of water of 4.25 feet = 4 feet 3 inches will satisfy the required condition very nearly, giving a width of flume of 8 feet 6 inches.

We will now verify the above dimensions to see if the flume will discharge the required amount of water.

The wetted perimeter is $2 \times 4\frac{1}{4} + 8\frac{1}{2} = 17$ feet, the area of the water cross-section is $8\frac{1}{2} \times 4\frac{1}{4} = 36.125$ square feet, and the hydraulic radius is $\frac{36.125}{17} = 2.125$. Substituting in formula **38,** we have

$$v = \sqrt{\frac{100{,}000 \times 2.125^2 \times .0016}{6.6 \times 2.125 + .46}} = 7.06 \text{ feet per second;}$$

therefore, the discharge will be $36.125 \times 7.06 = 255$ cubic feet per second, which satisfies the conditions of the problem very well.

62. Other Forms of Flume.—Besides the wooden flumes already described, there are some in which the lumber is cut in the form of staves and put together somewhat in the form of a semicircle. Other flumes are made of sheet iron or steel, some very large and of complicated construction. These will generally be set on iron or steel trestles, when it becomes necessary to cross depressions, and their consideration involves a knowledge of structural ironwork. In calculating the capacity of such flumes, formula **38** may be used.

TRESTLES

63. Trestles are sometimes used in hydraulic engineering to carry the flumes over depressions. It will rarely be found expedient to carry an irrigation flume or a pipe line upon an embankment, as the almost inevitable settling of the earth seriously endangers the conduit.

Pile trestles are mostly confined to moderate heights and are, perhaps, less frequently used than *framed trestles*.

64. Pile Trestles.—In driving piles to sustain loads, it is necessary to know what weight the pile can bear without sinking farther into the ground and what weight it can bear without crushing.

To determine the first point, there are many formulas given, all more or less empirical and approximate, as must naturally be the case in such problems. The factors in these formulas are: the weight of the hammer with which the piles are driven; the height of fall of the hammer; and the **refusal** of the pile, that is, the average distance it sinks under the last few blows. As these formulas are all approximate, what is wanted is some simple rule, founded upon experience, that will give safe results. The following is based on the condition that the final set or "refusal" of the pile does not exceed 1 inch.

$$S = W H. \qquad \textbf{(39.)}$$

In this formula,

$S =$ the weight which the pile will safely bear without settlement;
$W =$ weight of hammer in same unit as S;
$H =$ height of fall of hammer in feet.

EXAMPLE.—If $W = 1$ ton and $H = 10$ feet, what load will the pile safely carry?

SOLUTION.—Substituting the data in the formula, we have

$$S = 1 \times 10 = 10 \text{ tons.}$$

If the weight of the hammer had been given in pounds, the value of S would be in pounds also.

65. Resistance to Crushing.—Although the formulas for the resistance of piles to being driven take no account of the crushing strength of the material of which the pile is composed nor of its cross-sectional area, they are generally considered as giving safe results against farther penetration and resistance to crushing. It is supposed that if the head of the pile resists the battering of the pile driver, it will also resist the permanent pressure of the quiescent load, the intensity of which is indicated by the formulas. When a pile of small section is subjected to the blows of a heavy hammer with a considerable fall, it is necessary to **ring** the head, that is, to shrink on an iron band or ring, which enormously increases its resistance to splitting and

brooming. It is evident, also, that a heavy hammer and low fall are preferable to the reverse conditions of a high fall and light hammer, because they more nearly approach the action of a quiet, permanent load.

It will be well, however, to compare the value of S, as obtained above, with that derived from multiplying the area in square inches of the pile by its safe crushing strength, as given in the tables of the crushing strength of the timber of which the pile is composed.

EXAMPLE.—What is the resistance to crushing of a round spruce pile 6 inches in diameter?

SOLUTION.—The area of the head of the pile is $6 \times 6 \times .7854 = 28.27$ square inches. The resistance to crushing of spruce may be taken as 800 pounds per square inch; hence, the resistance of the pile to crushing $= 28.27 \times 800 = 22{,}616$ lb. Ans.

If the pile had been driven as in the last example, its resistance, by formula **39**, would be $10 \times 2{,}000 = 20{,}000$ pounds, which would be well within its resistance to crushing.

66. Framed Trestles.—These trestles are framed so as to stand upon a sill, which should rest upon a proper

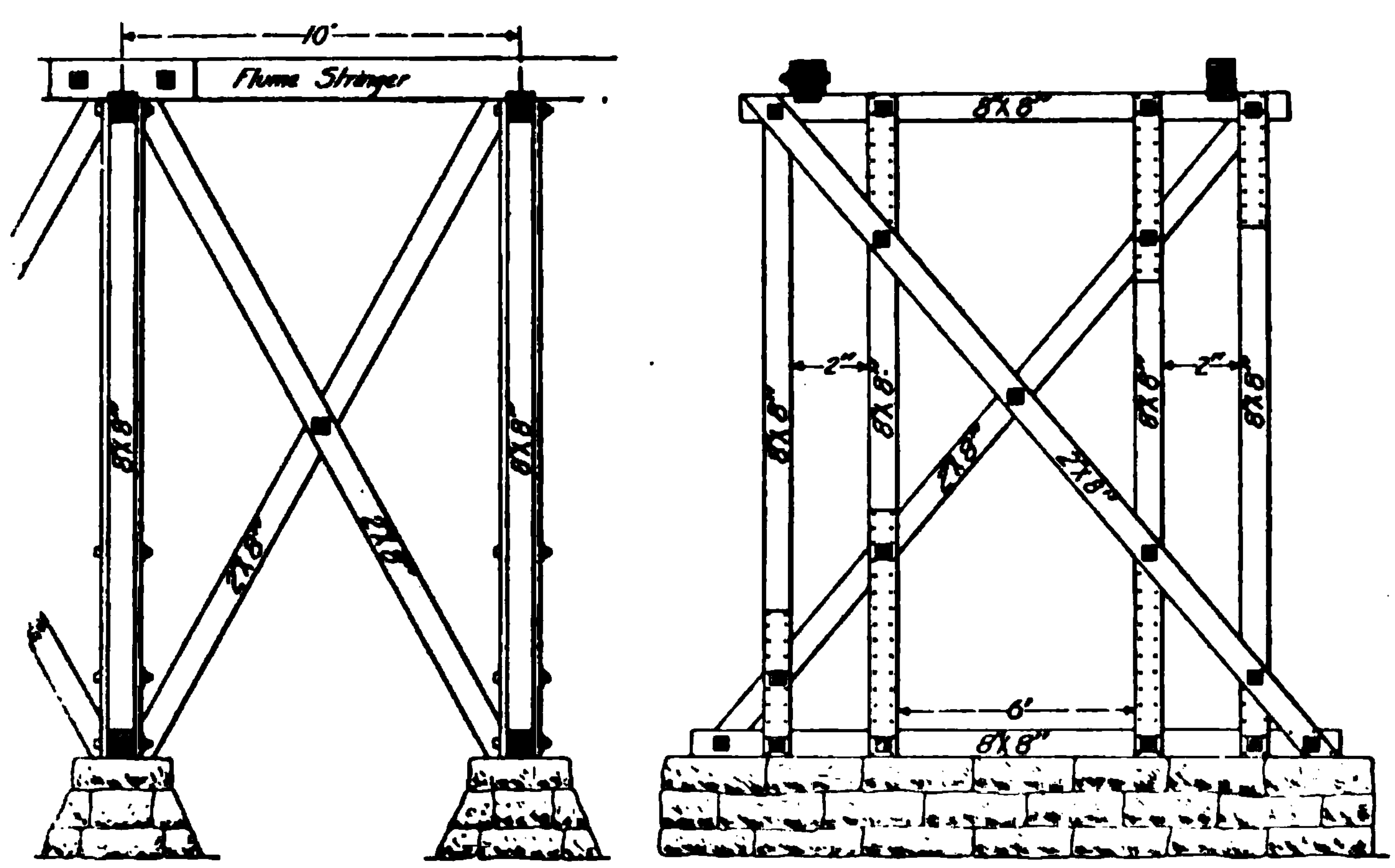


FIG. 39

foundation, either of piles or of masonry, and not directly upon the ground or on mudsills.

The guiding principle for trestles carrying a steady load should be to avoid inclined posts, mortising, and, as far as possible, different sizes of timber. Figs. 39 and 40 represent the general features of a good system of trestling for supporting flumes at moderate heights.

Fig. 40

The 8″ × 8″ posts rest upon 8″ × 8″ sills, either flush or notched in ½ inch into the sills. They are held in place by **plaster plates,** of 8″ × 2″ stuff, bolted and spiked to posts and sills. Fig. 40 shows one of the posts and sills connected in this way, drawn to a larger scale and in isometrical projection. The caps are connected with the upper ends of the posts in the same way. The posts are steadied by means of the 8″ × 2″ **X** bracing, as shown in the right-hand view in Fig 39. The two braces are bolted together at the center against an 8″ × 8″ block set between them; they are also bolted and spiked to posts, caps, and sills. These connections can be more perfectly made by first spiking the pieces together in place and then boring the bolt holes. The flume stringers are notched over the sills and are so disposed that joints will occur over the caps. These joints are secured by plaster plates bolted and spiked. The trestle is stiffened longitudinally by 8″ × 2″ **X** bracing, as shown in the left-hand view in Fig. 39, butting under the flume stringers and against the sills, and secured laterally by plaster plates, chocks, mortising, or otherwise.

This trestle will carry a 10′ × 5′ flume, and is about as light as will be perfectly satisfactory under this loading.

Fig. 39 represents a trestle suitable for moderate heights, say up to 20 feet. Beyond this height some other system must be used. The building of very high trestles, whether of wood or iron, constitutes an interesting and complex

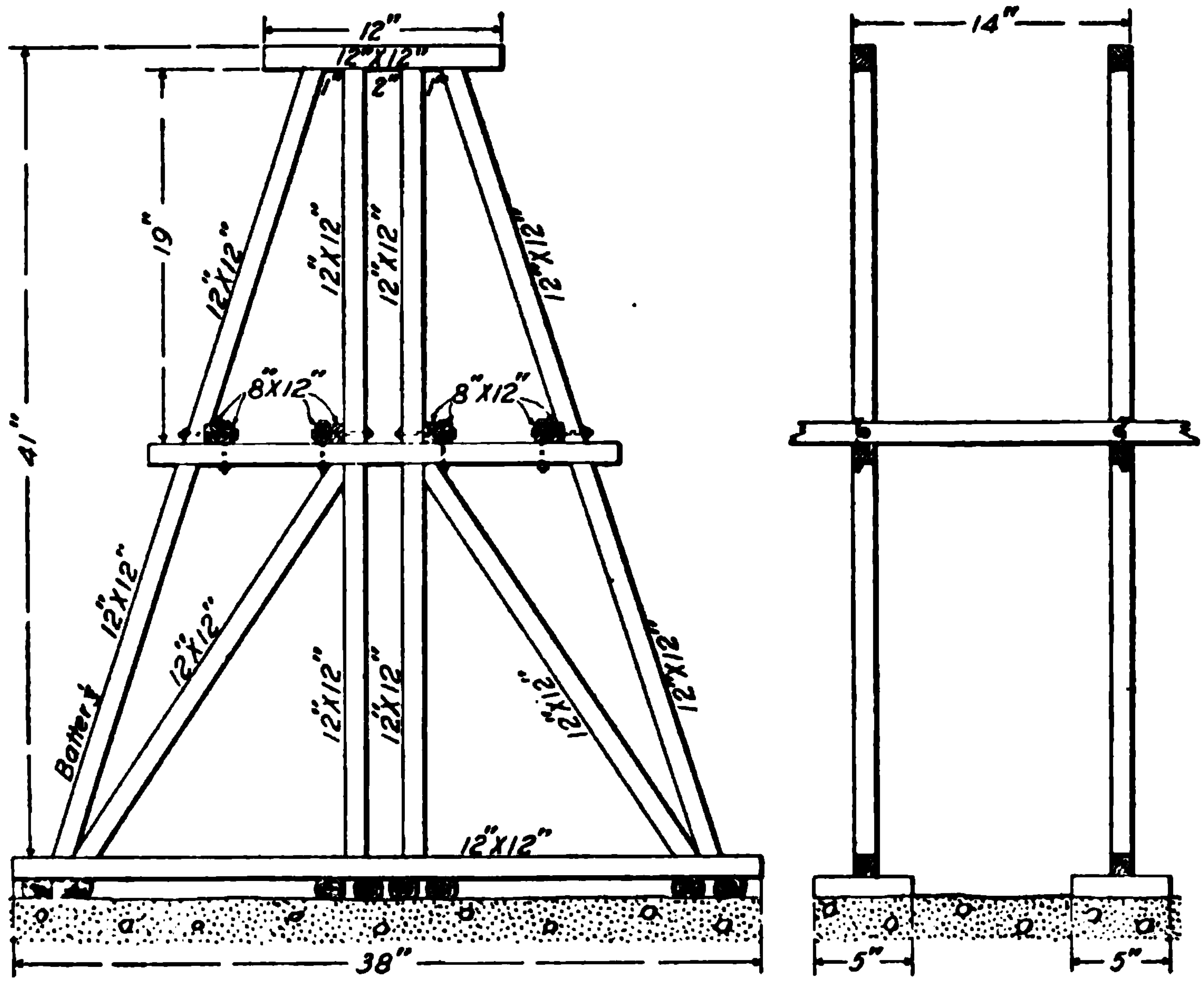


FIG. 41

subject belonging to structural engineering and cannot properly be discussed here. As an example of trestles of medium height, however, Fig. 41 shows a system of framing for a trestle 41 feet high. The dimensions are marked on the figure.

HYDRAULICS AND HYDRAULIC MACHINERY

(PART 2)

WATER SUPPLY

OPEN CANALS

1. Preliminary Work.—Surveys are necessary to the design and construction of either a canal or a pipe line; especially the former, because the course of the canal is confined to narrower limits than that of the pipe line by the topography of the country through which it passes.

The survey will begin at some point very near the stream that furnishes the water and will generally follow the same valley for a considerable distance. It will be found that it pays well to do a good deal of surveying in all such operations, because more work can frequently be done in a few days with transit, chain, and level than in many weeks with pick and shovel. It must be impressed upon the engineer that, as far as alinement is concerned, this survey does not call for any great degree of accuracy, the leveling being of much more importance. Errors are liable to occur unperceived in leveling, and no line of levels can be trusted that has not been checked. Therefore, when the alinement has been completed and leveled, check levels should be run back over the entire line; it will not be necessary, however, to verify the entire profile, a check on the benches being

sufficient. All important tributaries should also be surveyed, carrying the survey up the valley until an elevation approximately equal to that of the starting point has been reached.

A rough estimate of the length of the canal can be made from this survey, in connection with the topographical notes taken at the same time; then the approximate length, together with the total fall to be overcome, will enable the engineer to make a preliminary design of the section and grade.

A trial line for the canal can now be run. One of the best and most expeditious methods is this: Suppose a grade of 4 feet to the mile is decided on for the slope of the canal. The tangent of the angle corresponding to this slope is $\frac{4}{5280} = .00075$, which corresponds to an angle of nearly 3 minutes. Having a transit provided with a vertical limb, let the telescope be depressed to this angle and clamped. When the transit is set up, let the target of a leveling rod be set at the height of the telescope of the transit from the ground. This can be sufficiently approximated by holding the rod alongside of the transit and sighting across the wyes. Let the rod now be taken as far ahead as possible and moved along the ground, up or down hill, until the center of the target is bisected by the horizontal cross-hair of the transit. The foot of the rod is then on ground falling at the desired rate, and a plug should be driven at this point and the distance measured. The direction will be ascertained by the needle, as this will be quite sufficiently accurate. From time to time measurements will be taken to convenient stations on the line of the river survey, if one has been made, as a check. It will be well to carry this line along, following all the indentations and tributary valleys, for in this way the length and characteristics of a line following the natural surface of the ground for its entire distance will be obtained. It will be very rare that this line is actually followed by the canal, as too great development would result. Valleys will be crossed by aqueducts and promontories will be *thorough cut*, or tunneled, but only in this way can a full estimate of the comparative advantages of alternative lines be compared.

When an approximate location of the line has thus been determined, it will be accurately rerun and leveled over, so as to establish the final location and make a more nearly exact estimate of cost.

2. Grade.—The character of the bottom and sides of the canal will place certain limits on the velocity of the water, which must be great enough to prevent the deposition of silt and not so great as to do injury to the canal itself. The grade necessary to maintain the velocity within the desired limits will depend on the character of the interior surface of the canal, being very much less for one having a smooth lining—of brick, for instance—than for one merely excavated in the earth. The area of cross-section also affects the question, for the water in a large and deep canal will move with a greater velocity under a given grade than that in a smaller and shallower one having the same slope. The form of the cross-section also exerts a considerable influence upon the velocity of flow, so the question of the determination of the grade becomes a complex one, depending on the desired discharge of the canal, its character and form, and the dimensions of its cross-section.

3. General Principles Affecting the Flow of Water Through Open Channels.—"Gravity is the sole force that acts on a mass of water left to itself in a channel of any form; it produces all the motion which takes place—the inclination of the surface of the water in the channel is the immediate cause of motion, being that which enables gravity to act" (*Downing*).

It is a matter of common observation that the steeper the slope, the greater the velocity; and as this steepness is determined by the ratio of the vertical height to the distance in which it is overcome, it is evident that the accelerating force producing velocity will be expressed by the ratio $\frac{h}{l}$, in which $h =$ the difference of level between the two extremities of the canal and $l =$ the distance, usually measured horizontally, separating the two.

If there were no resistance to the flow of water running through the canal, the constant accelerating force would cause the velocity to go on increasing indefinitely. But observation shows that water under these circumstances very soon acquires a constant velocity, provided the ratio $\frac{h}{l}$ remains constant. It is evident, therefore, that there are resistances at work which increase in intensity with the increase of velocity, so that after a certain time the increasing resistance just equals the increasing acceleration, and the velocity then becomes constant. This constant velocity is sometimes known as the **permanent regimen** of the canal.

4. Resistance to the Flow of Water Through Conduits. — The laws governing the resistance to the passage of water over the interior surface of a conduit are almost directly opposite to those governing the resistance of friction when one solid body slides over another.

The laws governing the flow of water that have the most important bearing on the subject now under consideration may be briefly expressed as follows:

I. *The resistance for any given velocity is proportional to the extent of the surface over which the water flows.*

II. *This resistance affects the entire volume of water flowing over the given surface, being greatest for the film in immediate contact with the surface, and becoming less and less for the films and threads more remote from that surface.*

III. *The greater the extent of the surface in contact with a given volume of water, the greater the resistance becomes; conversely, the greater the volume subject to a given resistance, the less will the velocity be affected.*

IV. *The resistance is nearly proportional to the square of the mean velocity of flow.*

V. *The resistance varies with the nature of the surface of the conduit, being greater for a rough surface and less for a smooth one.*

Let

h = difference in level between ends of the canal or any two cross-sections of the canal;

l = horizontal length of that portion of the canal included between the sections whose difference of level is h;

s = slope = ratio $\frac{h}{l}$;

a = area of water cross-section;

p = wetted perimeter;

r = hydraulic radius = ratio $\frac{a}{p}$;

c' = coefficient depending on the nature of the surface of conduit;

v = the mean velocity of flow.

Then, the laws for the resistance to flow may be expressed by the relation $h\,a = c'\,l\,p\,v^2$, from which we have the formula

$$v = \sqrt{\frac{h}{c'\,l} \times \frac{a}{p}} = \sqrt{\frac{1}{c'} \times s \times r}. \qquad \textbf{(1.)}$$

By replacing the factor $\sqrt{\frac{1}{c'}}$ in this formula by an equivalent factor c, such that $\sqrt{\frac{1}{c'}} = c$, then $v = c\sqrt{r\,s}$.

5. Importance of the Hydraulic Radius.—It is evident from formula **1** that the velocity increases with the hydraulic mean radius $r = \frac{a}{p}$, and that therefore the most favorable shape of cross-section will be the one in which a given area is enclosed by the smallest wetted perimeter. In the case of an open canal, this section would be a half circle, since the circle is that geometrical figure

which encloses the greatest area within a given perimeter. In the case of the circle, the value of the hydraulic radius is $r = \frac{a}{p} = \frac{\frac{1}{4}\pi d^2}{\pi d} = \frac{d}{4}$, and, since both the area and the wetted perimeter of a half circle are, respectively, equal to one-half of the area and wetted perimeter of a circle when running full, the ratio $\frac{a}{p}$ for the half circle is also equal to $\frac{d}{4}$.

The half-circular form of conduit is impracticable for a canal, since the form could not be constructed and maintained unless the inside were lined with brick or some other permanent material, and even then the constructional difficulties would generally render this form inadvisable, as entailing a considerable expense of labor without a corresponding economy of material. An approximation to this best form is half a regular hexagon, in which $r = \frac{D\sqrt{3}}{8} = \sqrt{\frac{3D^2}{8}}$, D being the diameter of the circumscribing circle. This form would also require a permanent revetment if it were applied to an earthen canal.

PRACTICAL FORMULAS FOR MEAN VELOCITY OF FLOW IN CONDUITS

6. Formula for Canals With Earthen Banks.—An approximate formula that may be used for canals with earthen banks in good condition is the following:

$$v = \sqrt{\frac{100{,}000\, r^2 s}{9r + 35}}, \qquad (2.)$$

in which v = mean velocity in feet per second;

r = hydraulic radius $= \frac{a}{p}$;

s = the slope $= \frac{h}{l}$.

EXAMPLE.—Let the fall in an earthen canal having the cross-section shown in Fig. 1 be 5.25 feet per mile. What is the mean velocity of flow ?

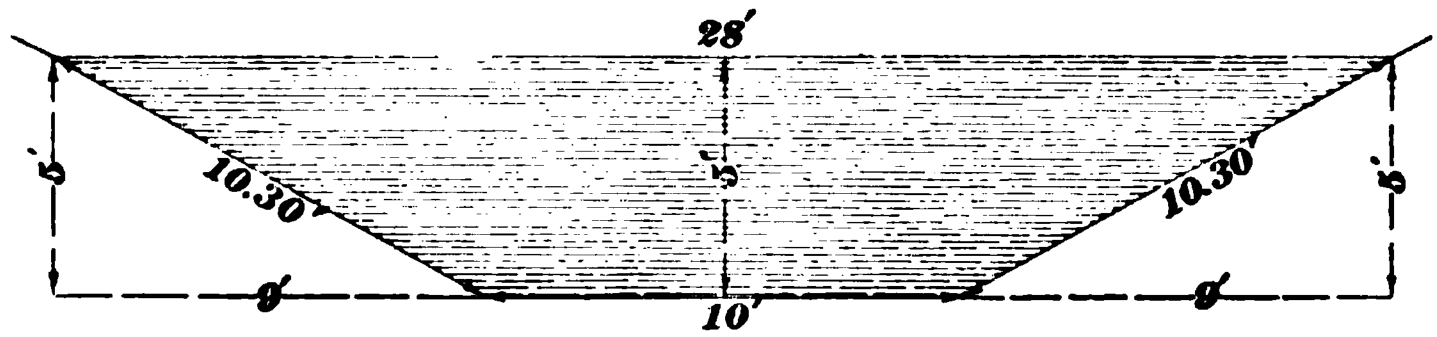


FIG. 1

SOLUTION.—Here, $s = \frac{5.25}{5,280} = .00099+$, which call .001. Also, $r = \frac{95}{30.6} = 3.105$.

Then, $v = \sqrt{\frac{100,000 \times 9.64 \times .001}{27.95 + 35}} = 3.91$ ft. per sec. Ans.

7. Limiting Velocity.—In the above example the question would be: Is the velocity, which is nearly 4 feet per second, too great for the earthen banks of the canal to resist without washing ? The answer to this question can only be given by referring to the results of experience. It has been found that light and sandy soils cannot safely resist a mean velocity greater than 2 feet per second, while at the same time this velocity is sufficient to prevent plant growth and remove silt. In firmer soil, velocities of 3 to 4 feet per second are permissible, but, except in hard pan or very resisting material, 5 feet seems to be the limiting velocity for earthen canals.

In almost any district where it is proposed to build such canals, there will be some examples of ditching, upon a greater or less scale, by observing which an approximate idea may be formed of the proper grade and side slopes to be given to the proposed canal.

8. Practical Considerations Limiting the Choice of Form of Cross-Section.—Besides the velocity there are other considerations that influence the choice of form of the cross-section of a canal. A certain ratio of side slope

will generally be necessary, according to the nature of the soil, and a certain depth will generally be found more convenient or desirable than another. The following illustrative example will make this plain:

It is desired to establish the proper cross-section and grade of an earthen canal under the following circumstances: The quantity of water to be conveyed is 250 cubic feet per second. A velocity of 2 feet per second is desired. The side slopes are to have an inclination of 1 vertical to $1\frac{1}{2}$ horizontal, and a depth of 6 feet of water is desired in the canal. What should be the form and area of the cross-section and what the grade of the canal?

Since the velocity is to be 2 feet per second and the discharge 250 cubic feet per second, the area of cross-section must be $\frac{250}{2} = 125$ square feet.

To determine the form in which this area must be put, it is necessary to know the bottom width of the canal, which

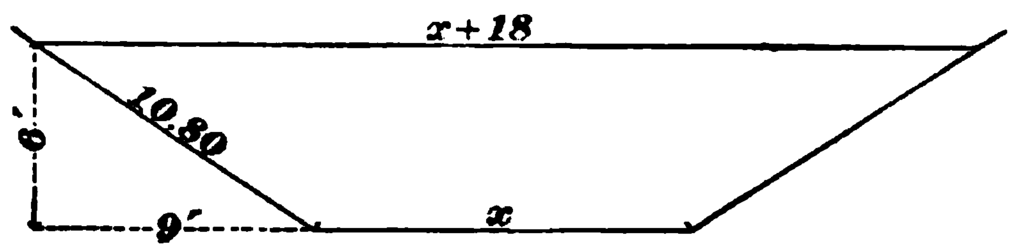


FIG. 2

is represented in Fig. 2 by x. From the data we have

$$6\,\frac{(x + x + 18)}{2} = 125, \text{ or } 6\,(x + 9) = 125.$$

Hence, $$x = \tfrac{71}{6} = 11.83.$$

Call this 12 feet, which will give the slightly greater area of 126 square feet. Next ascertain the hydraulic radius, which (since $p = 10.80 + 10.80 + 12$) is $\frac{126}{33.6} = 3.75$.

Everything is now known but the slope s, to obtain which we insert all the data in formula **2,** thus:

$$2 = \sqrt{\frac{100{,}000 \times 14.06 \times s}{9 \times 3.75 + 35}}.$$

Squaring and solving for s,

$$4 = \frac{100{,}000 \times 14.06 \times s}{9 \times 3.75 + 35},$$

from which $$s = \frac{275}{1{,}406{,}000} = .000195.$$

This represents a grade of $.000195 \times 5{,}280 = 1.03$ feet per mile.

9. Influence of Depth on Velocity of Flow.—The depth of the canal exercises a considerable influence on the velocity of flow. Thus, in the above illustration, assume a depth of 8 feet; then, all the other data remaining the same and using the same area of 126 square feet, the bottom width should be $8\ (x + 12) = 126$, or $x = 3.75$.

The depth being 8 feet and the ratio of slope being 1 to $1\frac{1}{2}$, the length of the side slope would be 14.42 feet and the wet perimeter 32.60. Therefore, the hydraulic radius is $\frac{126}{32.60} = 3.87+$, the square of which is nearly 15.

Then, $$2 = \sqrt{\frac{100{,}000 \times 15 \times s}{9 \times 3.87 + 35}},$$

and $$s = \frac{4}{21{,}481} = .000186.$$

This represents a grade of .98 foot to the mile as against 1.03 for the previous depth.

These examples show that with a given grade and area of cross-section, the velocity becomes greater as the depth increases, because, within certain limits, the hydraulic radius increases with the increase in depth. The limit is reached when the width of the canal is equal to twice its depth. This condition is most perfectly fulfilled in the case of a semicircular cross-section, as has already been shown. The following illustrative example will be useful in making this plain:

What will be the value of s in the previous examples if the form of cross-section is a half circle whose area is 126 square feet, the velocity to remain at 2 feet per second?

The diameter of the half circle will be $\sqrt{\frac{2 \times 126}{.7854}} = 17.92$ feet, which will be the width of the canal at the surface of the water. Its depth will consequently be equal to the radius, or half the above diameter, or surface line. The hydraulic radius, as already shown, will be equal to one-fourth the diameter, $\frac{17.92}{4} = 4.48$, the square of which is 20.07.

Then, $$2 = \sqrt{\frac{100,000 \times 20.07 \times s}{9 \times 4.48 + 35}},$$

and $$s = \frac{4 \times 75.32}{2,007,000} = .00015.$$

This represents a grade of .792 foot per mile.

10. All other data being as before, what is the value of s when the cross-section is that of a semi-hexagon?

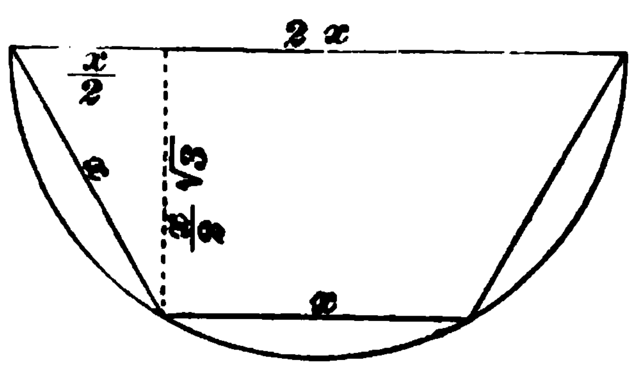


FIG. 3

Let Fig. 3 represent the semi-hexagon inscribed in a semi-circle. Since the side of a regular hexagon is equal to the radius of the circumscribed circle, the relation between the various parts shown in the figure exists. The side of the hexagon, x in the figure, is required first, and the depth, which, as will be seen, is $\frac{x}{2}\sqrt{3}$. Since the area is 126 square feet, $\frac{3x}{2} \times \frac{x}{2}\sqrt{3} = 126$, $\frac{3x^2}{4}\sqrt{3} = 126$, $\sqrt{3}x^2 = 168$, $x = \sqrt{\frac{168}{1.73}}$, $x = 9.85$.

It has been stated in Art. **5** that in the case of the semi-hexagon the hydraulic radius is $r = \frac{D\sqrt{3}}{8}$, in which $D =$ the

diameter of the circumscribing circle $= 2x$. Therefore, in the present instance, $r = \frac{19.70 \times 1.73}{8} = 4.26$,

and $$2 = \sqrt{\frac{100{,}000 \times 18.15 \times s}{9 \times 4.26 + 35}}.$$

Whence, $s = .000162$, or .85 foot per mile.

Although the last two forms of section are not adapted to unrevetted banks, their properties have been introduced here to show still more strikingly the effect of depth on velocity.

11. General Remarks on Earthen Canals.—Earthen canals, particularly in light, sandy soils, often give a great deal of trouble, even when properly side-sloped and graded, by reason of the tendency to wash; their use is, therefore, mostly confined to those very large works where the use of pipes or flumes would be out of the question. When lined with masonry they are much more efficient, and the greater velocity which they can then safely sustain, and their consequently greatly reduced cross-section, makes their relative expense, as compared with canals having unprotected interior surfaces, less than might be imagined. Sometimes cheap substitutes for masonry lining are used, and it has been found in California that a good and quite durable lining can be made by coating the sides and bottom with a plastering $\frac{3}{4}$ inch thick composed of 1 part of Portland cement and 4 parts of sand, the sides and bottom having previously been accurately trimmed and moistened.

Earthen canals are best when built entirely in excavation. It is impossible, however, to obtain this result unless the ground is exceptionally favorable and the location very carefully selected. Even then such a canal would have a greatly increased length, owing to the necessity of many deviations, in order to keep it on suitable ground. Practically, for a large proportion of their length, canals will be formed partly in excavation and partly in embankment, the

material thrown out of the excavation being used, if suitable, in the embankment. Great care must be taken to trim the banks to true lines.

The proper inclination to give to the side slopes is a point requiring very careful consideration. Very steep or very flat slopes both lead to deterioration by wash. If too steep,

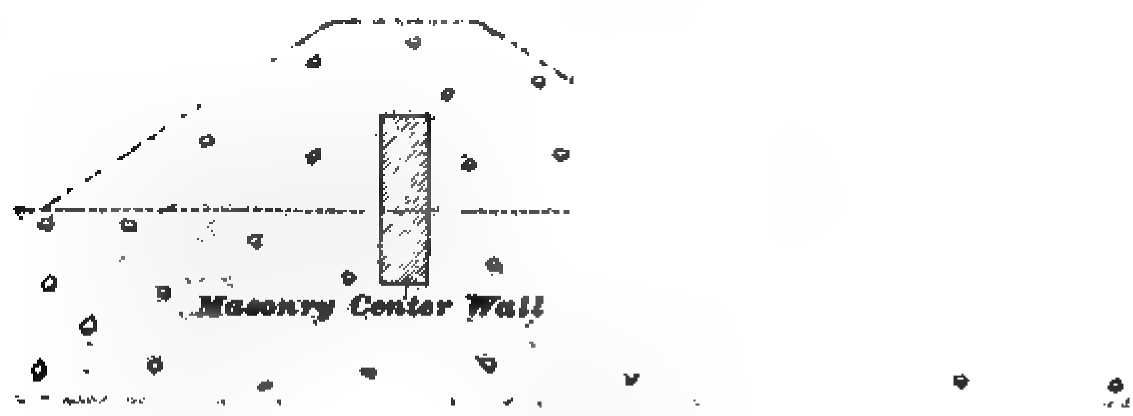


FIG. 4

they fall by the undermining effect of the flow of water in the canal, and if too flat, the exposed surfaces are damaged by rain. The best guide is a careful examination of any canals or ditches in the district.

When the embankment is high, it is better to keep a narrow berm between the foot of the bank and the edge

FIG. 5

of the ditch. When very high, a center wall should be used. Fig. 4 shows a half section in which these features have been carried out.

It has been observed in the case of canal and railroad banks and excavations that the effect of time and wash is always to reduce the original straight lines and sharp angles to curves and rounded edges. It would undoubtedly be an

advantage to anticipate this result by giving to such work at the start a form somewhat similar to that which it will eventually assume.

Thus, in Fig. 5, if the heavy straight lines represent the original form of the cross-section, it will gradually assume the shape shown in the shaded portions. It will be better, therefore, to favor this form in shaping the slopes of the excavation and embankment.

12. Canals Revetted With Dry Stone.—So much trouble is occasioned by the deterioration, slow or rapid, of canals with unprotected banks, and so much uncertainty exists regarding their probable discharge, which latter

FIG. 6

consideration frequently leads to giving them unnecessarily large dimensions or steepness of grade, that it is often good policy and economy to line them at least with dry stone. Such an arrangement is shown in Fig. 6, which represents a canal cut in sloping ground. Any stones that can be obtained will be used for the purpose, preferably of a flat form, but frequently nothing but cobblestones can be procured. It is generally best to lay the pavement continuously under the side walls and to build these upon it, as shown in the figure. Some rough hammer dressing is usually required at the corners, where the side walls connect with the pavement. In laying the pavement, if flat stones can be procured, they should all

be laid on edge, with a slight inclination down stream, and packed as closely as possible.

13. Formula for Velocity of Flow in a Canal Lined With Dry Stone.—It is difficult to adapt a formula for lined canals, because the velocity of flow will be dependent on the character of the lining. The following will be an approximately correct formula for a well-laid dry wall, without pointing or plastering:

$$v = \sqrt{\frac{100,000\, r^2 s}{8\, r + 15}}. \qquad (3.)$$

EXAMPLE.—Referring to Fig. 6, let the bottom width of a canal lined with dry stone be 8 feet, the batter of the side walls being 1 vertical to $\frac{1}{2}$ horizontal. Let the depth of water be 8 feet and the desired velocity 7 feet per second. What is the value of s?

SOLUTION.—Here the breadth of the waterway at the surface of the water is 16 feet. The area a is, therefore, $\frac{16+8}{2} \times 8 = 96$ square feet. The length of the wet line on a section of the side wall, with the given batter and depth of water, is $\sqrt{4^2 + 8^2} = 8.94$ feet, and the value of p, or the wet perimeter, is, consequently, $2 \times 8.94 + 8 = 25.88$ feet. Therefore, $r = \frac{96}{25.88} = 3.71$. Inserting these values in formula **3**,

$$7 = \sqrt{\frac{100,000 \times 13.76 \times s}{8 \times 3.71 + 15}}.$$

Whence, $s = .00159$, or 8.40 ft. per mi. Ans.

14. Formula for Canals Lined With Rubble Masonry.—The canal lined with masonry laid in cement is far more permanent in its character than those described and permits a higher velocity of flow without injury to itself; with a given grade and hydraulic radius, it offers less resistance to a rapid flow.

The formula suitable for a canal lined on the sides and bottom with a good class of rubble masonry, pointed but not plastered, is

$$v = \sqrt{\frac{100,000\, r^2 s}{7.3\, r + 6}}. \qquad (4.)$$

EXAMPLE.—In a rubble-lined canal, whose section is shown in Fig. 7, let the permissible velocity be 10 feet per second; what is the value of s?

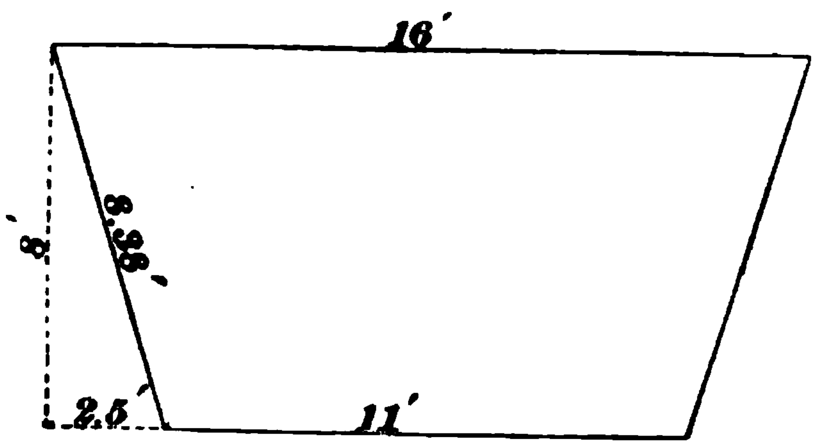


FIG. 7

SOLUTION.—The data are $a = 108$; $p = 27.76$; $r = \frac{108}{27.76} = 3.89$. Substituting in formula **4**,

$$10 = \sqrt{\frac{100{,}000 \times 15.13 \times s}{7.3 \times 3.89 + 6}}.$$

Whence, $s = .002273 = 12.00+$ ft. per mi. Ans.

DARCY'S FORMULAS

15. Open Channels.—The formulas for water running through open channels differ from those applied to closed conduits or pipes under pressure. The data required for water running through open channels are:

U = mean velocity of flow in feet per second;
S = water section in square feet (= area a, b, d, c in (a), Fig. 8);
WP = wet perimeter in feet (= the sum of the lines $a\,c$, $c\,d$, $d\,b$ in (a), Fig. 8);
R = mean hydraulic radius $= \frac{S}{WP}$;
I = slope of free water surface $a\,b$ per foot of length = total fall of surface divided by total length.

Darcy, in experimenting on this class of conduit, found that the nature of the lining of the channel exercised the same influence as that of the interior surface of pipes in modifying the velocity of flow. He therefore adopted a series of coëfficients suited to different degrees of smoothness or roughness of lining.

16. Brick-Lined Channels.—For the ordinary case of a tunnel or channel lined with well-laid brick, the formula is

$$U = R\sqrt{\frac{100{,}000\,I}{6.6\,R + .46}}. \qquad (5.)$$

EXAMPLE.—Referring to Fig. 8 (a), what is the velocity of flow where $ab = 26$ feet; $cd = 16$ feet; ac and $bd = 10$ feet each, and $e = 9$? Let the longitudinal slope of the surface $ab = \frac{1}{1000}$.

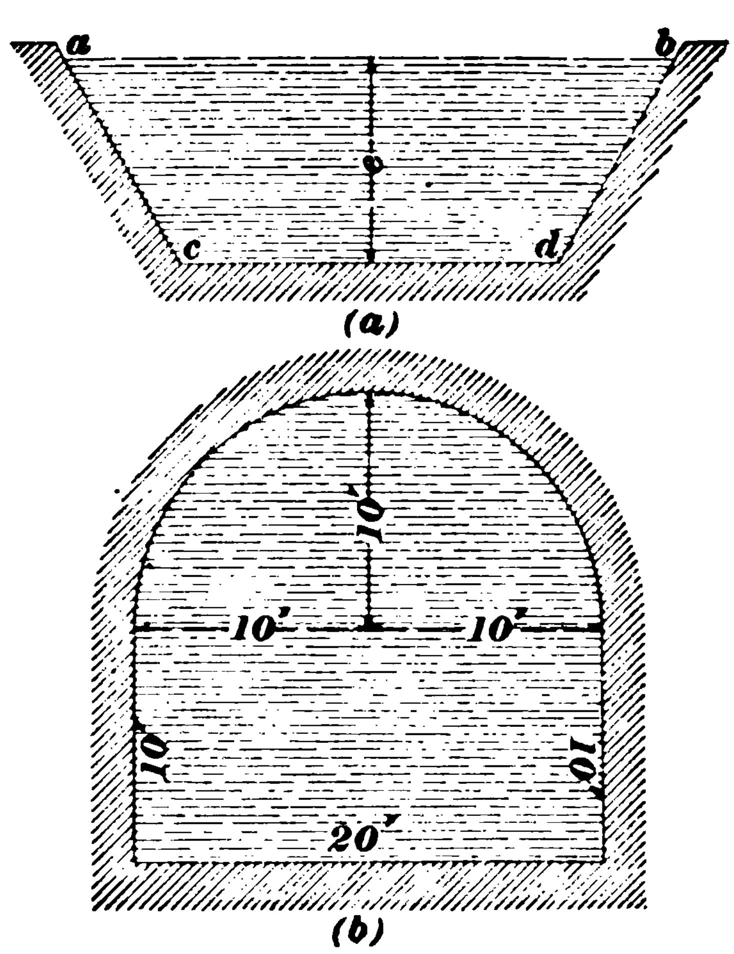


FIG. 8

SOLUTION.—

U = mean velocity in feet per second.

S = 189 square feet.

WP = 36 feet.

$R = \frac{189}{36} = 5.25.$

$I = .001.$

Then,

$$U = 5.25\sqrt{\frac{100}{35.11}} = 8.87 \text{ ft. per sec. Ans.}$$

NOTE.—The "mean velocity" is used in these calculations because there is frequently a considerable difference between the surface velocity and the velocity at various depths and at various distances from the sides. The mean velocity is that which when multiplied by S gives the quantity discharged by the channel in cubic feet per second.

17. In the case of a **circular, brick-lined conduit, running full,** R is always $= \frac{D}{4}$, D being interior diameter of conduit in feet. The formula for such is

$$U = \frac{D}{4}\sqrt{\frac{100{,}000\, I}{1.65\, D + .46}}. \qquad \textbf{(6.)}$$

It is to be understood that the conduit, though running full, is not under pressure.

EXAMPLE.—Let $D = 4$ feet and $I = \frac{1}{1000}$, and find the velocity.

SOLUTION.— $U = \sqrt{\frac{100}{6.6 + .46}} = 3.76$ ft. per sec. Ans.

This result is somewhat in excess of the velocity through a smooth cast-iron pipe of same diameter with the same fall per 1,000. Formula **6** agrees well with observed velocities, however, and the greater velocity in circular

brick-lined conduits is probably due to the fact that they are laid always to a true descending grade, with few and slight lateral deflections, and are quite uniform throughout as to interior surface.

18. Maximum Flow Through Brick-Lined Conduits of Circular, or "Horseshoe," Section.—This occurs when the conduit is not running quite full (unless it is under pressure), but when running to within about $\frac{1}{10}$ of the radius at the crown. This singular result is due to the fact that, although the cross-section of the body of water moving through the conduit is somewhat reduced, the velocity is more than proportionately increased, owing to the favorable value of the hydraulic mean radius.

EXAMPLE.—Given a cross-section as shown in Fig. 8 (*b*) and $I = .001$; what is the discharge (*a*) when the conduit is running just full, but not under pressure, and (*b*) when running to within 1 foot of the crown?

SOLUTION.— (*a*) When running full, $S = 357.08$ square feet; $WP = 71.41$ feet; $R = \frac{357.08}{71.41} = 5.00$.

Then, $$U = 5.00\sqrt{\frac{100}{33.46}} = 8.63 \text{ feet per second.}$$

The discharge is $357.08 \times 8.63 = 3{,}081.6$ cu. ft. per sec. Ans.

(*b*) When running to within 1 foot of the crown, $S = 351.18$ square feet; $WP = 62.38$ feet; $R = \frac{351.18}{62.38} = 5.63$.

Then, $$U = 5.63\sqrt{\frac{100}{6.6 \times 5.63 + .46}} = 9.18 \text{ feet per second.}$$

And the discharge is $351.18 \times 9.18 = 3{,}223.83$ cu. ft. per sec. Ans.

This is greater than when the conduit is running full.

NOTE.—The section shown in the figure is only given as furnishing an easy numerical example and not as one to be used in practice. Such sections—"horseshoe," so called—should have battering sides and a curved invert.

19. Conduits With Rough Sides.—All tunnels or open channels for the conveyance of water are ordinarily lined with brick. When of other material—rubble masonry,

for instance—the flow will be proportionately diminished. It is difficult to calculate the flow in such cases, because, for one thing, it is impossible to accurately determine the area and wet perimeter, owing to the projecting points of the masonry. The best way is to make all the calculations as for brick lining and then apply a coefficient of from 85 to 90 per cent., according to the degree of roughness of the sides.

SOURCES OF WATER

20. The water supply for milling operations is obtained from running streams, melting snows, and rains. Snow accumulates on the mountains during the winter; the heavy rains of spring rapidly thaw the snow and cause enormous volumes of water to rush down the gullies and ravines. In case the country is timbered, the streams may furnish a water supply throughout the year; but near the timber line or in a sparsely wooded country, the water usually rushes off in the spring and the streams become dry later in the year. In order to provide a supply of water for milling in such a region, it becomes necessary to collect the water in a reservoir, from which it is drawn during the dry season.

To determine the available power that may be obtained from a given stream, one must first find the available head, i. e., the vertical distance through which the water can be made to act on the waterwheel or turbine. In general, this requires a careful survey of the site for the location of the waterwheel, so that the difference in level between the surface of the water in the stream, dam, or reservoir from which the supply is to be drawn to the surface of the tail-water where the discharge from the wheel takes place, may be known. In many cases, the head will vary at different seasons of the year, and, since the available power varies with the head and the discharge, it is necessary in such cases to make observations on the various stages of high, mean, and low water.

The value of a water-power depends on the steady power it can be made to furnish at all seasons of the year. When a stream is so located that the surplus water of freshets and storms can be easily stored by means of reservoirs, it can be made to furnish a steady power at all times, whose maximum value depends on the mean discharge of the stream. In some locations, a stream that would be of no practical value as a source of power, owing to the fact that the rainfall from which it is supplied varies greatly at different seasons of the year and the watershed is of such a nature that the water flows off rapidly, may be made to furnish a uniform and reliable supply of water by the means of reservoirs. It is seldom that these reservoirs can be located where the water can be used from them directly, but they serve to store a surplus during periods of high water, which can be drawn on to furnish a supply during low water.

21. The value of a proposed location for a water-power plant must be based on the following considerations: (1) The available fall. (2) The minimum flow of the stream. (3) The effect of high water on the available fall. (4) The possibility of building storage reservoirs for the purpose of regulating the flow and furnishing a supply of water during periods of drought.

It is very important that the pipes or channels leading the water from dam or weir to the waterwheel be made amply large and so arranged that as little head as possible will be lost through friction. If a canal or sluice is to lead water from a dam to a flume or waterwheel and the channel is too small to carry the required amount of water, great loss of power will result. In the case of a long pipe, the frictional resistances may be so great that the pressure head at the end of the pipe will be greatly reduced, and this will result in a serious loss of power. In short pipes leading from dams to wheel cases, flumes, or penstocks, the velocity of flow is often comparatively high and the loss of head due to resistances at entrance of pipe, bends, sudden changes of section, etc. are more serious than in the case of long pipes

and low velocities. A careful computation, in which all the conditions are considered, should always be made for each case, in order to provide against losses of head between the supply and the wheel.

RESERVOIRS

LOCATION FOR DAMS

22. The selection of a proper site for a storage reservoir is a matter of the utmost importance. The first desideratum is that the dam should control a sufficiently large area of watershed lying above it to furnish the requisite volume of water. It should also be situated at such an elevation that water will flow by gravity to the mill. It should be at a narrow part of the valley, so as to decrease the length of the dam, and it should offer, if possible, a rock foundation. The valley should be flat and wide above the site, so that a dam of moderate height and length may impound a large volume of water.

Naturally, a location embracing all these features can rarely be found, and the skill and judgment of the engineer are exercised in selecting a site which, bearing all the desired features in view, will most satisfactorily fulfil those that, under the circumstances, may be the most important.

23. Examination of the Ground.—The first step in selecting a proper site on any given stream is to make a thorough reconnaissance of the entire valley, armed with pocket compass, aneroid barometer, thermometer, hand level, and sketch and notebook. If possible, an assistant should remain at some fixed point of which the elevation is known, making hourly, or more frequent, observations with another barometer and thermometer, and the explorer should note the time at which he makes his barometrical observations. Without this precaution, his elevations will be much less trustworthy, because the barometer fluctuates

with the changes of atmospheric density as well as by changes in altitude.

24. Generally speaking, in thus exploring a valley, one or more points will be plainly indicated as suitable for the building of a dam. The next step will be to survey the watershed lying above each of these points, so as to ascertain, as before mentioned, the probable amount of water that can be furnished at each location. Sometimes it will be possible to estimate, from maps or by simple observation, this area with sufficient accuracy for a preliminary study without a survey, particularly when the size of the stream indicates that the drainage area of the valley is very large.

25. These preliminaries having been accomplished, a careful survey of the site of the proposed dam is made. Cross-sections are taken not only at the point which appears to the eye to be the most favorable, but for some distance above and below such point. These cross-sections are then used in making a contour map of the territory covering the site of the dam, and on this map a "paper location" of the dam is made more understandingly than would be possible by mere inspection of the ground. This paper location is then laid out on the ground and carefully examined to see if any small changes can be made to improve it.

Next, an elevation is assumed for the level of the overflow, which will determine the height of the dam. A "flow line" is then run around from one end of the proposed dam to the other, which will establish the shape and area of the ground to be covered with water when the dam is built and the reservoir filled.

26. Foundations.—The foundation for a dam must be solid to prevent settling and must be water-tight to prevent leakage under the base of the dam and wear in front by water running over the top. Whenever possible, the foundation should be solid rock. Gravel is better than

earth, but when gravel is used, it will be necessary to drive sheet piling under the upper toe of the dam to prevent water from seeping through the formation under the dam. Vegetable soil is unreliable, and all porous matter, such as sand, gravel, etc., should be stripped off until hard pan or solid rock is reached. In case springs occur in the area covered by the foundation of the dam, it will be necessary to trace them up and confine their flow to the inner or upper side of the dam, so that they may not become passageways for water from the upper face of the dam, which might ultimately wash holes through the foundation and destroy the structure.

WOODEN DAMS

27. Wooden dams are constructed of round hewn or sawed logs 1 or 2 feet in diameter, laid in a series of cribs 8 or 10 feet square. The logs composing the cribs are pinned together by means of treenails, and the individual cribs are attached by the same means or by bolting. The cribs are usually filled with loose rock to keep them in place, and in many cases are secured to the bed rock by means of bolts. A layer of plank on the upper face of the dam makes it water-tight.

28. Aprons.—Where water discharges over the top or crest of a dam, it will be necessary to provide some surface to receive the impact of the falling water, for if this is not done, the dam may be undermined and thus destroyed. If the dam is on firm bed rock, the upper surface can simply be extended slightly and the water allowed to fall on the bed rock, which will not be badly cut; but where there is danger of the foundation being washed away, it will be necessary to provide some form of apron or water cushion. An apron may be formed by providing small cribs, which are set on the lower side of the dam and are covered with a plank floor, on to which the water falls and from which it is discharged into the stream below. Sometimes the plank of the floor

is made to pitch back towards the dam in such a manner as to form a tank, into which the water falls, the impact of the fall being taken by the water cushion in the tank. A similar result may be accomplished by building a low dam just below the main dam, so as to form a small pond between the two, which acts as a water cushion and protects bed rock or foundation.

STONE DAMS

29. Dry-Stone Dams.—In regions where cement or lime is expensive and large quantities of suitable rubble

BEDROCK.

FIG. 9

stone can be obtained, dams are frequently constructed without the use of mortar. They are rendered water-tight by a plank facing on the upper side of the dam. Fig 9 illustrates the main dam for the Bowman reservoir in California. Originally the dam consisted of unhewn cedar and tamarack logs notched and firmly bolted together and solidly filled with loose stones of small size. A skin of pine plank was constructed on the water face to form a water-tight lining. This dam was originally 72 feet high, but was increased subsequently to a height of 100 feet by building below the main dam a dry-stone structure composed mainly of angular stones taken from the mountain side and

carefully laid up in irregular range work, so as to break joints. This range work was faced with quarried stones and the lower face made practically vertical, being given a batter of only 15 per cent. for 17½ feet in height. This vertical wall was composed of heavy stones carefully laid and securely bolted together and to the structure behind them. The face of the dam above was given less inclination, as shown, and was also built of quarried stones laid up dry. If water ever flows over the top of such a dam, a great deal of it is liable to pass through the interstices in the slanting stonework, thus subjecting the lower portion of the structure to considerable hydrostatic pressure. To overcome this, the vertical portion is provided with openings through which this portion of the water would find a ready exit.

EARTH DAMS

30. Earth dams are used for reservoirs of moderate depth. They should be at least 10 feet wide on the top, and a height of more than 60 feet is unusual.

31. Puddle Wall.—Where the material of which the dam is to be constructed is not of itself water-tight, as, for instance, gravel, sand, etc., it is sometimes necessary to construct what is called a puddle wall. This consists of a narrow dam made of clay mixed with a certain proportion of sand. If the foundation of the dam is open gravel or sand, the puddle wall should be carried to bed rock or to an impervious stratum. The puddle wall should not be less than 6 or 8 feet thick at the top of the dam and should be given a slight batter on each side, so that it will be somewhat wider at the base. It is constructed during the building of the dam and should be protected from direct contact with the water on the upper face by a considerable thickness of earth, for water will slightly dissolve and wash it away. The upper face of an earthen dam is usually protected by means of a plank lining or a pavement of stones.

32. Masonry Core.—Sometimes earth dams are provided with a masonry core, in place of a puddle wall, to render them water-tight. This consists of a masonry wall carried to an impervious strata and up through the center of the dam. This masonry core should not be less than 2 or 3 feet thick at the top and should be given a batter of at least 10 per cent. on each side.

CENTER WALL

33. In earthen dams, the **center wall** should be carried down to a water-tight foundation, if possible, and its ends should be deeply embedded in the sides of the valley. Its object is twofold: first, to afford an impermeable and indestructible cut-off to any water that might otherwise percolate through the bank; secondly, to afford a means of making water-tight connections for the culverts or pipes used in conveying water from the reservoir and which run through the embankment from one side to the other. Without such water-tight connection as the cut-off wall offers, there is always danger that the water may follow along *outside* of these pipes and finally create a channel in the bank, allowing the water to escape thereby. When this occurs, the destruction of the dam unprovided with a center wall is prompt and certain.

The center wall should be carried up as high as the level of the highest water in the reservoir. Its thickness at the surface of the ground may be one-quarter of its height, and this thickness may be reduced by a batter of 1 inch to the foot on both sides, or, preferably, by building the wall with vertical sides and stepping in 2 feet (1 foot on each side) every 10 feet, which amounts to about the same thing. As the foundation ascends the sides of the valley, it is stepped up, care being taken to keep the bottom of the wall well below the surface of the ground and well embedded in good material.

EMBANKMENT

34. The center wall, as above described, forms the *core* about which the **earthen embankment** is formed. This embankment rises to a certain height above the top of the wall, depending on circumstances to be considered hereafter. It is flat on the top and has a gentle slope on each side, the rate of slope depending greatly on the material of which the bank is formed. The best material is undoubtedly a fine gravel or coarse sand, such as would be proper for making mortar. Clay, although perfectly water-tight when confined, is a treacherous material in a bank, because it is so acted upon by water that it is liable to run away in the form of semi-fluid mud.

FIG. 10

A fair average for the outside slope, i. e., that on the lower or down-stream side (see *c*, Fig 10), is 1 vertical to 2½ horizontal. A somewhat flatter slope is advisable on the water side of the bank (marked *a* in the figure), say 1 vertical to from 2½ to 3½ horizontal. In a high embankment, say 50 feet and upwards, it is better to divide the inside slope into two or more steps, as shown in Fig. 10; that is, starting from

the top, to carry a slope of, say, 1 to 2½ down for 25 or 30 feet, and then introduce a level "berm" 8 to 10 feet wide, continuing the slope with a somewhat flatter grade, say 1 to 3, for another 25 or 30 feet when another level berm is introduced, and the slope then continued either with the same grade as before or, preferably, somewhat flatter, say 1 to 3½.

The inside slope, next to the water, should be carefully paved with stone, or "riprapped," for a thickness of from 1 foot to 2 feet from the bottom to a point well above high-water line. The greatest thickness of the riprap should be at the level of high water in the reservoir. The stone composing this paving must be placed and packed by hand, not dumped at random. The outside slope should be sodded or, at least, sown to grass and carefully tended until a good sod is formed. The earthen embankment should be carried up in horizontal layers and kept constantly moist by sprinkling. In many specifications, it is provided that these layers must be consolidated with a roller.

Before placing the embankment, the natural surface upon which it is to stand must be carefully stripped of all sods, roots, and vegetation, so that the first layer of the earthen embankment may rest upon and incorporate itself with clean earth. This is particularly necessary under the inner slope.

Fig. 10 shows all these features in a general way and represents a correct type of earthen dam, but requires some further explanation. In this figure, *a* is the inner slope of the embankment, *b b* the earthwork embankment, *c* the outer slope, *d d* loose material under the embankment from which the vegetable soil was stripped, as shown by the lighter shading at *e e*. The original surface of the ground is shown by the dotted line *m m*, and *k* is the center wall, carried well down into the compact sand and gravel *n n*. The line *r* shows the high-water level or level of freshet overflow and *s* is the low-water level or level at which overflow begins. It will be perceived that several of the dimensions depend on *H*, *H being the vertical distance from the*

natural surface of the ground to the level of the overflow or lip of the spillway. If the dam crosses a deep and wide stream, then H must be taken *equal to the vertical distance from the bottom of the stream to the lip of the spillway.*

SPILLWAY, OR OVERFLOW

35. The **spillway** is one of the most important features of a dam. It is the means by which the surplus water, when the reservoir is full, is allowed to run to waste, and want of sufficient discharging capacity in this particular has been probably the most prolific cause of destruction of earthen dams. If such dams are once overtopped by a flood, especially when not provided with a proper center wall, they are rapidly cut down and destroyed by the water running over them. Frequently, a natural overflow can be found in some lateral depression of the ground, by which the surplus water can be passed into another valley. Sometimes it is possible to form such an overflow by cutting into some rocky ridge leading either into another valley or into the same one across which the dam is built. In the majority of cases, however, it is found necessary to provide a special piece of masonry construction for the purpose.

The dimensions of this spillway must be proportioned to the amount of water liable to go over in times of freshets. It may be comparatively long and shallow or short and deep. A convenient length is given by the formula

$$L = 20\sqrt{A}, \qquad \textbf{(7.)}$$

in which L = length of lip of spillway in feet and A = area of watershed above the dam in square miles.

For the depth, or vertical distance of the lip of spillway below the level of high water in the reservoir, a convenient formula (the length having been determined by the formula just given) is

$$D = \frac{\sqrt[3]{Q^2 \times A}}{16} + C, \qquad \textbf{(8.)}$$

in which D = depth in feet of notch of spillway, Q = cubic feet of water per second per square mile, A = area of watershed in square miles, as before, and C = a certain additional height above high-water level, depending on the character of the dam, being less for a rock-founded masonry dam than for an earthen one.

EXAMPLE.—What are (*a*) the proper length and (*b*) the depth of the spillway of an earthen dam 50 feet high, built as already described, the area of watershed above the dam being 16 square miles and the maximum freshet flow being estimated at 70 cubic feet per square mile per second?

SOLUTION.— (*a*) Substituting the value of A in formula **7**,

$$L = 20\sqrt{16} = 80 \text{ ft. Ans.}$$

(*b*) Substituting given values in formula **8**,

$$D = \frac{\sqrt[3]{70^2 \times 16}}{16} + C = \frac{\sqrt[3]{78{,}400}}{16} + C = 2.68 + C. \text{ Ans.}$$

36. To determine the value of C, the height and character of the dam and the area of watershed must be taken into consideration. Under no circumstances must the dam be overtopped. In the present case, it is probable that C = 4 to 6 feet would answer. It must be borne in mind that adding a few feet to the top of the embankment increases the amount of material very slightly, as will be seen from Fig. 10, where the embankment could be continued to a considerably greater height with but little additional volume of material by somewhat steepening the slopes above the water-line. In such dams, therefore, a great additional security against overtopping can be obtained with but little extra cost.

If we assume $Q = 64$, which corresponds to a little over 41 million gallons per 24 hours per square mile and represents a very powerful freshet flow, although not, perhaps, the maximum, then formula **8** reduces in round numbers to

$$D = \sqrt[3]{A} + C. \qquad \textbf{(9.)}$$

37. We can now examine Fig. 10 more closely. H being the height from lowest point of natural surface at the dam

to the lip of spillway, the thickness of the earth embankment at the level of the lip is taken equal to H, evenly divided on each side of the center line of the dam.

The thickness of the base of the center wall $= \frac{H}{4}$, and at every 10 feet it is stepped in 1 foot on each side. There is also a small offset, or footing, given to the foundation, which is carried down to a secure formation of fine sand and gravel. The top of the center wall will be carried up to the height of the notch of the spillway or to a height $= D - C$, as determined by formula **8**, above the lip.

The following illustrative example will show how the above principles are applied:

Referring to Fig. 10, let $H = 48$ feet and let $D - C = 2.5$ feet. The center wall commences with a thickness of 12 feet. At a height of 10 feet, it is drawn in by offsets to 10 feet; at 20 feet, to 8 feet; at 30 feet, to 6 feet; at 40 feet, to 4 feet, which thickness is carried through to the top, the total height of center wall above foundation being $48 + 2.50 = 50.50$ feet. The embankment is carried up 8 feet above level of spillway. At the level of the spillway or low-water line, it has a thickness of 48 feet, the top width being 8 feet. At a depth of 27 feet below the spillway, a berm 10 feet wide is introduced and the slope continued at the rate of 1 to 3.

38. Cross-Section of Spillway.—When no natural overflow, as already mentioned, is available, an artificial one must be built. This will generally be in the line of the stream across which the dam is built, although it may sometimes be placed at or near one end of the dam, in case rock is found there, or if for any cause the ground seems more favorable. If rock is not found upon which to build the structure, then the foundations must be carried well down, perhaps deeper than those of the adjacent center wall.

39. Form of Spillway.—The form and dimensions of the cross-section of the spillway vary greatly. This is

sometimes formed in steps and sometimes in a concave curve. The steps constitute the more economical form, because the curved face requires cut stone. The object sought—besides economy—in using steps is to break the force of the falling water, so that it shall reach the bottom with very little velocity, either vertical or horizontal. The idea embraced in the curved face is exactly the contrary, the object sought—or at least obtained—being the greatest possible horizontal velocity when the water has reached the bottom of its fall, urging it rapidly forwards away from the foot of the spillway. For low dams, say up to 15 or 20 feet, the face may be nearly vertical, giving a clear fall to the water upon the apron at the bottom.

For higher dams, say up to 60 or 70 feet, the form shown in Fig. 11 is a very good one. Let $AB = H$, or height to lip of spillway, as already described. Take $AD = \frac{9}{10}H$; also $AC = \frac{9}{10}H$. Join C and D. Take $BE = \frac{H}{5}$. From E draw EF with a face batter of $\frac{1}{12}$, intersecting CD in F. Give EG a slope of 1 in 4 or 5 and round off the corner at E, taking care not to lose height. This is accomplished by starting the slope a little back of E. Draw steps of convenient height, following the line FD.

Fig. 11

Example.—Referring to Fig. 11, let $AB = H = 60$ feet; find the other dimensions.

Solution.— $AD = AC = \frac{9}{10} \times 60 = 54$ feet. $BE = \frac{60}{5} = 12$ ft. Ans.

Spillways for still higher dams require a special study and can be best considered after a study of high masonry dams.

Fig. 12

40. Accessories of Spillway. — The sides of the spillway, where it cuts through the embankment, must necessarily be protected by wing walls to prevent the earth falling into it. These wing walls need not have as flat a slope as the exterior embankment, which would make them unnecessarily large and expensive. They may have a slope of, say, 1 to 1½, and the bank can be graded down to them. The top of the wall may have a coping well doweled into the stones or be left with the horizontal courses forming steps, a construction which, though not so neat in appearance, is more substantial. When the bed of the stream is not rocky, the foot of the spillway must be protected by an apron composed of very heavy stones, if they can be obtained, laid in cement upon a bed of concrete. This apron must be extended well beyond the foot of the spillway, the distance depending

on the height of the dam, the volume of water passing over it, and the nature of the bed of the stream. Probably it should never extend less than a distance equal to the height of the dam. Beyond this, it will be well to protect the bed of the stream between the banks for some distance further with heavy dry-stone paving. All these features are shown in Fig. 12. No dimensions are given in this sketch, because the conditions of each particular case will greatly modify the details, but the general arrangement and proportions will always resemble that shown. In this sketch, the wing wall of ashlar masonry is shown at *A*; *M* is the exterior slope of the embankment and *S* the bank of the stream below the dam. Heavy paving stones laid in cement are shown at *L*, these stones, as well as the masonry composing the apron, being laid on a bed of concrete *K*. The bed of the stream is also protected by the dry paving stones *P*, to prevent cutting by the current.

APPLIANCES FOR DRAWING OFF THE WATER

41. In all cases, a communication must be established between the inside and the outside of the dam, and it is of the utmost importance that no water shall follow along the *outside* of the appliance, whether tunnel, gallery, or pipe line, by means of which this communication is established. As these features of the dam call for a considerable amount of heavy and expensive masonry, it is important to so design them that the necessary degree of solidity may be secured with the least possible volume of masonry.

In Fig. 13, (*a*) shows a plan and (*b*) a vertical section on the line *A B* of a general system of design which experience proves to be strong and satisfactory. In the same figure, (*c*) shows a front elevation and section through gate house *n* looking towards the spillway from outside the reservoir, the embankment being supposed to be removed.

On the inside of the dam, adjacent to the spillway *d*, is built a water tower *i*, one side of which is formed by the prolongation of the spillway itself. This tower communicates

(b) *(c)*

FIG. 18

with the inside of the reservoir by means of an arched gallery or tunnel *j* passing under the interior embankment and terminating in an open portal *a* with wing walls. The top of the tower is level with the top of the embankment. The reducer *s* leading to the cast-iron pipe *p* is securely built into the opposite wall of the tower, and the pipe itself is placed in an arched gallery *n*, under the exterior embankment, one wall of this gallery forming part of the main wing wall *r* of the spillway. The reducer *s* is a special casting having a rectangular opening at the face of equal area with the pipe to which the other end is fitted.

The pipe *p* discharges in some convenient way into the channel of the stream below the dam. In this way, a clear communication is established between the inside and the outside of the reservoir, through which the water can freely pass.

The water passing through this system can be controlled in several ways. The best way is to close the mouth of the reducer on the inside of the tower with a sliding sluice gate *z*, as shown in the figure. Besides this, there should be a stop-cock or valve upon the pipe inside of the exterior gallery *n*, by which the letting on or shutting off of the water will be ordinarily effected, the sluice gate *s* remaining open and only closed upon some emergency occurring, such as an accident to the valve. If no such sluice gate is provided, then there should be two stop-cocks on the pipe. In the figures, two valves *v*, *v* are shown, in addition to the sluice gate, but this would not generally be considered necessary.

Whichever of the two systems is used—and the single valve and sluice gate is probably the preferable one—there should always be a set of grooves *g* cut in the masonry of the tower, in which in an emergency stop-planks may be placed. These are slipped one by one into the grooves, being so reenforced with iron plates as to sink readily in the water. These iron plates may be made advantageously in the form of angle irons, which will afford means to withdraw them,

when required, with suitable hooks. Stop-planks constitute a very valuable means of shutting off the water so as to gain access to the mouth of the discharge pipe without emptying the reservoir. Means of access to the gallery *n* is provided by the manhole *q*, and the gallery is kept free from water by the small drain *e*. The whole structure rests on the hydraulic rubble foundation *k k*, which must extend down to rock or solid ground *m m*.

WING DAMS

42. For turning streams from their courses, wing dams are sometimes used. Wing dams may extend partly across the stream and then down one bank, forming a course for the stream, or it may simply extend partly across the stream, so as to turn the water into a sluice or flume. In the case of comparatively small streams, wing dams are usually of a temporary nature and are constructed of brush or light cribs filled with stones, which are subsequently backed up with earth, stones, or some timber work. Fig. 14 illustrates one method by means of which wing dams may be constructed. In this case, bags filled with sand were piled across the course of the stream and backed up with gravel and pebbles. The sand bags and earth turned the course of the stream and enabled the operators to construct triangular wooden bents or frames behind the bank, as shown in the illustration. These frames were subsequently weighted down with stones and their lower face covered with a riprap of drift boulders, etc., so that in case the water ever came over the dam, it would not wash it away and destroy the work.

Fig. 14

MASONRY DAMS

43. General Considerations.—In the structures hitherto mentioned, there has been no attempt made to determine their dimensions by calculation. In earthen dams, all dimensions are fixed by empirical rules; that is, experience has taught that certain thickness of bank and certain ratios of slope lead to safe results; by adopting these dimensions, the stability of the work is insured.

In masonry dams the case is different. With a wall sustaining a certain head of water, the character, intensity, point of application, and direction of action of the destructive force or forces can be calculated and also the resistance that the wall presents in opposition can be calculated with a great degree of precision.

44. Elementary Principles.—The principal property of quiet water is its *horizontal thrust* upon any surface against which it presses. Referring to Fig. 15, the horizontal thrust T, expressed in pounds, against any surface $A\ B$,

Fig. 15

whether vertical as at (a), inclined as at (b), or curved as at (c), is the same, and is equal to half the square of the height or head of water H in feet pressing against $A\ B$, multiplied by 62.5, which is the weight in pounds of a cubic foot of water in round numbers and under ordinary conditions. Thus,

$$T = \frac{62.5\ H^2}{2} = 31.25\ H^2. \qquad \textbf{(10.)}$$

Also the point of application of this thrust is the same for (a), (b), and (c); namely, at one-third of the height H from the bottom of the wall. Hence, the overturning moment $M\,T$ of the thrust about the point C, expressed in static foot-pounds, is:

$$M\,T = 31.25\,H^2 \times \frac{H}{3} = 10.42\,H^3. \qquad \textbf{(11.)}$$

NOTE.—The value of $10.42\,H^3$ will always be slightly greater than the true value, as the decimal is a little too great.

EXAMPLE.—The depth of water H pressing against the curved surface $A\,B$ [Fig. 15 (c)] is 23 feet 7 inches. (a) What is the intensity in pounds of the horizontal thrust? (b) What is the overturning moment in foot-pounds about the point C per foot of length of wall?

SOLUTION.—(a) $T = 31.25\,(\frac{283}{12})^2 = 17{,}380.43$ lb. Ans.
(b) $M\,T = 10.42\,(\frac{283}{12})^3 = 136{,}673.18$ ft.-lb. Ans.

45. Action of the Thrust Against the Dam.—The first effort of the water against the wall or dam against which it presses is to push it bodily forwards by causing it to slide upon its base. The force tending to produce this effect is the horizontal thrust T alone. If the wall, from its stability in this respect, refuses to move, the next effort of the water is to endeavor to overturn it by causing it to rotate about its outer toe C, Fig. 15. The force tending to produce this effect is the overturning moment $M\,T$ or the combination of the intensity of the stress into the lever arm with which it acts.

46. Resistances of the Dam.—The dam opposes the thrust of the water by its weight and its coefficient of friction. Its total resistance is, therefore, its weight multiplied by its coefficient of friction. An ordinary estimate places the last-named factor at about 75 per cent. It must be noted that friction is now considered as if the dam were merely standing upon a level base, with no mortar joint intervening, and the resistance only that of the friction of stone against stone. The adherence of the mortar and the

bond of work are both neglected. The resistance thus estimated is, therefore, much below the truth. It is, however, at least *safe* to take the coefficient of friction in what follows as .75.

The resistance R of the wall to sliding is the area A of its vertical cross-section multiplied by its density D, or the weight of a cubic foot of the material of which it is composed multiplied by its coefficient of friction, which we have agreed to call .75. Hence,

$$R = .75\, A\, D. \qquad \textbf{(12.)}$$

EXAMPLE 1.—A trapezoidal wall, Fig. 16, 12 feet high, 3 feet wide on top, and 8 feet at bottom has a density of 115 pounds. (*a*) What is its resistance to sliding and (*b*) what is its factor of safety?

SOLUTION.—(*a*) Substituting in formula **12**,

$$R = .75 \times \frac{8+3}{2} \times 12 \times 115 = 5{,}692.5 \text{ lb.} \quad \text{Ans.}$$

(*b*) To ascertain the factor of safety F, it is necessary to find the amount of thrust to be resisted. From formula **10**,

$$T = 31.25 \times 144 = 4{,}500 \text{ pounds.}$$

Hence, $$F = \frac{5{,}692.5}{4{,}500} = 1.26. \quad \text{Ans.}$$

EXAMPLE 2.—Let the wall in example 1 be built of granite, with a density of 170. Determine (*a*) the thrust and (*b*) the factor of safety.

SOLUTION.—(*a*) $$R'' = 49.5 \times 170 = 8{,}415 \text{ lb.} \quad \text{Ans.}$$

(*b*) $$F = \frac{8{,}415}{4{,}500} = 1.87. \quad \text{Ans.}$$

47. The **resistance to overturning** will be the **moment of resistance** of the wall and will be the product of its weight multiplied by the horizontal distance of its center of gravity from the point about which rotation tends to take place. This point, in the case of a dam, is the exterior toe.

If the wall were a *plumb* wall, i. e., one with vertical sides, as Fig. 17, its weight (considering a length of one foot, as already stated) would be $D\,H\,B$, and as the vertical

line passing through the center of gravity would cut the base in the center, its moment of resistance MR would be

$$MR = DHB \times \frac{B}{2} = \frac{DHB^2}{2}. \qquad \textbf{(13.)}$$

But dams are very rarely built plumb on both faces. The almost invariable section, or "profile" as it is generally termed, except for very high dams, is trapezoidal.

FIG. 16 FIG. 17 FIG. 18

To calculate the moment of resistance of such a wall, Fig. 18, it is divided into a rectangle HA and a triangle of height H and base b. The moment of resistance mr of the rectangle about the toe C is

$$mr = DHA\left(b + \frac{A}{2}\right);$$

and of the triangle, since the center of gravity is horizontally distant $\frac{2b}{3}$ from C,

$$mr' = \frac{DHb}{2} \times \tfrac{2}{3} b = \frac{DHb^2}{3}.$$

Adding the two together,

$$MR = DHA\left(b + \frac{A}{2}\right) + \frac{DHb^2}{3}$$

$$= DH\left(Ab + \frac{A^2}{2} + \frac{b^2}{3}\right) = \frac{DH}{6}(6Ab + 3A^2 + 2b^2).$$

Observing that $b = B - A$,

$$M R = \frac{D H}{3}\left(A B - \frac{A^2}{2} + B^2\right). \qquad \textbf{(14.)}$$

EXAMPLE 1.—Referring again to Fig. 16 (*a*), what is the moment of resistance of the wall when $D = 115$? (*b*) What is its factor of safety?

SOLUTION.—(*a*) Substituting in formula **14**,

$$M R = \frac{115 \times 12}{3}\left(3 \times 8 - \frac{9}{2} + 64\right) = 38{,}410 \text{ ft.-lb.} \quad \text{Ans.}$$

(*b*) To ascertain the factor of safety, calculate the overturning moment of the water thrust. From formula **11**,

$$M T = 10.42 \times 1{,}728 = 18{,}005.76.$$

Hence,
$$F = \frac{38{,}410}{18{,}006} = 2.133. \quad \text{Ans.}$$

EXAMPLE 2.—Suppose the same wall to have a density of 170. Determine (*a*) the moment of resistance and (*b*) the factor of safety.

SOLUTION.—(*a*) $M R = \frac{170 \times 12}{3}\left(3 \times 8 - \frac{9}{2} + 64\right) = 56{,}780$ ft.-lb. Ans.

$$(b)\ F = \frac{56{,}780}{18{,}006} = 3.15. \quad \text{Ans.}$$

48. Designing Profiles.—The above formulas and examples show how to ascertain the resistances of walls of which the dimensions, etc. are given. But they give no help in designing a wall to fulfil certain requirements, except by "trial and error."

Supposing now it were desired to design a wall of height H, of density D, of top width A, and it were desired that it should have a factor of safety C as against sliding and overturning. What should be the bottom width B?

Considering first the case of sliding, since the factor of safety is C, the thrust must be taken as

$$31.25\, C H^2.$$

The resistance, from formula **12**, must be

$$R = .75\, H\left(\frac{A + B}{2}\right) D.$$

These two values must be equal. Hence,

$$31.25\,C H^2 = .75\,H\left(\frac{A+B}{2}\right)D;\quad 31.25\,C H = .75\left(\frac{A+B}{2}\right)D;$$

$$83.33\,C H = (A+B)\,D;\quad B = \frac{83.33\,C H}{D} - A. \qquad \textbf{(15.)}$$

Having now established the general formula, any example where A, C, D, and H are given can be worked.

EXAMPLE.— $H = 30$ feet; $A = 6$ feet; $D = 140$ pounds; and $C = 2.5$; what must be the bottom width of the wall?

SOLUTION.— $B = \frac{83.33 \times 2.5 \times 30}{140} - 6 = 38.64$ ft. Ans.

49. To determine the breadth of base to resist overturning, consider formula **11** (using its more exact form of $\frac{31.25}{3}H^3$).

$$\frac{31.25}{3}\,C H^3 = \frac{D H}{3}\left(A\,B - \frac{A^2}{2} + B^2\right);$$

$$62.50\,C H^2 = 2\,D\,A\,B - D\,A^2 + 2\,D\,B^2;$$

$$B^2 + A\,B = \frac{62.50\,C H^2 + D\,A^2}{2\,D};$$

which, solved for B, gives

$$B = \sqrt{\frac{62.50\,C H^2 + D\,A^2}{2\,D} + \frac{A^2}{4}} - \frac{A}{2};$$

or, $$B = \tfrac{1}{2}\sqrt{\frac{125\,C H^2}{D} + 3\,A^2} - \frac{A}{2}. \qquad \textbf{(16.)}$$

EXAMPLE.—What must be the width of base of the wall in the last example to resist overturning?

SOLUTION.—Substituting the given data in formula **16**,

$$B = \sqrt{\frac{62.50 \times 2.50 \times 900 + 140 \times 36}{280} + \frac{36}{4}} - 3 = 20 \text{ ft.} \quad \text{Ans.}$$

50. In examining the results given by formulas **15** and **16,** one is struck with the fact that a much greater width of base is required to insure security against sliding than is required to guard against overturning. It must be borne in mind, however, that, as has already been mentioned, only the mere friction of stone upon stone is taken into the account when calculating the resistance to sliding, such as might occus if two level surfaces of stone were brought in contact. When it is remembered that a well-bonded piece of masonry is by no means in this condition, but is knit together in a more or less homogeneous mass, it will be seen that the tendency to move forwards is counteracted not by mere friction alone, but also by the resistance to shearing of the stonework.

As, however, all masonry dams should stand on a rock foundation, into which the footing course is well embedded, no danger of their moving bodily forwards need be apprehended if the stability is satisfactory as regards overturning.

51. Average Dimensions.—Calculations made with various practical values for density D and top width A show that a bottom width equal to from $\frac{2}{3}H$ to $\frac{3}{4}H$ will always give a satisfactory factor of safety, and in nearly all cases the smaller of these two values, i. e., $B = \frac{2H}{3}$, will give a perfectly secure profile.

MEASUREMENT OF FLOWING WATER

52. Meters.—Various forms of meters are used for this purpose. Some of them are so devised that they measure the actual quantity of water flowing through the weir; others simply give the velocity of the current, and from this and the size of the channel through which the water flows, the quantity is determined.

53. Gauging by Right-Angled V Notch.—A right-angled **V** notch, cut from thin sheet iron, is frequently used for gauging comparatively small flows. The notch is fitted

in one end of a box, as shown in Fig. 19. The edge of the plate forming the notch must be sharp and the bevel must be on the lower side of the plate, the inside face being at right angles to the surface of the still water. To prevent

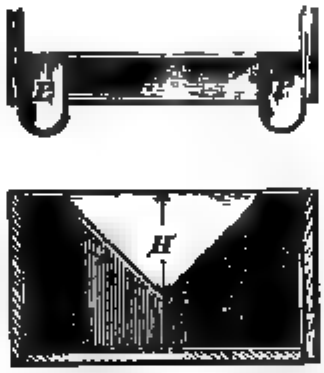


FIG. 19

surface currents in the box, baffle boards are placed some distance back, as shown in the illustration. The distance *a* of the surface of the water below the top of the weir and below the top of the box is taken at a point some distance back from the notch (at least 18 to 24 inches), where the water surface is level. This distance subtracted from the total depth *H* of the notch gives the head *h* of water passing over the notch. This head may be obtained as follows: A straightedge or level is placed on the weir plate *P*, so as to extend back over the surface of the water in the box. The distance *a* between its lower edge and the surface of the water is measured. This distance subtracted from *H* leaves *h*, which is the depth or head of water in the notch. This head may also be obtained by measurements from the bottom of the box, in which case the height of the bottom of the notch above the bottom of the box will be subtracted from the depth of water in the box. This depth may be obtained by measuring with a rule or scale which extends to the bottom or by means of a hook gauge, as explained later.

The discharge in cubic feet per second is equal to .0051 times the square root of the fifth power of the head expressed in inches. Table I gives the discharge in cubic feet per second through the right-angled **V** notch for heads *h* varying from 1.05 inches up to 12 inches.

TABLE I

DISCHARGE OF WATER THROUGH A RIGHT-ANGLED V NOTCH

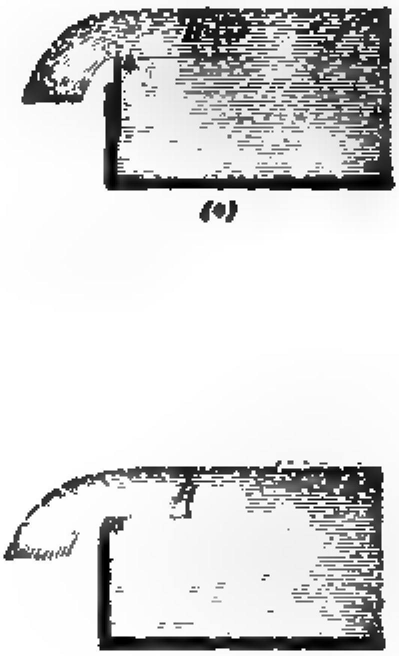
h Head. Inches	Q Quant. per Min. Cu. Ft.	h Head. Inches	Q Quant. per Min. Cu. Ft.	h Head. Inches	Q Quant. per Min. Cu. Ft.	h Head. Inches	Q Quant. per Min. Cu. Ft.	h Head. Inches	Q Quant. per Min. Cu. Ft.
1.05	.3457	3.25	5.827	5.45	21.22	7.65	49.53	9.85	93.18
1.10	.3884	3.30	6.054	5.50	21.71	7.70	50.34	9.90	94.37
1.15	.4340	3.35	6.285	5.55	22.20	7.75	51.16	9.95	95.56
1.20	.4827	3.40	6.523	5.60	22.70	7.80	51.99	10.00	96.77
1.25	.5345	3.45	6.765	5.65	23.22	7.85	52.83	10.05	97.98
1.30	.5896	3.50	7.012	5.70	23.74	7.90	53.67	10.10	99.20
1.35	.6480	3.55	7.266	5.75	24.26	7.95	54.53	10.15	100.43
1.40	.7096	3.60	7.524	5.80	24.79	8.00	55.39	10.20	101.67
1.45	.7747	3.65	7.788	5.85	25.33	8.05	56.26	10.25	102.92
1.50	.8432	3.70	8.058	5.90	25.87	8.10	57.14	10.30	104.18
1.55	.9153	3.75	8.332	5.95	26.42	8.15	58.03	10.35	105.45
1.60	.9909	3.80	8.613	6.00	26.98	8.20	58.92	10.40	106.73
1.65	1.0700	3.85	8.899	6.05	27.55	8.25	59.82	10.45	108.02
1.70	1.1530	3.90	9.191	6.10	28.12	8.30	60.73	10.50	109.31
1.75	1.2400	3.95	9.489	6.15	28.70	8.35	61.65	10.55	110.62
1.80	1.3300	4.00	9.792	6.20	29.28	8.40	62.58	10.60	111.94
1.85	1.4240	4.05	10.100	6.25	29.88	8.45	63.51	10.65	113.26
1.90	1.5220	4.10	10.410	6.30	30.48	8.50	64.45	10.70	114.60
1.95	1.6250	4.15	10.730	6.35	31.09	8.55	65.41	10.75	115.94
2.00	1.7310	4.20	11.060	6.40	31.71	8.60	66.37	10.80	117.29
2.05	1.8410	4.25	11.390	6.45	32.33	8.65	67.34	10.85	118.65
2.10	1.9550	4.30	11.730	6.50	32.96	8.70	68.32	10.90	120.02
2.15	2.0740	4.35	12.070	6.55	33.60	8.75	69.30	10.95	121.41
2.20	2.1960	4.40	12.420	6.60	34.24	8.80	70.30	11.00	122.81
2.25	2.3230	4.45	12.780	6.65	34.89	8.85	71.30	11.05	124.21
2.30	2.4550	4.50	13.140	6.70	35.56	8.90	72.31	11.10	125.61
2.35	2.5900	4.55	13.510	6.75	36.23	8.95	73.33	11.15	127.03
2.40	2.7300	4.60	13.890	6.80	36.89	9.00	74.36	11.20	128.45
2.45	2.8750	4.65	14.270	6.85	37.58	9.05	75.40	11.25	129.90
2.50	3.0240	4.70	14.650	6.90	38.27	9.10	76.44	11.30	131.35
2.55	3.1770	4.75	15.040	6.95	38.96	9.15	77.49	11.35	132.81
2.60	3.3350	4.80	15.440	7.00	39.67	9.20	78.56	11.40	134.27
2.65	3.4980	4.85	15.850	7.05	40.38	9.25	79.63	11.45	135.75
2.70	3.6660	4.90	16.260	7.10	41.10	9.30	80.71	11.50	137.23
2.75	3.8380	4.95	16.680	7.15	41.83	9.35	81.80	11.55	138.73
2.80	4.0140	5.00	17.110	7.20	42.56	9.40	82.90	11.60	140.23
2.85	4.1960	5.05	17.540	7.25	43.30	9.45	84.01	11.65	141.75
2.90	4.3820	5.10	17.970	7.30	44.06	9.50	85.12	11.70	143.28
2.95	4.5740	5.15	18.420	7.35	44.82	9.55	86.24	11.75	144.82
3.00	4.7700	5.20	18.870	7.40	45.58	9.60	87.37	11.80	146.36
3.05	4.9710	5.25	19.320	7.45	46.36	9.65	88.52	11.85	147.91
3.10	5.1780	5.30	19.790	7.50	47.14	9.70	89.67	11.90	149.48
3.15	5.3880	5.35	20.260	7.55	47.92	9.75	90.83	11.95	151.05
3.20	5.6050	5.40	20.730	7.60	48.72	9.80	92.00	12.00	152.64

1 cubic foot contains 7.48 U. S. gallons; 1 U. S. gallon weighs 8.34 pounds.

WEIRS

54. A **weir** is an obstruction placed across a stream for the purpose of diverting the water so as to make it flow through the desired channel. This channel may be a notch or opening in the obstruction itself, and it has been found that when properly constructed and carefully managed, such a weir forms one of the most convenient and accurate devices for measuring the discharge of streams. Many careful experiments have been made to determine the quantity of water that will flow over different forms of weirs under various conditions. As the result of these experiments, two forms have come into general use, and the amount of flow over either can be determined by simple formulas and coefficients that depend on observed conditions.

55. A Weir With End Contractions.—Such a weir is shown in Fig. 20 (*a*). The notch is narrower than the



(*a*)

(*b*)

FIG. 20

channel through which the water flows, thus causing a contraction at the bottom and two sides of the issuing stream.

56. A Weir Without End Contractions.—This is also called a weir with end contraction suppressed and is shown in Fig. 20 (*b*). In this case, the notch is the full width of the channel leading to it, and, consequently, the stream issuing is contracted at the bottom only.

57. Crest of the Weir.—The edge *a*, Fig. 20 (*c*) and Fig. 20 (*d*), is called the crest of the weir. The inner edge of the crest should be made sharp, so that the water in passing over it touches only a sharp edge. For very accurate work, both vertical and horizontal edges should be made from thin plates of metal having a sharp inner edge, as shown at *a*, Fig. 20 (*c*), but for ordinary work, the edges of the board in which the notch is cut may chamfer off, as shown in (*b*). Frequently this edge is not made absolutely sharp, but is left flat for about $\frac{1}{8}$ inch, so as to increase the strength of the edge and to decrease the liability of its being damaged. The bottom edge of the notch must be straight and set perfectly level; the sides must be at right angles to the bottom. The inside or upper edges of the notch must always be in a plane at right angles to the surface of still water. The head H producing the flow is the vertical distance from the crest of the weir to the surface of the water, as shown in Fig. 20 (*c*) and Fig. 20 (*d*). This head must be measured at a point sufficiently back from the crest so that the surface of the water is not affected by the curvature of the stream flowing over the weir.

58. Fig. 21 shows a simple form of weir located in a small stream for the purpose of measuring the discharge.

A plank dam is put across the stream at a convenient point, care being taken to prevent any leakage around or under the dam. The length of the notch is great enough to provide for a flow having a head of from .5 to 1.5 feet, but at the same time the length of the crest for accurate work should never be less than 3 or 4 times the head. If it is desired to measure smaller amounts of water, the **V** notch can be used. A stake E is driven firmly into the ground at

a point about 6 feet up stream from the weir and near the bank, as shown. The stake is driven down until its top is exactly level with the top of the weir. The head is the vertical distance from the top of this stake to the surface of the water, and it may be measured by means of a 2-foot rule

FIG. 21

or a square, as shown in the illustration. For very accurate work, the hook gauge is used.

HOOK GAUGE

59. For accurate measurements, such as are made when testing the efficiency of waterwheels, the head is measured with an instrument called the hook gauge, shown in Fig. 22. A hook a is attached to the lower end of a sliding scale b; the scale is graduated to hundredths of a foot and is provided with a vernier, by means of which readings

can be made to thousandths of a foot. The scale and the hook can be raised or lowered by means of a screw *s*. The hook is so set that the gauge will read to zero when the point of the hook is at the same level as the crest of the weir. The instrument is fastened securely to some substantial structure at a point a few feet up stream from the weir and where the surface of the water is quiet and protected from the influence of winds or eddies. When the point of the hook is raised to the surface of the water, it lifts the surface slightly before breaking through. To use the gauge, start with the hook slightly below the surface of the water and raise it slowly until a slight pimple, caused by the lifting of the surface, appears over the point of the hook. It is the difference in level between the zero at the weir and the reading where the instrument is secured that gives the measurement of the head of water.

DISCHARGE OF WEIRS

60. When the dimensions of the notch and the head on the crest of a weir are known, the discharge can be computed by means of the following formulas and tables of coefficients.

FIG. 22

Let l = length of the weir in feet;
H = measured head in feet;
v = velocity with which the water approaches the weir in feet per second;
h = head equivalent to the velocity with which the water approaches the weir, or a head which would produce a velocity equal to v;
c = coefficient of discharge;
Q = actual discharge in cubic feet.

The actual discharge for weirs with end contractions is given by the formulas:

$$Q = 5.347\, c\, l\, (H + 1.4\, h)^{\frac{3}{2}}, \qquad \textbf{(17.)}$$

which is used where the water approaches the weir with a velocity equivalent to the height h, and

$$Q = 5.347\, c\, l\, H^{\frac{3}{2}}, \qquad \textbf{(18.)}$$

where the water has no velocity of approach.

The actual discharge for weirs without end contractions is given by the following formulas:

$$Q = 5.347\, c\, l\, (H + \tfrac{4}{3}\, h)^{\frac{3}{2}}, \qquad \textbf{(19.)}$$

which applies in cases where the water has a velocity of approach, and

$$Q = 5.347\, c\, l\, H^{\frac{3}{2}}, \qquad \textbf{(20.)}$$

which applies where the water has no velocity of approach.

61. Velocity of Approach.—By this term is meant the velocity with which the water flows through the canal leading to the weir. This may be obtained by finding, approximately, the amount of water discharged in a given time and the area of the cross-section of the canal leading to the weir. Then the velocity of approach will be equal to the given amount of water divided by the area; or,

A = area of the cross-section of the canal in square feet;
v = velocity of approach in feet per second;
Q = quantity of water in cubic feet.

Then, $$v = \frac{Q}{A}.$$

Q may be obtained approximately by assuming that v is equal to zero and applying the formula for the class of weir in question, as given above. Having obtained this quantity Q and from it the value of v, the equivalent head h may be found by the formula

$$h = .01555\, v^2. \qquad \textbf{(21.)}$$

Since v is small with a properly constructed weir, it is usually neglected unless great accuracy is required.

62. Table II gives the values of the coefficients of discharge c for weirs with end contractions and different values of H and l. In this table, the head given is the effective head $H + \frac{4}{3}h$. When the velocity of approach is small, h is neglected and the head becomes simply H, but this change will not affect the coefficients in the table.

63. Table III gives the values of c for weirs without end contractions. Weirs with end contractions are more often used and are to be recommended in most cases. Values of c corresponding to values of H and l between those given in the tables can be found by interpolating or taking an average between the desired figures, assuming that the variation is uniform between the values given. In Table III, the head given is the effective head $H + \frac{4}{3}h$, which, when h is neglected, becomes simply H. This does not affect the value of the coefficient in the table.

64. The tables and formulas thus far given require the measurements to be taken in hundredths of a foot.

TABLE II

VALUES OF THE COEFFICIENT OF DISCHARGE FOR WEIRS WITH END CONTRACTIONS

Effective Head in Feet	Length of Weir in Feet						
	.66	1	2	3	5	10	19
.10	.632	.639	.646	.652	.653	.655	.656
.15	.619	.625	.634	.638	.640	.641	.642
.20	.611	.618	.626	.630	.631	.633	.634
.25	.605	.612	.621	.624	.626	.628	.629
.30	.601	.608	.616	.619	.621	.624	.625
.40	.595	.601	.609	.613	.615	.618	.620
.50	.590	.596	.605	.608	.611	.615	.617
.60	.587	.593	.601	.605	.608	.613	.615
.70		.590	.598	.603	.606	.612	.614
.80			.595	.600	.604	.611	.613
.90			.592	.598	.603	.609	.612
1.00			.590	.595	.601	.608	.611
1.20			.585	.591	.597	.605	.610
1.40			.580	.587	.594	.602	.609
1.60				.582	.591	.600	.607

Frequently, operators are not provided with apparatus for accomplishing this, and hence Table IV is given, which gives the cubic feet of water per minute for every inch in length of the weir corresponding to the depths given in the table, the table giving the depths by eighths of an inch from 1 inch to 25 inches. This table is not accurate for weirs whose depths are great compared with their lengths, and should not be used unless the length of the crest of the weir is at least 3 or 4 times the depth of the water on the weir.

TABLE III

VALUES OF THE COEFFICIENT OF DISCHARGE FOR WEIRS WITHOUT END CONTRACTIONS

Effective Head in Feet	Length of Weir in Feet						
	19	10	7	5	4	3	2
.10	.657	.658	.658	.659			
.15	.643	.644	.645	.645	.647	.649	.652
.20	.635	.637	.637	.638	.641	.642	.645
.25	.630	.632	.633	.634	.636	.638	.641
.30	.626	.628	.629	.631	.633	.636	.639
.40	.621	.623	.625	.628	.630	.633	.636
.50	.619	.621	.624	.627	.630	.633	.637
.60	.618	.620	.623	.627	.630	.634	.638
.70	.618	.620	.624	.628	.631	.635	.640
.80	.618	.621	.625	.629	.633	.637	.643
.90	.619	.622	.627	.631	.635	.639	.645
1.00	.619	.624	.628	.633	.637	.641	.648
1.20	.620	.626	.632	.636	.641	.646	
1.40	.622	.629	.634	.640	.644		
1.60	.623	.631	.637	.642	.647		

65. As an illustration of the use of Table IV, suppose that we had a weir 5 feet in length and that the water upon it stood at a depth of 10⅝ inches. Then the amount of water passing for each inch of length of the weir would be found by noting the figure 10 in the left-hand column of the table and passing along this horizontal line until the value under ⅝ was obtained, which would be found to be 13.93 cubic feet per minute, and as the weir was 60 inches long, the total amount passing per minute will be 13.93 × 60 = 835.8 cubic feet per minute.

TABLE IV

WEIR TABLE GIVING CUBIC FEET DISCHARGED PER MINUTE FOR EACH INCH IN LENGTH OF WEIR FOR DEPTHS FROM 1-8 INCH TO 25 INCHES

Inches		1/8	1/4	3/8	1/2	5/8	3/4	7/8
		.01	.05	.09	.14	.20	.26	.33
1	.40	.47	.55	.65	.74	.83	.93	1.03
2	1.14	1.24	1.36	1.47	1.59	1.71	1.83	1.96
3	2.09	2.23	2.36	2.50	2.63	2.78	2.92	3.07
4	3.22	3.37	3.52	3.68	3.83	3.99	4.16	4.32
5	4.50	4.67	4.84	5.01	5.18	5.36	5.54	5.72
6	5.90	6.09	6.28	6.47	6.65	6.85	7.05	7.25
7	7.44	7.64	7.84	8.05	8.25	8.45	8.66	8.86
8	9.10	9.31	9.52	9.74	9.96	10.18	10.40	10.62
9	10.86	11.08	11.31	11.54	11.77	12.00	12.23	12.47
10	12.71	13.95	13.19	13.43	13.67	13.93	14.16	14.42
11	14.67	14.92	15.18	15.43	15.67	15.96	16.20	16.46
12	16.73	16.99	17.26	17.52	17.78	18.05	18.32	18.58
13	18.87	19.14	19.42	19.69	19.97	20.24	20.52	20.80
14	21.09	21.37	21.65	21.94	22.22	22.51	22.79	23.08
15	23.38	23.67	23.97	24.26	24.56	24.86	25.16	25.46
16	25.76	26.06	26.36	26.66	26.97	27.27	27.58	27.89
17	28.20	28.51	28.82	29.14	29.45	29.76	30.08	30.39
18	30.70	31.02	31.34	31.66	31.98	32.31	32.63	32.96
19	33.29	33.61	33.94	34.27	34.60	34.94	35.27	35.60
20	35.94	36.27	36.60	36.94	37.28	37.62	37.96	38.31
21	38.65	39.00	39.34	39.69	40.04	40.39	40.73	41.09
22	41.43	41.78	42.13	42.49	42.84	43.20	43.56	43.92
23	44.28	44.64	45.00	45.38	45.71	46.08	46.43	46.81
24	47.18	47.55	47.91	48.28	48.65	49.02	49.39	49.76

66. The following examples are given to illustrate the use of the formulas and tables:

EXAMPLE 1.—What is the discharge of the stream in Fig. 21 if the length of the weir is 5 feet, the head $10\frac{1}{2}$ inches, the coefficient of discharge .603, and the velocity of approach 0?

SOLUTION.—Applying formula **20**,

$$Q = .603 \times 5.347 \times 5 \times .875^{\frac{3}{2}} = 13.1934 \text{ cu. ft. per sec.} \quad \text{Ans.}$$

EXAMPLE 2.—What is the discharge from a weir with end contractions under the following conditions? The length of the weir is 4 feet $1\frac{1}{2}$ inches and the measured head $10\frac{1}{16}$ inches. Assume that there is no velocity of approach.

SOLUTION.—The length l of the weir = 4 feet $1\frac{1}{2}$ inches = 4.125 feet and the head $H = 10\frac{1}{16}$ inches = .84 foot. From Table II, we find the coefficient $c = .600$ for a weir 3 feet long and a head of .8 foot and $c = .604$ for a weir 5 feet long with the same head. This is an increase in the coefficient of $(.604 - .600) \div 2 = .002$ for each increase of 1 foot in length. The coefficient for a weir 4.125 feet long is, therefore, $.600 + (1.125 \times .002) = .60225$. The rate of increase for a head of .9 foot is $(.603 - .598) \div 2 = .0025$ and the coefficient for a weir 4.125 feet long is $.598 + (1.125 \times .0025) = .60081$. The decrease in the coefficient for an increase in head of .1 foot is $.60225 - .60081 = .00144$, and for an increase in head of .04 foot, the decrease is $.00144 \times \frac{.04}{.1} = .000576$. This subtracted from the coefficient for .8 foot gives $.60225 - .000576 = .601674$ as the coefficient of discharge for a weir 4.125 feet long and a head of .84 foot. Using but four decimal places, the discharge by formula **20** is

$$Q = 5.347 \times .6017 \times 4.125 \times .84^{\frac{3}{2}} = 10.22 \text{ cu. ft. per sec.} \quad \text{Ans.}$$

EXAMPLE 3.—If the canal leading to the above weir is 10 feet wide and 3 feet deep below the crest of the weir, what is the head equivalent to the velocity of approach?

SOLUTION.—The depth of water in the canal is the depth below the crest plus the head = 3.84 feet. The area of the cross-section of the water in the canal is $A = 3.84 \times 10 = 38.4$ square feet and the velocity is $v = \frac{10.22}{38.4} = .266$ foot per second. The head h equivalent to the velocity v is, according to formula **21**,

$$h = .01555 \times .266^2 = .0011 \text{ ft.} \quad \text{Ans.}$$

NOTE.—This value of h is so small that its influence on the discharge is much less than the probable errors in measuring the head H, and so need not be considered in finding the discharge.

MINER'S INCH

67. The **miner's inch** varies in different districts and is by no means a definite quantity of water, as the methods of deriving it vary in different places. Merriman states that the miner's inch may be roughly defined as the quantity of water that will flow from a vertical standard orifice 1 inch square when the head on the center of the orifice is $6\frac{1}{2}$ inches. He determines this amount of water to be equal to 1.53 cubic feet per minute and states that the mean value of the miner's inch may, therefore, be taken at 1.5 cubic feet per minute. Bowie states that in different counties in California the value of the miner's inch varies from 1.20 to 1.76 cubic feet per minute. The reason for these variations is mainly due to the fact that when water is bought in large quantities it is discharged through large areas; thus, at Smartsville, a vertical orifice or opening 4 inches deep and 250 inches long, with a head of 7 inches above the top edge, is said to furnish 1,000 miner's inches. At Columbia Hill, an opening 12 inches deep and $12\frac{3}{4}$ inches wide, with a head of 6 inches above the upper edge, is said to furnish 200 miner's inches In Montana, the common method of measurement was formerly through a vertical rectangle 1 inch high, with the head on the center of the orifice 4 inches. The number of miner's inches was said to be the same as the number of linear inches in the rectangle; thus, under the given head an orifice 1 inch deep and 60 inches long would be 60 miner's inches.

FLOW IN BROOKS AND RIVERS

68. Weirs and orifices furnish the best and most accurate means of measuring the discharge of pipes, conduits, brooks, rivers, or channels of any kind whose volume is too great to be measured in a tank.

The best method of finding the discharge of streams where weirs cannot be used is to make a careful measurement of the cross-section and then find the mean velocity at that cross-section by one of the methods described below.

Experiments have proved that if a cross-section of a stream, as shown at (*a*), Fig. 23, is taken, the maximum velocity is at some distance below the surface where the stream is deepest. The velocity of any film surrounding

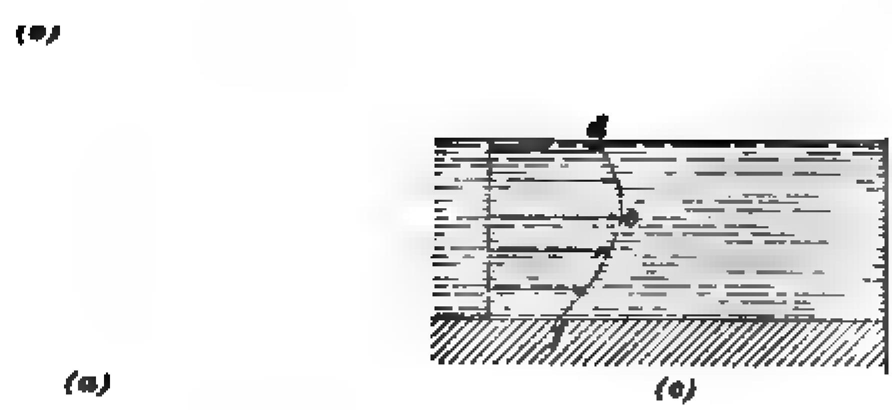


FIG. 23

the point of maximum velocity is less the greater its distance from that point, and the least velocity is in the film bordering on the bed of the stream.

At (*b*) is shown a plan of the stream, with a curve *a b c* showing the relative velocities at different points on the surface; and at (*c*) a longitudinal section in the deepest part, with a curve *d e f* showing the relative velocities from the surface to the bottom.

In order to measure the average velocity, select a part of the stream where the flow is regular and not too rapid and where the channel is as smooth and regular as possible for a distance of 100 feet or more Divide the cross-section into a convenient number of parts, as shown in Fig. 23 (*d*) and find the average depth of each of these parts by sounding. If possible, it is well to mark the cross-section by a wire *o o'* stretched across the stream, with bits of cloth or string tied to it to mark the points of division.

The area of each of the parts of the cross-section is its breadth multiplied by its average depth, and the quantity of water passing a division is the product of its area and its mean velocity of flow.

THE CURRENT METER

69. The best method of measuring the velocity is by means of a **current meter,** an instrument provided with vanes like a windmill that turn when the instrument is held

FIG. 25

in a current of water. Fig. 24 shows the original current meter of this class, known as the **tachometer.** The

instrument is fastened to a rod *d*, by which it can be held in the stream where the velocity of the current is to be measured. The vanes are attached to a spindle which is provided with a worm *w*. A frame carries a train of wheels that can be made to gear with the worm by pulling the cord *e*.

To use the instrument, it must be held steadily in the water with the wheel facing the current. When it is in the proper position, throw the recording device into gear by pulling the cord *e*, and hold it there for a certain time; then release it, taking great care to note the exact time during which the cord is held. The instrument can then be taken out of the water and a record made of the time and the number of revolutions shown by the recording device.

Fig. 25 shows one of the latest forms of current meter, in which the number of revolutions is shown by an electric register *a*, which is connected to the instrument by an insulated wire. The register may be placed on the shore or in a boat. A vane *v* holds the wheel to the current, and for deep, swift currents, a heavy weight *b* is used to sink and hold the meter in place. For shallow streams, the instrument is best used on a rod similar to that shown for the tachometer.

USE OF THE CURRENT METER

70. To find the mean velocity for a given division of any cross-section of a stream (for example, the division marked *2* in (*d*), Fig. 23) hold the meter for a given length of time at different successive depths in the middle of the division and note the velocity for each position; then take the average of these velocities as the mean velocity of the given section. Thus, suppose the mean depth of division *2* in Fig. 23 (*d*) is 8 feet and the meter shows a velocity of 1.8 feet per second at the bottom, 2.1 feet per second at 2 feet, 2.5 feet per second at 4 feet, and 2.3 feet per second at 6 feet from the bottom, and 2.2 feet per second just below the surface; then the average of the five readings is

2.18 feet per second, which may be taken as the mean velocity for this division. If this division is 6.25 feet wide, the quantity of water flowing through it is $8 \times 6.25 \times 2.18 = 109$ cubic feet per second. The quantity flowing in each division can be found in the same way and the sum of the values found for all the divisions will be the total discharge of the stream.

MEASURING THE VELOCITY BY MEANS OF FLOATS

71. If a current meter cannot be had, the velocity of a stream may be measured by means of **floats.** For this purpose, the best is the **rod float,** which consists of a rod of wood or a closed tin tube, with one end so weighted that it will float in the water in a nearly vertical position and with the lower end as near the bottom as possible without touching at any point. See *1*, *2*, *3*, etc., Fig. 23 (*d*). It is best to use a number of tin tubes about 2 inches in diameter and of such lengths that each will float in the division whose velocity is to be measured with only enough above the surface to be plainly seen. Fill the lower ends of the tubes with sand or shot until they float at the required depth. Mark two stations on the stream at least 100 feet apart in such a manner that the exact moment at which a tube passes each station can be noted. A good way is to mark the stations by a wire, as shown in Fig. 23 (*d*). Another and more accurate method is to have a transit at each station and note the time when each tube passes the cross-hair. Good results may be obtained by range stakes placed on the opposite banks.

Start the tubes far enough up the stream from the first station so that they will have the velocity of the water when they pass that station, and carefully note the time it takes each tube to pass over the distance between the two stations. The distance between the stations in feet divided by the time in seconds gives the velocity in feet per second for the division in which the tube floated. From this velocity and

the area of the division, the quantity of flow can be computed in the same manner as described for the current meter.

72. Surface floats are sometimes used for obtaining the velocity of flow. Find the average velocity in feet per second with which the float passes between two stations in the same way as has been described for rod floats. Then, if v' is the observed velocity for any division of the stream, the mean velocity of that division may be taken as

$$v = .9\,v'. \qquad \textbf{(22.)}$$

EXAMPLE.—If two stations on a stream are 200 feet apart and a surface float in a given section passes over the distance between the two stations in 7 minutes and 25 seconds, what is the mean velocity of flow in that section?

SOLUTION.—The velocity of the float is $v' = \frac{200}{445} = .449$ foot per second; therefore, the mean velocity is $v = .9\,v' = .9 \times .449 = .404$ ft. per sec. Ans.

For a rough approximation, a single surface float passing along the axis of the stream may be used. For this case, the mean velocity of the whole stream may be taken as .8 of the velocity of the float. Surface floats should be of such a form as to rise but little above the surface of the water, so as to be but little affected by currents of air, and they should be used only when there is a calm, since the wind has a great influence on the surface velocity of a stream.

73. Submerged Floats.—A float that gives better results than the surface float is shown in Fig. 26. A body heavy enough to sink and at the same time present as large a surface to the water in proportion to its weight as possible, is attached to a float on the surface by a fine cord or wire, whose length is just sufficient to allow the submerged body to float half way between

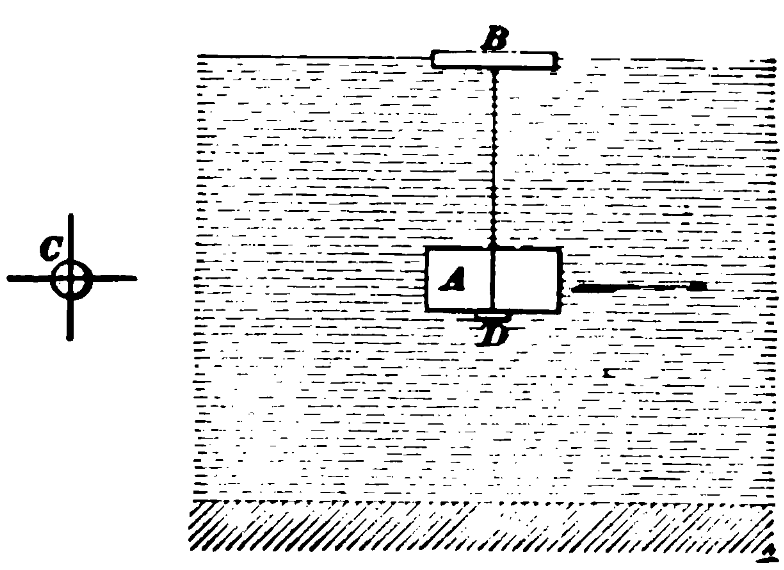


FIG. 26

the surface and the bottom. The surface float B should be made to offer as little resistance as possible to its passage through the water, the object being to get the velocity at mid depth, as nearly as possible. In the figure the submerged float A is made of two strips of tin or sheet metal fastened together in the form of a cross, as shown in plan at C. A small weight D is attached to the bottom, as shown, to assist in keeping the float in a vertical position.

EXAMPLE FOR PRACTICE

The following table gives the results of a survey for the purpose of finding the flow of a river:

Divisions of the Cross-Section	Width of Divisions	Mean Depth of Divisions	Mean Velocity in Each Division as Determined by Current Meter
No. 1	6 feet	2.12 feet	.315 foot per second
No. 2	10 feet	5.17 feet	1.227 feet per second
No. 3	10 feet	8.27 feet	2.080 feet per second
No. 4	10 feet	7.46 feet	2.049 feet per second
No. 5	10 feet	4.72 feet	1.156 feet per second
No. 6	5.25 feet	3.35 feet	.720 foot per second

What is the discharge (*a*) in each division, and (*b*) in the whole stream?

Ans. (*a*)
Division 1, 4.0068 cu. ft. per sec.
Division 2, 63.4359 cu. ft. per sec.
Division 3, 172.0160 cu. ft. per sec.
Division 4, 152.8554 cu. ft. per sec.
Division 5, 54.5632 cu. ft. per sec.
Division 6, 12.6630 cu. ft. per sec.

(*b*) 459.5403 cu. ft. per sec.

HYDRAULICS AND HYDRAULIC MACHINERY

(PART 3)

WATERWHEELS

GENERAL PRINCIPLES

ENERGY, THEORETICAL WORK, AND POWER OF FALLING WATER

1. A given weight of water W, with an available head h, may have its *energy* K expressed by the formula

$$K = Wh = W\frac{v^2}{2g}, \qquad (1.)$$

in which v is the velocity the water would attain if it fell freely through the height h. The **theoretical work** that the water can do is equal to its energy, and is the same whether this energy is the *potential energy* due to the weight W at an elevation h or the *kinetic energy* that the same weight of water would have when moving with a velocity $v = \sqrt{2gh}$.

It will be remembered that energy is absorbed only where some resistance is overcome; while falling freely through

the height h the water has done no work, consequently, none of its energy has been absorbed, and it has the same amount of energy stored in it in the form of motion at the end of its fall that it had before falling, by virtue of its elevated position.

2. Horsepower.—When water falls from a higher to a lower level, it may be made to do work by acting on the buckets of a waterwheel in such a manner that its energy is absorbed in imparting motion to the wheel. If 10 pounds of water per second fall from a height of 55 feet, the theoretical work that it can do is equal to its energy, which, according to formula **1**, is $Wh = 10 \times 55 = 550$ foot-pounds; and since 550 foot-pounds of work per second equals 1 horsepower, the water is capable of doing a theoretical amount of work equal to 1 horsepower.

Rule.—*To find the theoretical horsepower that a given quantity of water will furnish, multiply the weight of water that falls in 1 second by the distance through which it falls, and divide this product by 550; the quotient will be the theoretical horsepower.*

EXAMPLE.—What will be the theoretical horsepower that can be obtained from 750 pounds of water falling a distance of 25 feet during each second?

SOLUTION.—Applying the rule,

$$\frac{750 \times 25}{550} = 34\tfrac{1}{11} \text{ H. P. Ans.}$$

The flow of water is measured in cubic feet per second, and since 1 cubic foot of water weighs 62.5 pounds, the theoretical horsepower of a given fall of water may be expressed by the formula

$$\text{H. P.} = \frac{Q \times 62.5 \times H}{550}, \qquad (2.)$$

where Q is the quantity of water in cubic feet per second and H the total available fall in feet.

ENERGY OF A JET

3. Let (*a*), Fig. 1, be a vessel that is supplied with water in such a way that the head on the orifice in one side is

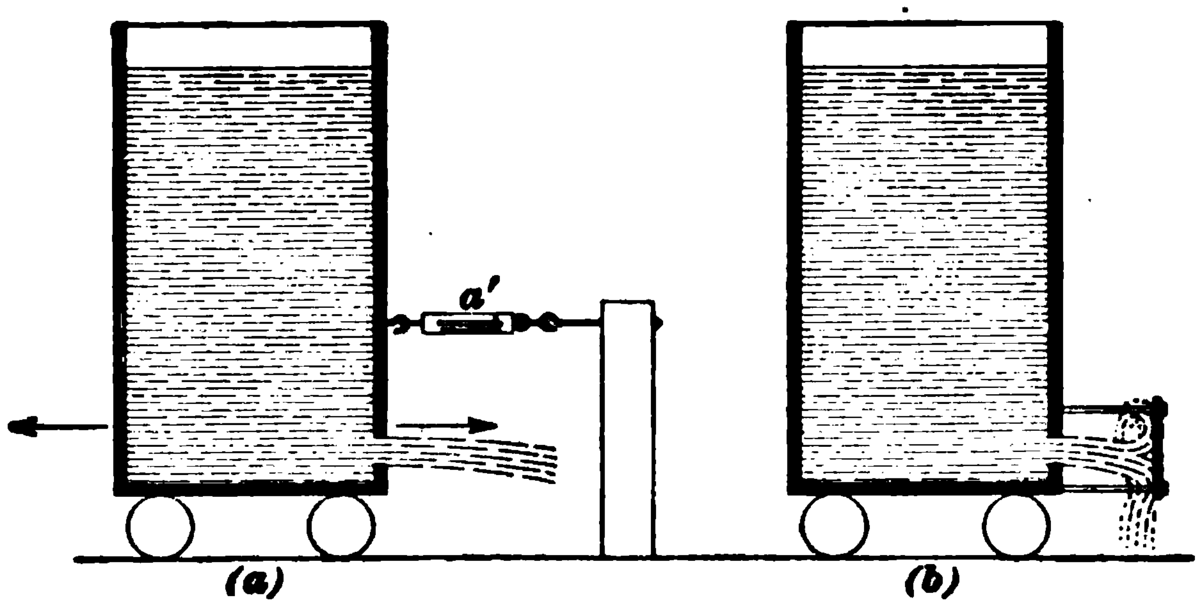


FIG. 1

constant. Water will flow from the orifice with a velocity that depends on the head of water and the form of the orifice. The *theoretical capacity of doing work* that the water has as it flows from the orifice is expressed by the formula

$$K = W\frac{v^2}{2g} = c\,W h. \qquad \textbf{(3.)}$$

Here h is the head on the orifice, v the velocity of flow from the orifice, c the coefficient of velocity for the given orifice, and W the weight of water that flows from the orifice in 1 second. If a is the area of the jet in square feet and w the weight of a cubic foot of water, then

$$W = w\,a\,v. \qquad \textbf{(4.)}$$

When this value of W is substituted in formula **3**, there is obtained

$$K = \frac{w\,a\,v^2}{2g} = c\,w\,a\,v\,h. \qquad \textbf{(5.)}$$

EXAMPLE.—What is the theoretical energy in a jet whose area is .10 square foot, if the head on the orifice is 25 feet and the coefficient of velocity .98 ?

SOLUTION.—The velocity of flow is $v = .98 \times 8.02 \times \sqrt{25} = 39.3$ feet per second; therefore, from formula **5**,

$$K = \frac{62.5 \times .10 \times 39.3^3}{2 \times 32.16} = 5{,}898 \text{ ft.-lb. per sec.} \quad \text{Ans.}$$

PRESSURE DUE TO IMPACT AND REACTION OF A JET

4. If a jet impinges against a vertical plane surface, as shown in Fig. 2, it will exert a pressure P on this surface whose intensity is expressed by the formula

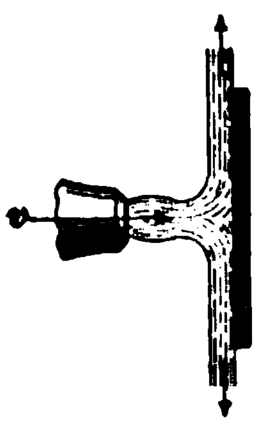


FIG. 2

$$P = w a \frac{v^2}{g} = 2 c w a h = W \frac{v}{g}. \qquad \textbf{(6.)}$$

As the jet issues from the orifice there is a reaction on the vessel equal to the pressure that is produced by the same jet as it strikes a vertical surface. If R is this reaction, its value is given by the formula

$$R = P = w a \frac{v^2}{g} = 2 c w a h = W \frac{v}{g}. \qquad \textbf{(7.)}$$

The hydrostatic pressure exerted on an area a by a head h is equal to $w\,a\,h$; therefore, with a coefficient of velocity c equal to unity, the reaction of a jet whose area is a and whose velocity of flow is produced by a head h is twice the hydrostatic pressure that would be produced on the given area by the same head.

EXAMPLE.—The area of an orifice in the side of a vessel is 2 square inches, the head on the center of the orifice is 10 feet, and the coefficient of velocity is .98. (*a*) What theoretical pressure will the jet exert when it impinges on a vertical plane surface? (*b*) What is the pressure on the vessel due to the reaction of the jet?

SOLUTION.—(*a*) From formula **7**, we have the pressure

$$P = \frac{2 \times .98 \times 62.5 \times 2 \times 10}{144} = 17.01 \text{ lb.} \quad \text{Ans.}$$

(*b*) Since the reaction is equal to the pressure due to impact, as shown by formulas **6** and **7**,

$$R = P = 17.01 \text{ lb.} \quad \text{Ans.}$$

5. The reaction and pressure of a jet may be shown by experiment as follows: Let a vessel be placed on rollers, as shown at (*a*), Fig. 1, in such a way that a very slight pressure will produce motion. When the water issues from the orifice, as shown, the vessel will begin to move in the opposite direction. If there were no friction, a spring balance attached to the vessel, as shown at a', would show a pull equal to $w a \frac{v^2}{g}$. Now, if we fasten a plate to the vessel, as shown at (*b*), Fig. 1, so that the jet strikes it, the pressure exerted by the jet on the plate will equal the reaction of the jet on the vessel, and there will be no motion.

If the plate is perfectly smooth, so that there is no loss from friction, the velocity of the water as it leaves it will be the same as the velocity with which it strikes and there will be no change in the energy contained in the water. The velocity in the direction of the jet has been entirely overcome and changed to pressure, but, since this pressure produces no motion, no work is done.

PRESSURE PRODUCED BY CHANGE OF DIRECTION

6. The amount of pressure exerted by a given jet on a surface depends entirely on the change in direction of the motion of the water.

Thus, in Fig. 3, let the jet strike the surface in such a way that it cannot spread sidewise and let the surface be perfectly smooth, so that there is no loss from shock or friction; then the water leaves the surface with the same velocity it had when it struck. If the angle between the jet and the water as it leaves the surface is $a°$, the pressure P on the surface in the direction of the jet is expressed by the formula

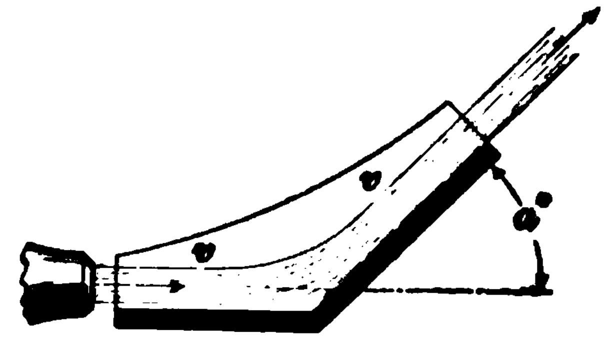


FIG. 3

$$P = (1 - \cos a°) W \frac{v}{g}. \qquad (8.)$$

If the surface is perpendicular to the jet, the angle $a°$ becomes $90°$ and its cosine is 0. The pressure, therefore, becomes $(1 - \cos 90°)\, W\frac{v}{g} = (1 - 0)\, W\frac{v}{g} = W\frac{v}{g}$, which is the same as given in formula **6.**

If the water strikes into a hemispherical cup, as shown in Fig. 4, the direction in which it leaves the cup makes an angle of 180° with the direction of motion of the jet. If the cup is smooth, so that there is no loss of velocity or energy, the pressure becomes $(1 - \cos a)\, W\frac{v}{g} = (1 - \cos 180°)\, W\frac{v}{g}$ $= [1 - (-1)]\, W\frac{v}{g} = 2W\frac{v}{g}$; that is, *the pressure is twice as great as the pressure produced when the jet struck a surface at right angles to the direction of its motion.*

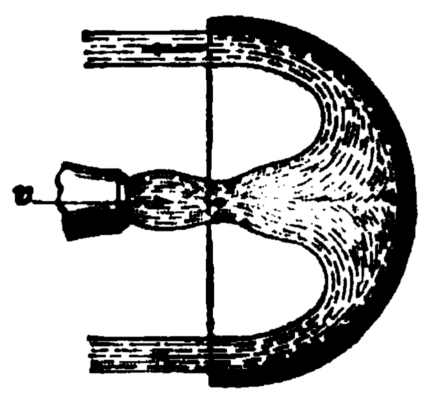


FIG. 4

EXAMPLE 1.—A jet whose cross-section is 1 square inch flows with a velocity of 75 feet per second and strikes a surface that changes its direction 35°. What pressure is exerted on the surface in the direction of the jet before striking?

SOLUTION.—From formula **8,**

$$P = (1 - .81915) \times \frac{75 \times 1 \times 62.5}{144} \times \frac{75}{32.16} = 13.73 \text{ lb. Ans.}$$

EXAMPLE 2.—If the jet in example 1 strikes a hemispherical cup, so that its direction is changed 180°, what is the pressure exerted?

SOLUTION.— $P = (1 - \cos 180)° \times \frac{75 \times 1 \times 62.5}{144} \times \frac{75}{32.16}$

$$= 2 \times \frac{75 \times 1 \times 62.5}{144} \times \frac{75}{32.16} = 151.83 \text{ lb. Ans.}$$

7. Effect When the Surface Is in Motion.—Let the surface against which the water impinges be moving in the direction of the motion of the jet with a velocity v'. The relative velocity with which the jet strikes the surface then becomes $v - v' = v_1$. The weight of water that strikes the surface is $W' = w\,a\,(v - v') = W\left(1 - \frac{v'}{v}\right)$, and the pressure P is given by the formula

$$P = (1 - \cos\ a)\ W' \frac{v_1}{g}$$

$$= (1 - \cos a)\ W\left(1 - \frac{v'}{v}\right)\frac{v - v'}{g}. \qquad \textbf{(9.)}$$

If the surface is a hemispherical cup, like the one shown in Fig. 4, this pressure is

$$P = 2\ W\left(1 - \frac{v'}{v}\right)\frac{v - v'}{g} = .0622\frac{W}{v}(v - v')^2. \qquad \textbf{(10.)}$$

Since there is motion, this pressure does work, and this work is equal to the pressure multiplied by the distance through which it acts. The maximum work will be done when the water leaves the surface with the least absolute velocity. Since the surface moves in the direction of the impinging jet, the velocity of the jet, relative to the surface, is $v - v'$, and the jet leaves the cup in the opposite direction with the same *relative velocity*. Its *absolute velocity* when it leaves the cup is evidently the relative velocity with which it leaves the cup minus the velocity of the cup $= (v - v') - v'$. If this is 0, the water has no absolute velocity and therefore no energy; all its energy has been expended in doing work; consequently the work done is a maximum. The equation $(v - v') - v' = 0$ gives us $v = 2\ v'$ or $v' = \frac{1}{2}\ v$; that is, *the theoretical work is a maximum when the velocity of the cup is one-half the velocity of the impinging jet and it is equal to the theoretical work that would be done by the energy due to the velocity of the water.*

If, instead of a hemispherical cup, the water were to strike against a flat surface, the maximum theoretical work would be only one-half as much; it will be remembered that the water must leave the surface with an absolute velocity equal to the relative velocity with which it strikes, and the energy due to this absolute velocity can do no work.

EXAMPLE.—What pressure will be exerted on a hemispherical cup by a jet of water $1\frac{1}{4}$ square inches in section moving with a velocity of

100 feet per second, if the cup moves in the same direction as the jet, with a velocity of 50 feet per second?

SOLUTION.—From formula **10**, the pressure is

$$P = .0622 \times \frac{1.5 \times 100 \times 62.5}{144 \times 100} \times (100 - 50)^2 = 101.24 \text{ lb.} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

1. If a stream discharges 120 cubic feet of water per minute, what is the theoretical work it can do with a fall of 50 feet?

Ans. 375,000 ft.-lb. per min.

2. What is the horsepower corresponding to the theoretical work in example 1? Ans. $11\frac{4}{11}$ H. P.

3. If 450 pounds of water are discharged each minute from an orifice under a head of 80 feet and the coefficient of velocity is .98, what is the horsepower equivalent to the energy in the jet?

Ans. 106.9 H. P.

4. If the jet in example 3 impinges on a plane surface at right angles to its direction of motion, what pressure does it exert?

Ans. 983.64 lb.

5. A jet of water flows from a nozzle under a head of 100 feet with a coefficient of velocity of .98 and impinges in a moving hemispherical cup. What must be the velocity of the cup in order that the water will leave it with no absolute velocity? Ans. 39.3 ft. per sec.

6. If .25 cubic foot of water is discharged from the nozzle in example 5 each second, (*a*) what is the pressure exerted on the cup? (*b*) What is the work done by this pressure in one second?

Ans. $\begin{cases} (a) & 19.097 \text{ lb.} \\ (b) & 1{,}500.86 \text{ ft.-lb.} \end{cases}$

EFFICIENCY

8. In practice, the energy of falling water is made to do work by acting on a motor in one or more of three different ways:

1. The weight of the falling water is made to press on the vanes or buckets of a wheel.

2. The pressure due to the velocity of moving water is made to act on the vanes of a wheel or turbine.

3. The hydrostatic pressure of water under a head acts on the moving parts of a turbine or other motor.

No motor can utilize all the theoretical power in the fall of a given weight of water. Part of the energy is lost in overcoming the resistances due to the friction of the water as it flows through the gates and channels leading it to the motor; part is absorbed in shocks, eddies, and foam, and in the friction of the water as it passes through the motor; and part is lost in the form of velocity as the water leaves the motor, besides the mechanical losses due to the friction of the motor itself.

9. The **efficiency of a motor** is the ratio of the actual work it will do to the theoretical work in the water used. Thus, if the actual work done by a waterwheel is equal to 750 horsepower when the theoretical work that the water would do is equal to 1,000 horsepower, the efficiency of the wheel is $750 \div 1{,}000 = .75 = 75$ per cent.

EXAMPLE 1.—What is the efficiency of a waterwheel that delivers 24 horsepower when using 660 pounds of water per second with a head of 25 feet?

SOLUTION.—The theoretical power is $\frac{660 \times 25}{550} = 30$ horsepower; therefore, the efficiency is

$$24 \div 30 = .80 = 80 \text{ per cent. Ans.}$$

Rule I.—*To find the amount of work or power that can be obtained from a given fall of water when the efficiency of the motor is given, multiply the theoretical work or power by the efficiency expressed as a decimal fraction, and the product will give the available work or power.*

EXAMPLE 2.—How many horsepower will be furnished by a turbine that uses 3,000 pounds of water per second with a head of 45 feet, if the efficiency is 60 per cent.?

SOLUTION.—The theoretical power is $\frac{3{,}000 \times 45}{550} = 245\frac{5}{11}$ horsepower; therefore, the available horsepower is

$$245\tfrac{5}{11} \times .60 = 147.27. \text{ Ans.}$$

Rule II.—*To find the quantity of water required to furnish a given amount of power with a given efficiency, divide the theoretical quantity of water by the efficiency; the quotient will be the quantity required.*

EXAMPLE 3.—How much water will be required to furnish 125 horsepower if the head is 20 feet and the efficiency of the turbine 75 per cent.?

SOLUTION.—The theoretical weight of water is $\frac{125 \times 550}{20} = 3{,}437.5$ pounds per second, and the required amount is

$$3{,}437.5 \div .75 = 4{,}583\tfrac{1}{3} \text{ lb. per sec.},$$

or

$$4{,}583\tfrac{1}{3} \div 62.5 = 73\tfrac{1}{3} \text{ cu. ft. per sec.} \quad \text{Ans.}$$

OVERSHOT WHEELS

10. Overshot waterwheels are most often applied to falls of from 10 to 50 feet. In an overshot wheel, a small amount of the work is done by the impact of the water as it

FIG. 5

enters the buckets, but much the greater part is done by the weight of the water as it descends in the buckets. Since, even under the most favorable circumstances, only one-half

of the energy due to the velocity of the entering water can be utilized by impact, it is always best to make the velocity of entry for an overshot wheel as small as is consistent with a proper filling of the buckets. For this reason the head that produces the velocity of entry is made small, and the greater part of the fall is taken up by the diameter of the wheel. Fig. 5 shows two views of an overshot wheel with curved iron buckets. The water is brought out to the crown C by a trough, or sluice, A, which may be curved towards the wheel. It should so be placed that the water will enter the first, second, or third bucket from the vertical center line of the wheel. The thickness of the sheet of water in the trough should not exceed 6 or 8 inches. The sides B of the trough are extended far enough beyond the vertical center line to insure the filling of several buckets when the wheel is to be started.

11. The first point that should be considered in the design of an overshot wheel is the velocity v of the circumference. This varies with the diameter of the wheel and ranges from $2\frac{1}{2}$ feet per second for the smallest diameters to 10 feet per second for the largest.

The diameter of the wheel is fixed by the total fall H and the head h necessary to produce the required velocity of entry v_e of the water into the bucket. The velocity of entry is always greater than the velocity of the circumference of the wheel, and its value may be taken from the formula

$$v_e = 1\tfrac{1}{2}\,v \text{ to } 2\,v. \qquad \textbf{(11.)}$$

Owing to the frictional losses in the sluice and gate, the head h required to produce the velocity v_e may be taken as

$$h = 1.1\,\frac{v_e^2}{2g}. \qquad \textbf{(12.)}$$

The diameter D of the outside of the wheel is made to correspond to the difference $H - h$ and the clearance required between the wheel and trough.

The number of revolutions per minute N is fixed by the diameter D and the circumferential velocity v, and is given by the formula

$$N = \frac{60 \times v}{3.1416\,D} = 19.1\,\frac{v}{D}. \qquad \textbf{(13.)}$$

The number of buckets Z is given by $Z = 10\,D$ to $12\,D$ and the depth of the buckets by $d = 10$ inches to 15 inches.

The breadth b of the buckets should be great enough so that they will be only partly filled, in order that they may retain the water as long as possible. It is good practice to give the breadth the value

$$b = 3\,\frac{Q}{d\,v} \text{ to } 4\,\frac{Q}{d\,v}, \qquad \textbf{(14.)}$$

where b is the breadth in feet, Q the quantity of water in cubic feet per second, d the depth of the buckets in feet, and v the velocity of the circumference of the wheel in feet per second.

For wheels in which the width is greater than 4 feet, the buckets are not properly supported by the crowns C, C, Fig. 5, and one or more intermediate supports I must be provided. The form of the buckets is a matter of great importance. They must be so made that the water will enter freely and with little shock and at the same time be retained as long as possible.

12. Fig. 6 shows a good method of laying out wood or iron buckets for an overshot waterwheel. The description given applies especially to iron vanes, as shown at (*a*); wooden vanes may be made to approximate to this form more or less closely, as shown at (*b*). First draw the center line $A\,B$ of the sheet of entering water. This curve will be a parabola and may be constructed according to the formula given for the path of a jet in *Hydraulics and Hydraulic Machinery*, Part 1. With the radius $R = \frac{1}{2}\,D$, draw the arc $C\,E$ cutting the parabola in a, so that the distance e is equal to one-half the thickness of

the sheet of entering water plus the thickness of the trough, plus the clearance between the crown of the wheel and the trough. From the same center, draw the arc $F G$ with the radius R minus d, which gives the surface of the sole of the wheel. From the point b, in which this arc cuts the parabola $A B$, draw the straight line $A b$ and mark the point a'

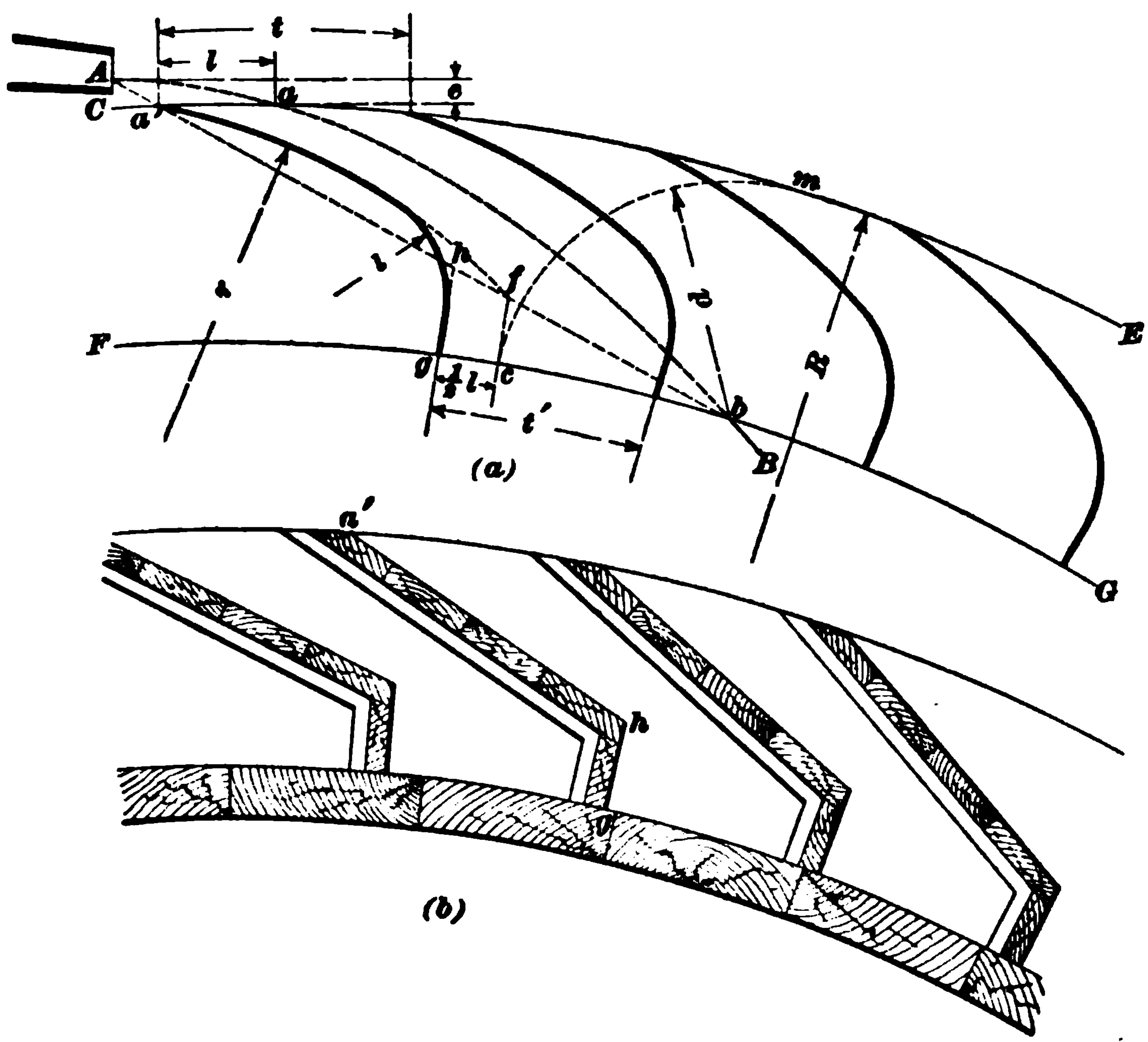


FIG. 6

where $A b$ cuts the arc $C E$. With b as a center and a radius equal to d, draw the arc $m c$ cutting $F G$ in c and draw $c f$, which is a prolongation of the radius of the arc $F G$ through c. Draw the outline $a' f$ of the bucket with the radius $r = a' b$; lay off $c g = \frac{1}{2} l$, l being the distance between the points a and a'; draw $g h$ parallel to $c f$; and finally join the curve $a' f$ and the line $g h$ with an arc whose radius is equal to l. This gives the outline for a bucket.

The pitch t is found by dividing the circumference of the wheel by the number of buckets. The pitch t' of the buckets at the sole of the wheel is found by dividing the circumference of the sole by the number of buckets. It will be noticed that in the wooden construction shown at (*b*), the points a', h, and g correspond with the points a', h, and g in (*a*). The outer edge of the buckets should be sharpened, so as to interfere with the entering water as little as possible.

Fig. 7

Fig. 7 shows an overshot wheel made mostly of wood and Fig. 8 one made entirely of iron. The power may be taken from the axle, as shown in Fig. 7, or it may be taken from gearing on the rim of the wheel, as shown in Fig. 8.

13. If the amount of water admitted to the wheel is greater than the capacity of the buckets, it will overflow and do no work. Fig. 5 shows that the water tends to leave the buckets before they reach the lowest point of the circumference of the wheel. This causes a loss of energy, since the water after leaving the buckets falls without doing work. A circular apron, arranged as shown by the dotted lines at EF in Fig. 5, tends to prevent this loss.

The wheel should not dip into the water in the tailrace, on account of the resistance this water would oppose to the passage of the buckets. If the level of the tailrace water varies, it may be better to place the wheel high enough to clear high water. The supply of water to the wheel is regulated

by a gate in the sluice, as shown at D', Fig. 5. This gate is generally operated by hand, but may be operated by an automatic governor.

14. The efficiency of the overshot waterwheel is high, ranging from 70 to 90 per cent. in well-constructed wheels. When the supply of water is small, as during a drought, the buckets are only partly filled; hence, the loss from the water leaving the buckets too early is reduced and the efficiency of the wheel increased.

Their large size makes them expensive, and for that reason they are now seldom used, since turbines furnish nearly the same efficiency at a much less cost.

FIG. 8

EXAMPLE.—Compute the principal dimensions of an overshot waterwheel to utilize 10 cubic feet of water per second with a total head of 25 feet.

SOLUTION.—If we make the circumferential velocity v of the wheel 8 feet per second and the velocity of entry $v_e = 2v$, the head required to produce the velocity of entry is $h = 1.1 \times \frac{16^2}{64.32} = 4.38$ feet. Since this corresponds to the maximum value of v_e for the assumed velocity v, we may take a value of h somewhat less, say 4 feet, and make the diameter of the wheel $D = H - h = 25 - 4 = 21$ feet.

The number of buckets may be taken as $10D = 210$, from which the pitch of the buckets at the crown becomes $t = \frac{3.1416 \times 21}{210}$ $= .31416$ feet $= 3$[illegible] inches.

If we make the depth d of the buckets 12 inches, the breadth b of the wheel may be made equal to $3 \times \frac{Q}{d\,v} = \frac{3 \times 10}{8 \times 1} = 3.75$ feet. In order that the water will enter the buckets freely, the width of the trough should be a little less than the breadth of the wheel, say 3.5 feet for this case.

The number of revolutions of this wheel with the assumed velocity v is $N = 19.1 \times \frac{8}{21} = 7.28$ per minute, nearly.

BREAST WHEELS

15. Breast wheels, shown in Fig. 9, are used for low falls, where overshot wheels would not be applicable. The

FIG. 9

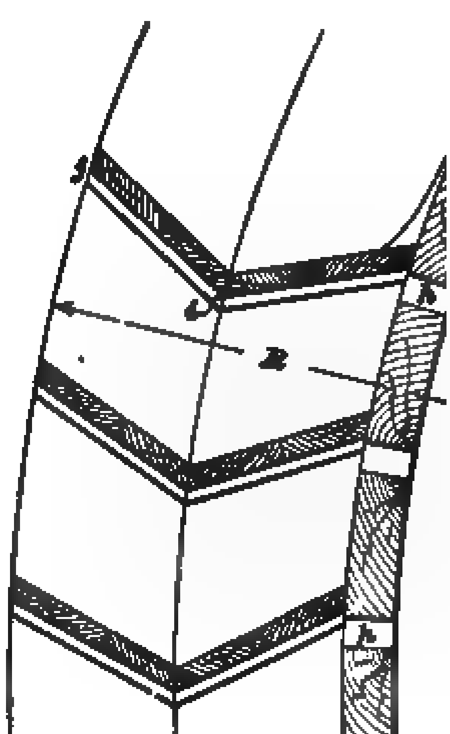
water is admitted, at a point below the horizontal center line, through an opening in the reservoir or sluice, which is regulated by a gate G. The opening should be so arranged as to lead the water to the wheel at an angle of from 10° to 25° to the radius at the point of entrance, and the inner edges of the opening should be well rounded in order to reduce the losses from friction and contraction as much as possible.

In order to permit a free entrance to the water, the buckets must be provided with holes h through the **sole**, or rim, to the interior of the wheel, for the exit of the air.

In breast wheels, the water acts more largely by its impulse than in overshot wheels, but generally the greater part of the action is due to the weight of the water.

16. The following rules may be used for the principal dimensions of a breast wheel: Velocity of circumference of wheel $v = 2$ feet per second to 8 feet per second; velocity of entry $v_e = 1\frac{1}{2}\,v$ to $2\,v$; depth of floats $d = 10$ inches to 15 inches; pitch of floats $t = d$; diameter of wheel, about twice the total head; breadth of wheel $b = 1\frac{1}{2}\frac{Q}{d\,v}$ to $2\frac{Q}{d\,v}$, where Q is in cubic feet per second, b and d in feet, and v in feet per second.

17. The floats are sometimes made nearly radial, but it is better to make them curved. Curved vanes may be



(b) (a)

FIG. 10

laid out according to the following method, which applies especially to iron floats; see Fig. 10 (a):

Draw the center line $A\ B$ of the path of the entering water, as was done in connection with Fig. 6; then draw the center line $m\ n$ of the floats so that it is nearly tangent to $A\ B$ and draw arcs of the outer and the inner edges of the floats, as $a\ c$ and $b\ e$. From A draw the radial line $A\ f$, and from the point of intersection g of this line and the inner edge $b\ e$ of the floats draw a line $g\ k$ tangent to $b\ e$. Through the point i where $A\ B$ cuts $m\ n$, draw a radial line $O\ P$ and through i draw $i\ j$ at an angle of 30° with this line; then join the line $i\ j$ and the tangent $g\ k$ by an arc whose radius is $\frac{1}{2}\ d$. As previously, t is the pitch of the floats.

Wooden floats may be made to approximate this form, as shown in Fig. 10 (*b*). Holes through the sole, to provide for the outflow of the air, are shown at *h*.

18. The breast or curb *S*, Fig. 9, is usually made either of masonry, lined with a smooth coating of cement to make it fit closely to the wheel, or of wood.

If made of the latter, it is liable to become distorted as the wood swells, and so cause friction against the wheel.

A clearance space is always necessary between the curb and wheel. This should be made as small as possible in order to prevent loss by leakage between the curb and floats. A distance of from $\frac{3}{8}$ inch to $\frac{3}{4}$ inch will be enough with well-constructed wheels.

19. The general construction of breast wheels is nearly the same as for overshot wheels. Their efficiency is less than for overshot wheels and is seldom more than 70 per cent. The usual range is from 50 to 70 per cent., the smaller value being for the smaller sizes.

EXAMPLE.—Compute the principal dimensions of a breast wheel to utilize 18 cubic feet of water per second with a total head of $9\frac{1}{2}$ feet.

SOLUTION.—Assume a velocity of the circumference of the wheel as $v = 6$ feet per second and the velocity of entry as $v_e = 1\frac{1}{2}\ v = 6 \times 1\frac{1}{2} = 9$ feet per second. The depth of the floats may be made 12 inches; then the pitch will be approximately 12 inches.

The diameter of the wheel $= 2 \times H = 2 \times 9\frac{1}{2} = 19$ feet. With this diameter and the assumed pitch, the number of buckets is $\frac{3.1416 \times 19}{1} = 59.69$, or in even numbers 60.

Make the breadth of the wheel $b = 1\frac{1}{2}\frac{Q}{d\,v} = 1\frac{1}{2} \times \frac{18}{1 \times 6} = 4.5$ feet.

The number of revolutions with the assumed diameter and velocity v is $N = \frac{3.1416 \times 19}{6} = 9.95$ per minute.

UNDERSHOT WHEELS

20. Undershot wheels are used for falls of 6 feet or less. The least efficient form has straight radial floats that

FIG. 11

are acted on directly by the current of a swiftly flowing stream (see Fig. 11). In this case, the water acts only by impulse and the efficiency is seldom greater than 25 per cent. The dimensions of these wheels may vary from 12 feet to 24 feet in diameter, with from 24 to 48 floats. The depth of the floats for best effect should be at least three times the depth of the stream. The velocity of the circumference of the wheel should be about one-half the velocity of the

water in the stream; the depth of the stream should be from 4 to 6 inches and the depth of the floats 12 to 20 inches.

There should be as little clearance as possible between the floats and the bottom and sides of the race.

21. A formula for the horsepower that may be developed by the use of an undershot wheel, such as has been described, is

$$\text{H. P.} = .0012\,(v - v_1)\,v_1\,Q, \qquad \textbf{(15.)}$$

where v is the velocity of the water in race in feet per second, v_1 the velocity of the circumference of the wheel in feet per second, and Q the quantity of water flowing in cubic feet per second.

22. For a **paddle wheel** suspended in an unconfined current, the horsepower may be computed from the formula

$$\text{H. P.} = .001\,(v - v_1)\,v\,v_1\,F, \qquad \textbf{(16.)}$$

where v is the velocity of the current in feet per second, v_1 the velocity of the circumference of the wheel in feet per second, and F the area of the immersed portion of the float in square feet.

In order to obtain the best effect from paddle wheels, the number of floats should be great enough so that at least two will always be immersed, the velocity of the circumference of the wheel should be about .4 of the velocity of the current, and the floats should have an inclination against the stream of about 60°.

The efficiency of an undershot wheel is increased by the use of a breast or curb similar to that described for the breast wheel.

IMPULSE WHEELS

23. The most simple form of **impulse wheel** consists of a wheel provided with a series of flat radial vanes around its circumference, similar to the paddle wheel of a steamboat. A wheel of this kind can never have a high efficiency,

since the water must leave the vanes with an absolute velocity nearly equal to the relative velocity with which it strikes. Experiments have shown a maximum efficiency of a little more than 40 per cent.

THE PELTON WATERWHEEL

24. The Pelton waterwheel, shown in Fig. 12, is an impulse wheel that is used for very high heads and comparatively small volumes of water. The jet from the nozzle A,

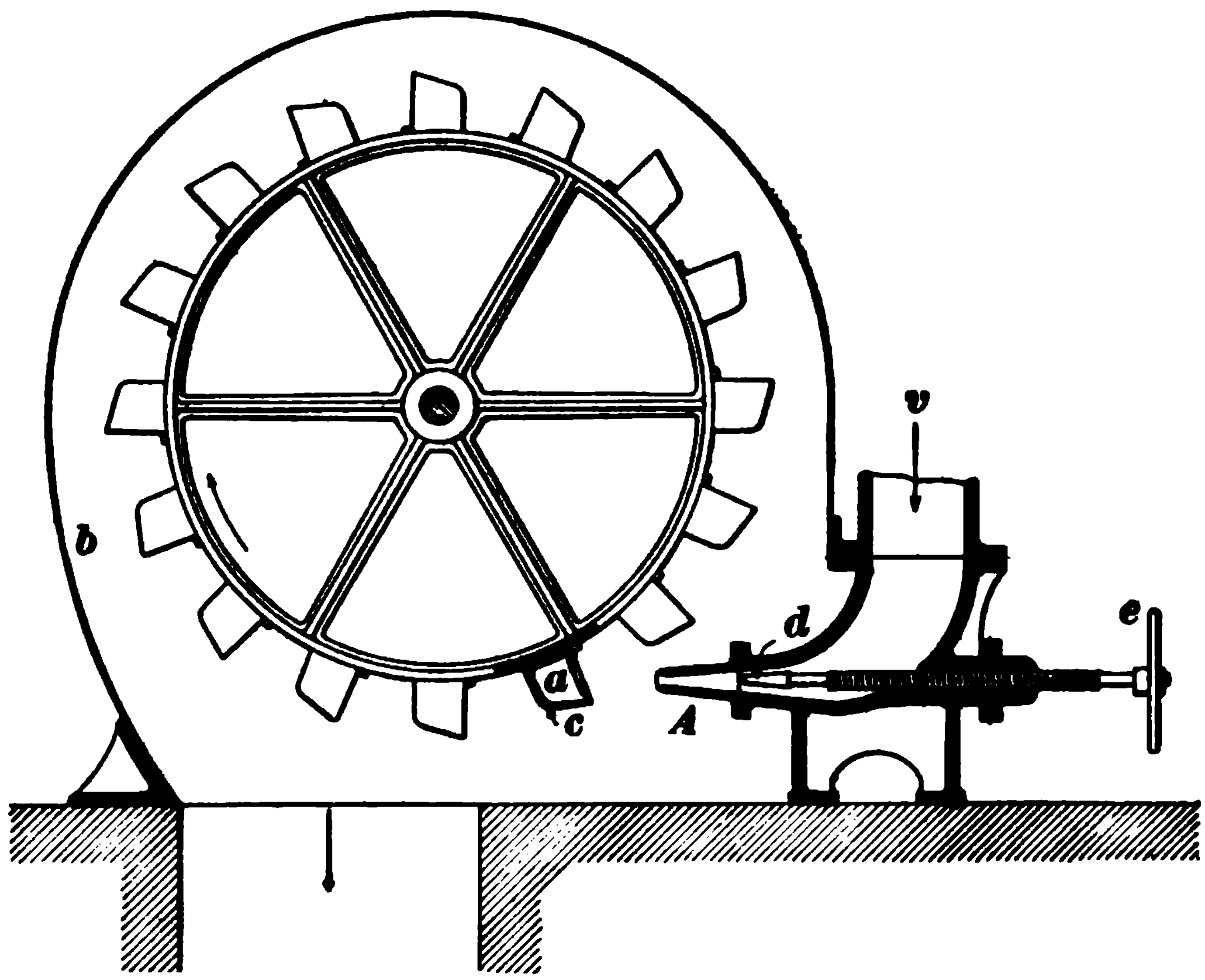


FIG. 12

which impinges on the raised center a of the cups c, is deflected to both sides, and finally leaves the cups in a direction tangent to their outer edges. In this way, the direction of the motion of the jet is changed nearly 180°; and when the velocity of the cup is equal to one-half the velocity of the jet, the theoretical efficiency of the wheel is 100 per cent. Experiments have shown that the actual efficiency is sometimes nearly 90 per cent. and that the best efficiency is obtained when the number of revolutions is such that the actual velocity of the cups corresponds nearly to the theoretical velocity.

The loss of efficiency is due to the friction of the water in the cups and the energy that is lost in the absolute velocity the water has when it leaves them.

25. Fig. 13 shows two sections of the cups and the common method of fastening them to the rim of a cast-iron wheel. The inclination of the edge *a* is such that the water as it leaves the cup flows clear of the wheel and offers no resistance to its motion. The faces of the cups are also inclined to the radius of the wheel, as shown, in order to give the water a slight

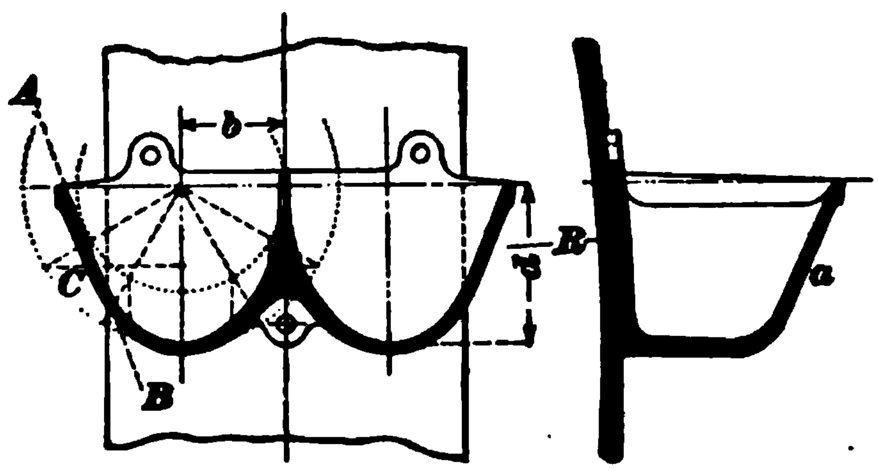


FIG. 13

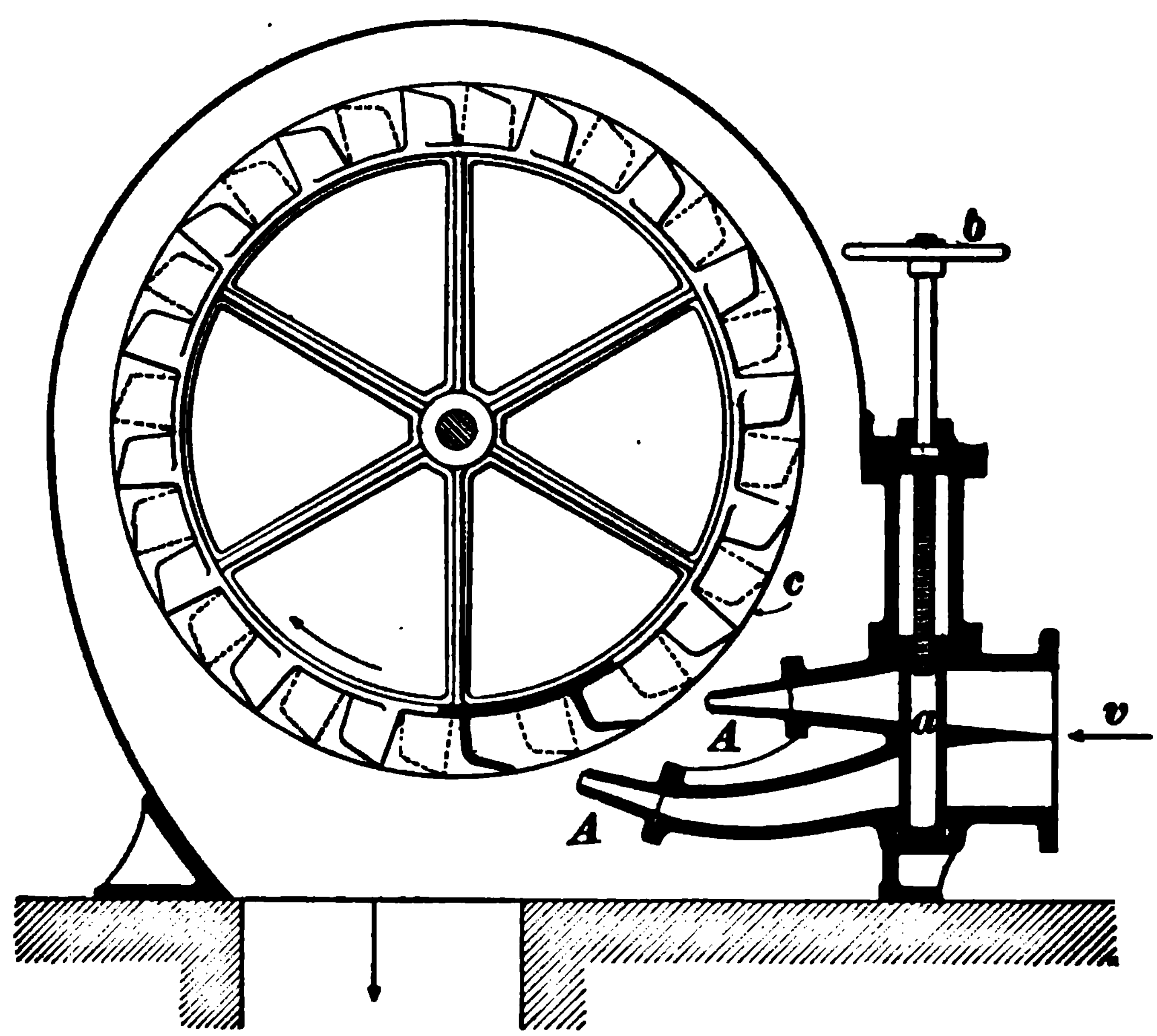


FIG. 14

tendency to flow from the center of the wheel as it reacts from the cups. The outer edges of the cups are made sharp,

so as to offer as little resistance to the water as possible, and the inside surface is sometimes finished for the purpose of reducing the loss by friction.

The wheel must be provided with a cover or casing *b* (see Fig. 12) to prevent spattering of the water.

THE LEFFEL CASCADE WHEEL

26. The **Leffel cascade wheel,** shown in Fig. 14, is another impulse wheel that acts on exactly the same principles as the Pelton waterwheel. The cups are cast solid with the rim of the wheel and are placed alternately on the two sides of a rib *c* that surrounds the rim. The jet strikes the sharp outer edge of this rib and is deflected to both sides through the cups, the action being the same as in the Pelton waterwheel.

Fig. 15 shows two sections of the rim and buckets of the Leffel cascade wheel.

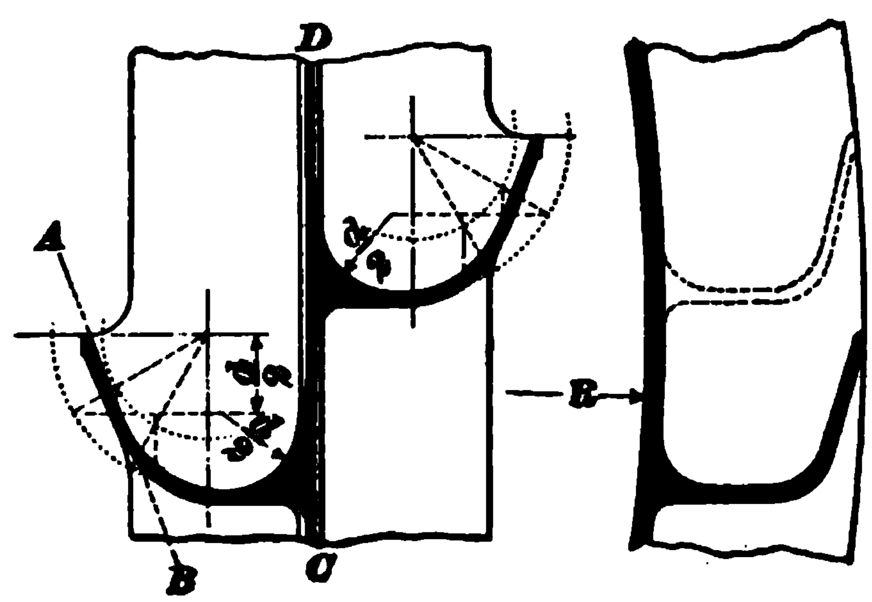


FIG. 15

NOZZLES

27. The water for impulse wheels is discharged through nozzles, and in order to secure a high efficiency, it is necessary that the pressure head in the pipe be converted into velocity in the issuing jet with as little loss as possible. The ordinary method of reducing the flow of water by means of a valve in the pipe necessarily causes extra resistances and reduces the head. Fig. 12 shows a nozzle applied to a Pelton wheel, in which the flow of water is regulated by a conical plug *d* operated by the hand wheel *e*. This has the practical effect of varying the size of the nozzle and, hence, the quantity of water without a corresponding change in the velocity.

The power of a given size of impulse wheel may be increased by increasing the number of nozzles. This increases the amount of water used and gives the same efficiency for all the nozzles that would be obtained if only one were used. Fig. 14 shows a double nozzle *A*, *A* applied to a Leffel cascade wheel, in which a gate valve *a*, operated by the hand wheel *b*, opens the orifices to the nozzles successively. By this means, as many of the nozzles may be opened as are required to furnish the necessary power and the water will be used without loss of head.

Another method, which is used when the supply of water is variable, is to have a number of nozzles of different sizes to correspond with the supply of water. When the water supply is small, a small nozzle can be used, and in this way the greatest efficiency can be obtained; and when the supply is increased, a larger nozzle enables the full power to be obtained without loss.

CALCULATIONS FOR IMPULSE WHEELS

28. The **circumferential velocity** of an impulse wheel, i. e., the actual velocity of the cups, depends on the head, and hence the velocity of the jet. With a properly designed nozzle, the velocity of the jet will be nearly that due to the pressure head in the end of the pipe, and the best efficiency is obtained when the velocity of the cups is about one-half the velocity of the jet.

29. The **number of revolutions,** with a given velocity at the circumference, varies inversely as the diameter of the wheel; it is, therefore, possible to make the number of revolutions correspond to the speed of the machinery to be driven within certain limits. In accordance with this principle, wheels are often designed so as to run at a speed that enables them to be connected directly to the shafts of dynamos, centrifugal pumps, or similar machinery, without the use of belts or gearing. The Pelton and Leffel cascade wheels are seldom used for heads of less than 50 feet, but

are applicable to falls of any greater height. A number of wheels are in use under heads of more than 2,000 feet.

EXAMPLE.—What should be the diameter of an impulse wheel that is to be directly connected to the shaft of a dynamo, if the pressure head is 275 feet? The dynamo is required to make 850 revolutions per minute and the coefficient of velocity of the jet is .98.

SOLUTION.—The velocity of the jet is $.98 \times 8.02\sqrt{275} = 130.84$ feet per second. The circumferential velocity of the wheel is, therefore, $130.84 \div 2 = 65.17$ feet per second, and the diameter required for 850 revolutions per minute is

$$d = \frac{65.17 \times 60}{850 \times 3.1416} = 1.464 \text{ ft., say 18 in.} \quad \text{Ans.}$$

TURBINES

IMPULSE TURBINES

30. Fig. 16 shows an **impulse turbine** with a horizontal shaft, in which the water is admitted to the **vanes,**

FIG. 16

or **buckets,** *A* through the opening *D* and curved **guide vanes** *E*. The water flows through the vanes from the

center towards the circumference, and it is, therefore, called a **radial outward-flow turbine.** The quantity of water is regulated by the revolving sleeve, or liner, *e*, which is operated by means of the hand wheel *G*. The casing *C* catches the water thrown from the wheel by centrifugal force and prevents spattering. Turbines of this form are adapted to high falls and small quantities of water and have been largely used in the mountain districts of Europe. In America, the wheels of the Pelton type are generally used for the same conditions, and it is probable they are to be preferred, since they give nearly the same efficiency with greater simplicity and smaller first cost.

31. Impulse or **Girard turbines** are built with either vertical or horizontal shafts, and the flow of water through

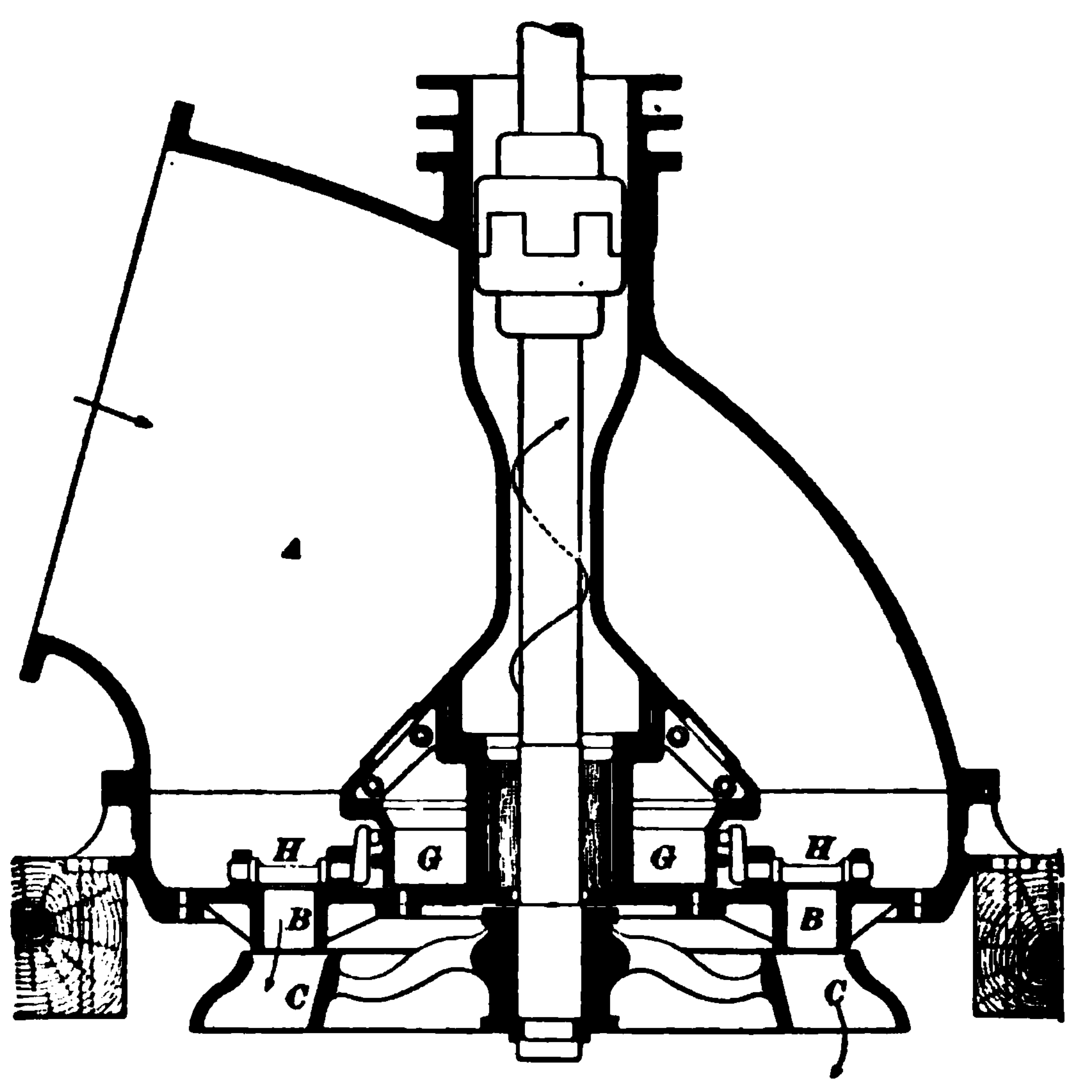


Fig. 17

the vanes may be either towards, away from, or parallel with the shaft. They are also adapted to various heads and quantities of water.

The space between two buckets should never be filled by the water and should always be open to the atmosphere. In order to secure the last condition, openings are often provided in the casing surrounding the vanes, which allow the free circulation of the air. Fig. 17 shows a section of an impulse turbine with a vertical shaft. The water is brought in through a pipe to the cast-iron casing *A*, flows downwards between the guide vanes *B*, and strikes the upper ends of the wheel vanes *C*. After passing over the wheel vanes, it falls to the tail-water below the wheel. This turbine has guide vanes around its entire circumference, and the passage between each pair of vanes is covered by a gate *H*. A ring-shaped cam *G*, which can be turned around the vertical axis by means of gearing not shown in the figure, operates the gates *H* one at a time. In this way, as many of the passages may be opened as are necessary to furnish the required power.

THEORY OF THE IMPULSE TURBINE

32. The action of a jet on the vane of an impulse turbine is as follows: The water flows from the guide passage *A*, Fig. 18, with a velocity v_e that is the result of the

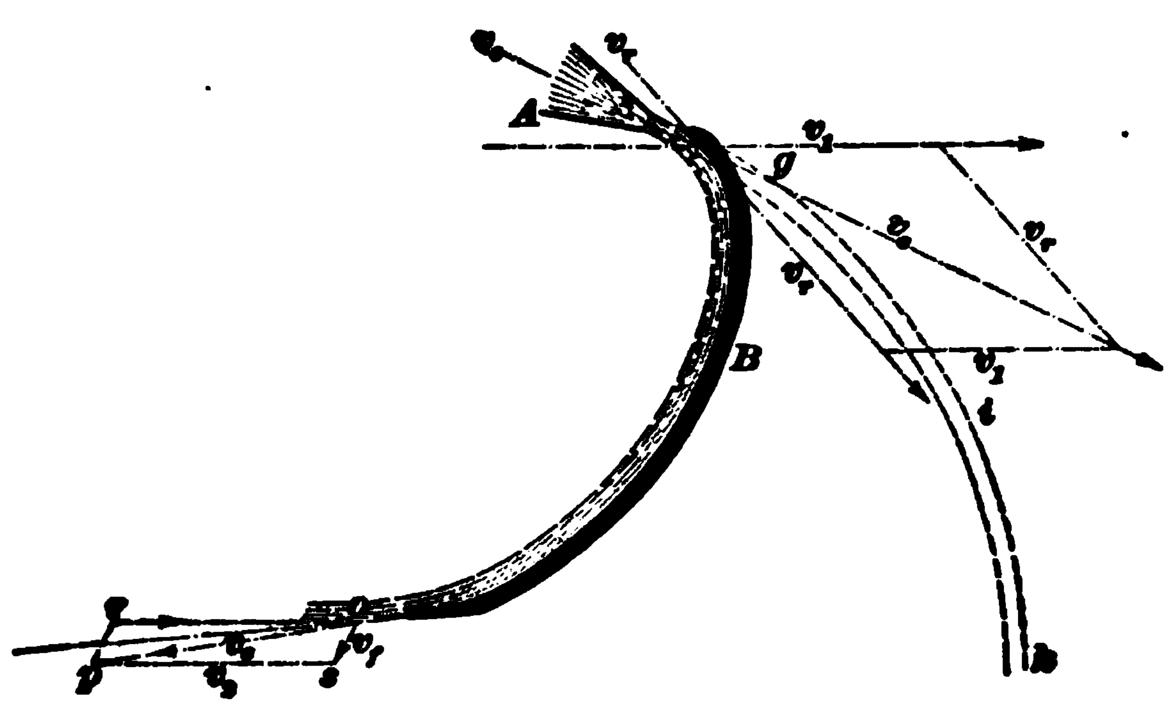


FIG. 18

pressure head h in the pipe from which A is supplied. This jet strikes one end of the curved vane B, which moves with a velocity of v_1. The relative velocity and direction with

which the jet strikes are given by the resultant v_r of the velocities v_e and v_1. It is evident from the figure that the form of the upper end of the vane and its velocity may be so proportioned that the relative motion of the jet as it strikes will be tangent to the surface of the vane.

In passing over the surface of the vane, the jet is gradually deflected from its original path and follows a path indicated by the dotted lines $g\,i\,k$. This path, called the **absolute path of the jet,** is the resultant of its relative motion over the surface of the vane and the motion of the vane. The water flows from the end of the vane in a relative direction that is tangent to the surface of the lower end of the vane, and with a direction and velocity represented by the line $o\,p = v_c$. The absolute direction and velocity are represented by the line $o\,s$, which is the resultant of the relative velocity v_c and the velocity v_2 of the lower end of the vane. If the surface of the vane could be perfectly smooth, so that there would be no loss from friction, the relative velocity v_c with which the water leaves the vane would be equal to the relative velocity of entrance v_e. This condition, however, can never be fulfilled in practice.

33. The change in the direction of the water as it passes over the vane produces a pressure on the vane that, since it produces motion, does work. Part of the kinetic energy of the water is absorbed in doing this work. Of what remains, part is absorbed in producing foam and shock as the water strikes the vane, part in friction, and part remains in the kinetic energy due to the absolute velocity v_f with which the water leaves the vane. In order that the efficiency will be high, these losses of energy must be made as small as possible.

The loss from shock is made small by making the upper edge of the vane sharp and the end of such a form as will cause the jet to strike with relative motion that is tangent to its surface. In order that the absolute velocity of the water leaving the vane will be as small as possible, it is evident that the angle $r\,o\,p$ that the tangent to the surface

of the vane at o makes with the direction of motion of the vane must be as small as possible. This angle cannot be made zero, however, as will be evident from an inspection of Fig. 16, since, if it were, the passages between the vanes would be closed.

REACTION WHEELS

34. In the simple **reaction wheel,** commonly known as **Barker's mill,** the motion is produced by the reaction of a jet of water that issues from an orifice.

Water is brought through the curved pipe *A*, shown in Fig. 19, into the revolving head *B*. From *B* the water flows through the nozzles *C*, and the pressure caused by the reaction of the issuing jets causes the head *B* to revolve. *B* is keyed to a shaft *S*, from which the power is taken, and a cup-leather packing *P* is provided to prevent leakage through the joint between *B* and the pipe *A*.

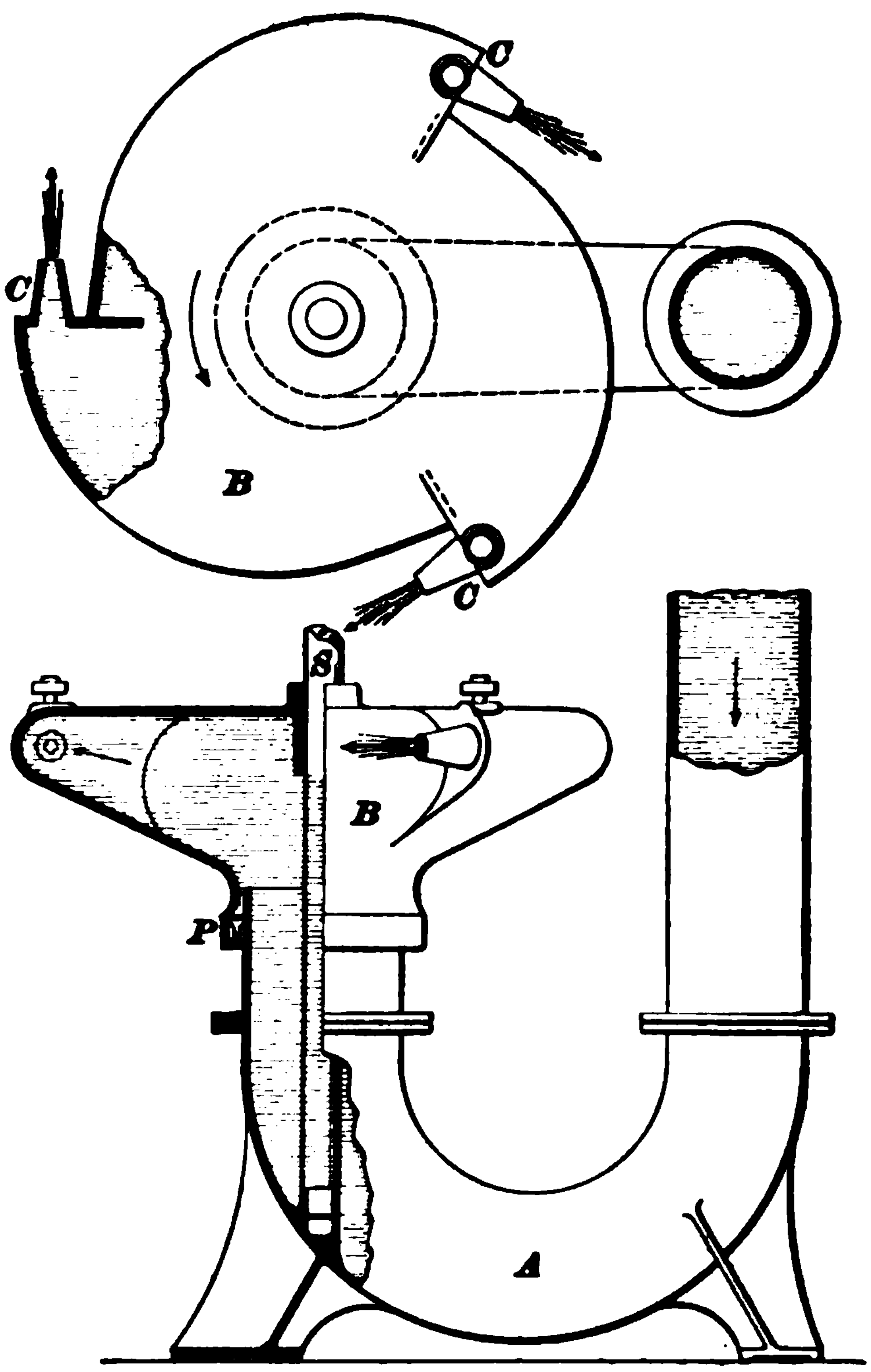


FIG. 19

The efficiency of the simple reaction wheel can be but little more than 50 per cent., even under very favorable conditions, and for that reason it is not now used as a motor. A familiar example of the simple reaction wheel is the revolving lawn sprinkler.

REACTION TURBINES

35. Reaction turbines are very similar in their principles of construction to the impulse turbines just described. The water passes through the space between a series of curved guides or vanes that deliver it on to the vanes of the wheel; the wheel vanes are curved in a direction opposite to the curvature of the guide vanes, so that they change the direction of motion of the water.

In an *impulse turbine*, the water flows freely into the wheel from the guide buckets in the form of a jet, whose velocity is produced by the head of water on the buckets, and it passes over the wheel vanes without filling the space between them; the passages through the wheel are open to the air, and the pressure in the space between the wheel vanes is, consequently, nearly equal to the atmospheric pressure; the acting force is almost entirely the pressure due to the impulse of the jets issuing from the guides.

In a *reaction turbine*, the passages between the wheel vanes are completely filled, so that the flow is said to be *continuous;* the pressure and the velocity of the water as it enters the wheel may, under different conditions, be equal to, greater, or less than the pressure and the velocity due to the head on the wheel; and the forces that act on the wheel vanes are: first, a certain amount of static pressure; second, the pressure caused by the change in direction of the moving water; and third, a pressure due to the reaction of the water as it issues from the wheel vanes. In most cases, the greatest of these forces is the pressure caused by the change in direction of the moving water in its passage through the wheel. If a reaction turbine is working open to the air and the flow from the guides is restricted so that the passages between the wheel vanes are only partly filled, it becomes an impulse turbine; hence, the same wheel under different conditions may work either as a reaction turbine or an impulse turbine.

TYPES OF TURBINES

36. Fig. 20 shows the general arrangement of an **outward-flow turbine,** also called a **Fourneyron turbine,** from its inventor, with a plan of the wheel vanes and guide vanes. The water is brought in at the center, passes *outwards* between the curved guide vanes *B*, *B* to the wheel vanes *C*, *C*, and is discharged at the *circumference* of the wheel. The flow of water is regulated in the wheel shown by a cylindrical gate that can be raised or lowered in an annular space between the wheel and guides. Various other methods of regulating the flow are also used, some of which will be described.

FIG. 20

37. Fig. 21 shows a vertical section of an **inward-flow** or **Francis turbine.** Here the water enters the guides *B* from the outside, passes *inwards* to the wheel vanes *C*, and is discharged near the *center* of the wheel. These wheels are often placed some distance above the level of the tail-water, as shown, and discharge into an air-tight tube, commonly called a *draft tube.* This places the wheel at a point where it can be easily inspected or repaired and at the same time utilizes the total fall.

FIG. 21

The supply of water in the wheel shown in the figure is regulated by a gate G at the outlet of the draft tube.

38. In Fig. 22 is shown a **downward-flow** or **Jonval turbine.** Here the general direction of the water is always parallel to the shaft A or axis; hence, wheels of this class are also known as parallel-flow turbines and axial turbines.

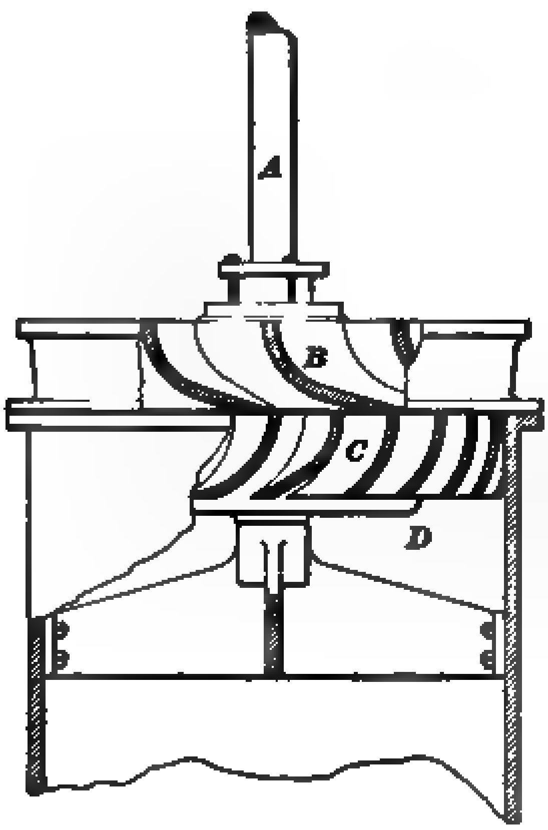


FIG. 22

The water usually enters the guides B from above and is discharged downwards through the wheel C into a draft tube D, as shown in the figure. The discharge may also take place into the air or tail-water without the use of a draft tube.

39. Mixed-Flow Turbines.—Many American turbines are made with the eel vanes so curved that the water enters the wheel in a radial direction, like an inward-flow turbine, and is discharged in a downward or axial direction. These are called *mixed-flow turbines.*

40. Fig. 23 shows the wheel of a **Risdon turbine** with the double curvature of the vanes. This wheel is cast in one piece. The band a serves the double purpose of strengthening the wheel and of making the proper form for the passage of the water through the lower part of the wheel, confining it on all sides.

FIG. 23

DRAFT TUBES

41. Let D, Fig. 24, be a turbine in a tight vertical penstock A that connects the reservoir B with the tail-water C. The total head, i. e., the difference in level between the surface of the water in the reservoir and the surface of the tail-water, is h. This is made up of the head h_1 above the turbine and the head h_2 between the turbine and the level of the tail-water. The pressure of the atmosphere acting on the surface of the water in the reservoir and also on the surface of the tail-water is equal to a head of 34 feet, which we will call h_a. Now, if the turbine is entirely closed, so that no water can pass through it, the pressure on the top is evidently equal to the pressure due to the head h_1 plus the atmospheric pressure; and the upward pressure on the under side of the wheel is equal to the pressure of the atmosphere minus the pressure due to the head h_2.

FIG. 24

The pressure that tends to produce a flow through the wheel is, according to the principles of hydromechanics, the difference between the pressures on its two sides; hence, if we substitute for the pressures their equivalent heads, we have the head that tends to produce the flow

$$(h_1 + h_a) - (h_a - h_2) = h_1 + h_2 = h. \qquad (17.)$$

42. Turbines are sometimes placed below the surface of the tail-water, as shown in Fig. 25, in which case they are

said to work *drowned*. Here the effective head is still the difference in level between the surface of the water in the reservoir and the surface of the tail-water, as will be made clear from the following: The total pressure on the top of the turbine is the pressure due to the head h_1 plus the pressure of the atmosphere, i. e., the pressure due to the heads $h_1 + h_a$; and the pressure on the under side of the turbine is the pressure due to the atmosphere plus the pressure due to the head h_2. The difference expressed in terms of the heads is, therefore,

FIG. 25

$$(h_1 + h_a) - (h_a + h_2) = h_1 - h_2 = h. \qquad (18.)$$

43. The theoretical limit of the distance h_2, Fig. 24, that the turbine may be placed above the surface of the tail-water and utilize the total head, is never greater than 34 feet, since this is the limit in the height of a column that will be supported by the pressure of the atmosphere. The expression for the pressure head under the wheel $(h_a - h_2)$ shows that the pressure under the wheel is always less than the pressure of the atmosphere, and is decreased as h_2 is made greater. Owing to this reduced pressure, there is a tendency for the air to leak into the draft tube; air will also separate from the water that passes through the wheel. If the tube is very long, this air will collect in the upper end, thus reducing the height of the column h_2, and, consequently, the total effective head, as will be plain from an inspection of Fig. 24 and formula **17.** In practice, turbines, unless they are very small, are seldom placed more than 20 feet above the level of the tail-water. The diameter

of draft tubes is generally fixed by the design of the wheel. They are best made of cast-iron or riveted plate and in all cases must be *thoroughly air-tight*. Wooden tubes are sometimes used, but they are not to be recommended on account of the difficulty in preventing leakage. The lower ends should extend at least 4 inches below the surface of the tail-water at its lowest stage and must open into the tail-race in such a manner that the outflow will be free. Any obstruction to the flow from the draft tube causes a loss of effective head and a consequent loss in efficiency.

44. Expanding Draft Tubes.—The efficiency of a turbine in which the absolute velocity of discharge from the wheel vanes is high may be increased by the use of a draft tube, the cross-section of which increases with the distance from the wheel. The area of the tube at the wheel should be nearly equal to the discharge area of the wheel buckets, in order to prevent a sudden change in velocity in the entering water, and its section should be gradually enlarged towards the outlet. Fig. 26 shows a draft tube for a mixed inward- and downward-flow turbine constructed on this principle. It consists of a bell-shaped, cast-iron tube A, provided with a flange B for bolting it to the lower end of the wheel case.

FIG. 26

Inside of this tube is a conical center C with its base, the diameter of which is equal to the inside diameter of the wheel buckets, placed upwards. This leaves an annular space for the passage of the water, whose area is nearly equal to the discharge area of the wheel buckets at its upper end and gradually increases towards the outlet. The tube has a broad base made to rest on the bottom of the tailrace, and a cone S in the base serves to change the direction of flow gradually from vertical to horizontal.

45. The **Boyden diffuser** shown in Fig. 27 is a device used on outward-flow turbines for the same purpose as the draft tube with enlarged section. It consists of a stationary annular casing C that surrounds the wheel, and into which the water from the wheel buckets B is discharged. The area of the passages through this casing gradually increases from the wheel outwards, as shown. The result is a decrease in the velocity of the outflowing water.

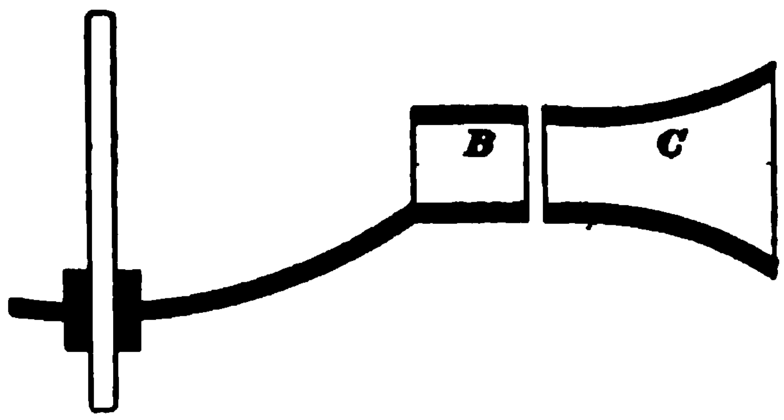


FIG. 27

The value of a diffuser or draft tube with increasing area depends on the absolute velocity of flow from the wheel buckets. If this velocity is small, the water carries very little energy with it and no device can be of much value. It sometimes happens, however, that, with a given diameter of wheel or number of revolutions, the velocity of outflow from the wheel cannot be made small; then the discharge opening must be given such a size that the passages between the vanes will not be too narrow. The following computation will serve to illustrate the effect of a device for reducing the velocity of the outflowing water: If the absolute velocity v_f is 10 feet per second, the corresponding loss of head is

$$h_f = \frac{v_f^2}{2g} = \frac{10^2}{64.32} = 1.55 \text{ feet.}$$

If, by the use of a diffuser, the final velocity of outflow is reduced one-half, the corresponding velocity head will be $h_f = \frac{5^2}{64.32} = .39$ foot, which shows a theoretical gain of 1.16 feet in the effective head. With a wheel so designed, however, that the velocity of the outflow v_f from the wheel is not more than 5 feet per second, it is seen that the loss in head will be so small as to make any saving by means of a diffuser difficult.

REGULATING TURBINES

46. The method of regulation has an important bearing on the efficiency of a turbine. In general, the best efficiency for a given head is obtained only when the wheel is running at the speed for which it was designed and at **full gate,** i. e., with the gate wide open. A partial closing of the gates reduces the available head or increases the frictional losses in the wheel itself, either of which results

Fig. 28

in a loss in the energy available for doing useful work. If the supply of water is unlimited, the loss in efficiency with partly closed gates is a matter of little or no importance, since the only object is a reduction of the power of the wheel to correspond to the work to be done. When, as is more often the case, however, it is desirable to obtain the greatest possible work from the stream at low water, and this work is less than the wheel is designed to furnish at full

gate, it becomes necessary to run the wheel at part gate, and then the loss in efficiency becomes more serious.

One of the simplest methods of regulation is shown in Fig. 28. Here the area of the passages through the wheel is not changed, but the head is reduced by varying the opening of the gate that admits the water to the flume or penstock *B*. The flow through the wheel is thus reduced, but there is a loss of head equal to the distance between the level of the water in the sluice and the level in the penstock.

When turbines are governed by regulating the discharge from the draft tube, as shown in Fig. 21, the head above the wheel is not reduced, but the pressure in the draft tube is increased, which has the same effect as raising the level of the tail-water. The result is a loss in the available head and a consequent loss of efficiency.

47. Regulation by means of a cylindrical gate between the guides and wheel vanes produces a sudden change in the cross-section of the passages at part gate. This absorbs energy by the production of eddies and foam and the contraction of the stream as it flows from the reduced section. When the turbine runs above the tail-water and without a draft tube, this method of regulation may reduce the flow through the wheel so much that the space between the buckets will be but partly filled, in which case it becomes an impulse wheel and its action similar to the action of an impulse turbine.

48. A method of regulation that gives better results is to have the wheel made in sections. Fig. 29 shows a double downward-flow turbine known as the **Geyelin-Jonval,** in which both guide and wheel vanes are in two

Fig. 29

independent sections. The inner section $A' B$ may be closed by lowering the cylindrical gate O, or the outer section $A B'$ by lowering the gate P. In this way, either section may be used alone or both may be used together, and a wide range of power may thus be obtained without a serious sacrifice of efficiency.

Fig. 30 shows an outward-flow turbine in three sections, *3*, *2*, and *1*, formed by dividing the spaces between the wheel vanes by horizontal partitions. D is a cylindrical gate between the wheel and guide vanes, which entirely closes the successive passages as it is lowered. This improves the action of the wheel at part gate, but the partitions offer extra resistance to the passage of the water at full gate.

FIG. 30

49. Fig. 31 shows the *Thompson vortex wheel*, an inward-flow turbine in which the flow is regulated by varying the opening between the guide vanes. The four guide vanes C are pivoted near their inner ends and the opening between them is regulated by the hand wheel D, which swings the outer ends of the guides by means of the combination of worm, worm-wheel, links, and levers. This forms what is known as the register gate, from its resemblance to a register used in regulating the flow of air through a heating

flue from a furnace. It will be seen that this change in the position of the guide vanes changes the angle of the entering water so that, with a given number of revolutions of

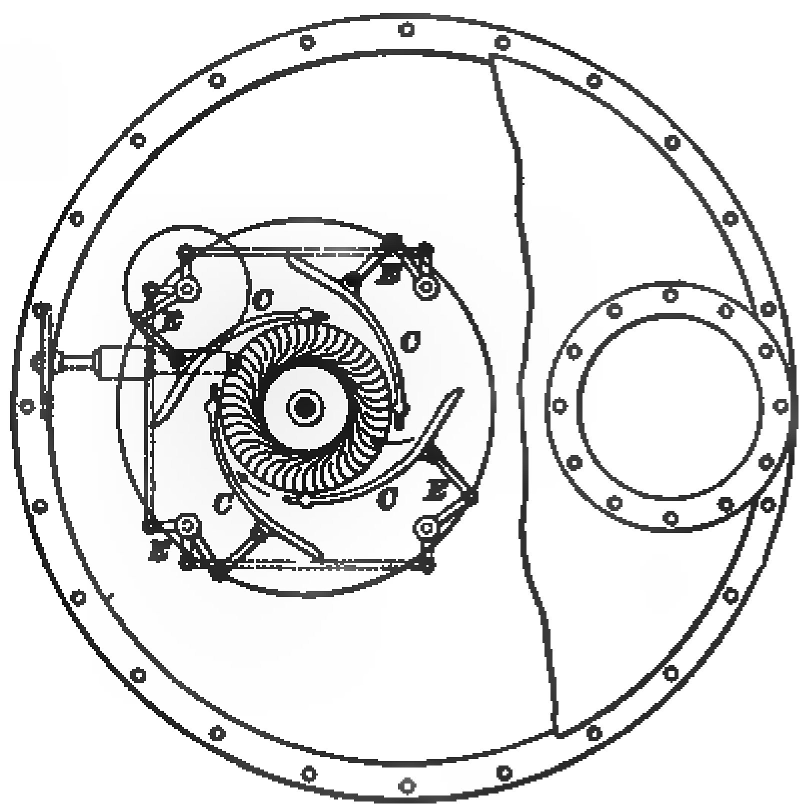


FIG. 31

the wheel, entrance of the water without shock can only occur for a certain gate opening.

EXAMPLES OF TURBINES

50. Fig. 32 shows a top view and a vertical section of the **Leffel turbine,** with a somewhat spherical-shaped cast-iron casing *h*. The wheel has two separate sets of vanes, one *f*, shown in the section, discharging inwards, and the other *g* discharging downwards. The wheel is a solid casting, and the two sets of buckets are separated by a partition that forms part of the rim of the wheel.

For these two sets of wheel buckets there is but one set of guides, and the water is admitted equally to each set of wheel vanes at all gate openings. The guide vanes are made to swing in a manner similar to the guide vanes of the Thompson vortex wheel. A pinion *p*, which is operated by a hand wheel or governor, gears into a sector *s*. This sector rotates a collar *d*, to the under side of which the rods *c* are attached, and the motion of the collar is transmitted to the vanes through

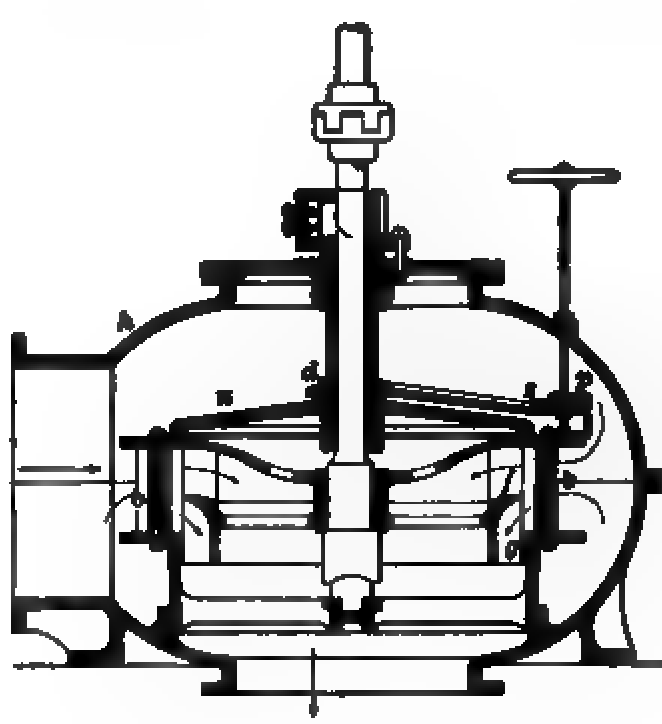


FIG. 32

these rods. In this way, the vanes, being pivoted near their inner ends, open and close in a manner similar to the vanes of a register; consequently, this belongs to the class of register-gate turbines.

51. Fig. 33 is a general view and Fig. 34 a section of a **cylinder-gate Risdon turbine.** This is a mixed inward- and downward-flow turbine, the wheel of which was illustrated in Fig. 23. The gate consists of a cylinder C that works in a space between the wheel A and the guide vanes B. Projections D are cast on the cylinder C and move with it up and down between the guides. They serve to guide the water into the wheel with less resistance and contraction than would occur if the water were forced to enter past the sharp edge of a thin cylinder.

FIG. 33

The gate is raised and lowered by means of a rack and pinion M operated by a hand wheel or governor acting through the shaft W and the bevel gearing. F, F are U-shaped pieces that support the crown plate E and rest on the guide vanes. P is a stationary cylinder supported by the crown plate and contains a piston $O\,O$ that serves to balance the weight of the gate by the action of the pressure of the water under it. The wheel shaft V is supported by the wooden step U and the bearing K.

52. Fig. 35 shows the **McCormick turbine,** a cylinder-gate wheel in which the gate is operated through the bars B, B by means of the two racks and pinions and the bevel gearing.

53. Fig. 36 is a perspective view, Fig. 37 a horizontal section through the guide vanes and wheel, and Fig. 38 a top view of the **New American turbine.** This is a modified form of register gate, in which the guide vanes consist

FIG. 36

of a fixed portion a and a swinging gate b. The gates are operated through the shaft s, the pinion and sector p and r, the collar c, and the rods d. An adjustable bearing o is provided for the wheel shaft. Fig. 39 is a perspective view of the wheel of the New American turbine,

showing the step bearing *e*, which rests on a conical wooden step in the wheel case.

54. Niagara Falls Turbines.—The 5,000-horsepower turbines for Niagara Falls (see Fig. 40) are the outward-flow or Fourneyron type. The water is brought to the wheel case through a steel penstock and is discharged through two wheels A, A and B, B. The flow is regulated by cylindrical gates C and C' outside of the wheels. Each set of guides and wheel vanes is divided into three parts by horizontal partitions, so that the flow at part gate will take place through the uncovered passages with no more resistance than at full gate. Holes a in the top of the wheel case permit the pressure of the water to act upwards on the web-plate b of the upper wheel. This serves to balance the weight of the wheels, shaft, and rotating parts of the dynamo. Owing to the varying pressure of the water, due to the different velocities of flow with changes of load, the upward pressure is sometimes a little greater and sometimes a little less than the opposing weight; this variation is provided for by a collar bearing near the upper end of the shaft. The number of wheel vanes in each wheel is 32 and the number of guide vanes 36.

FIG. 35

FIG. 36

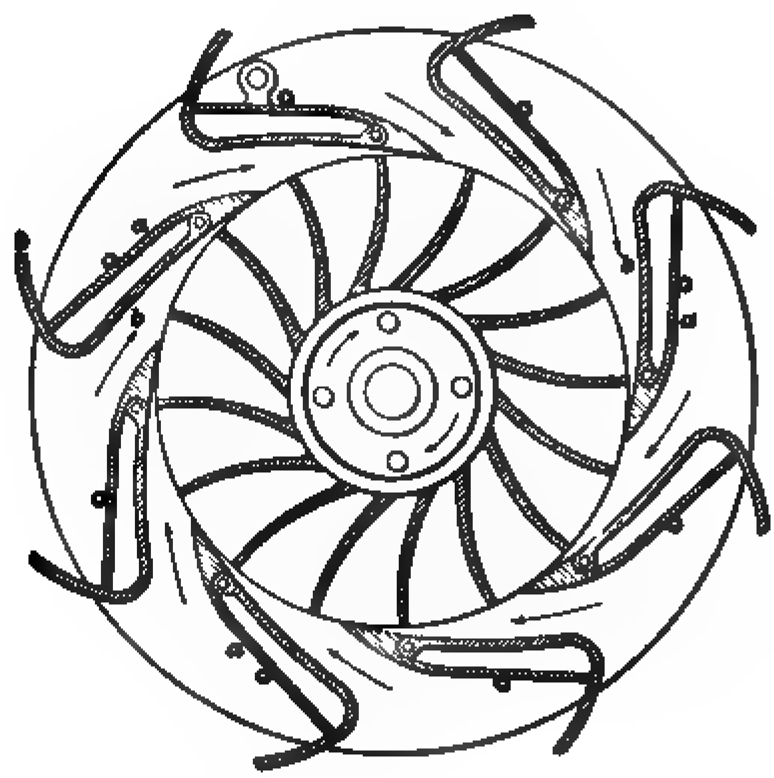


FIG. 37

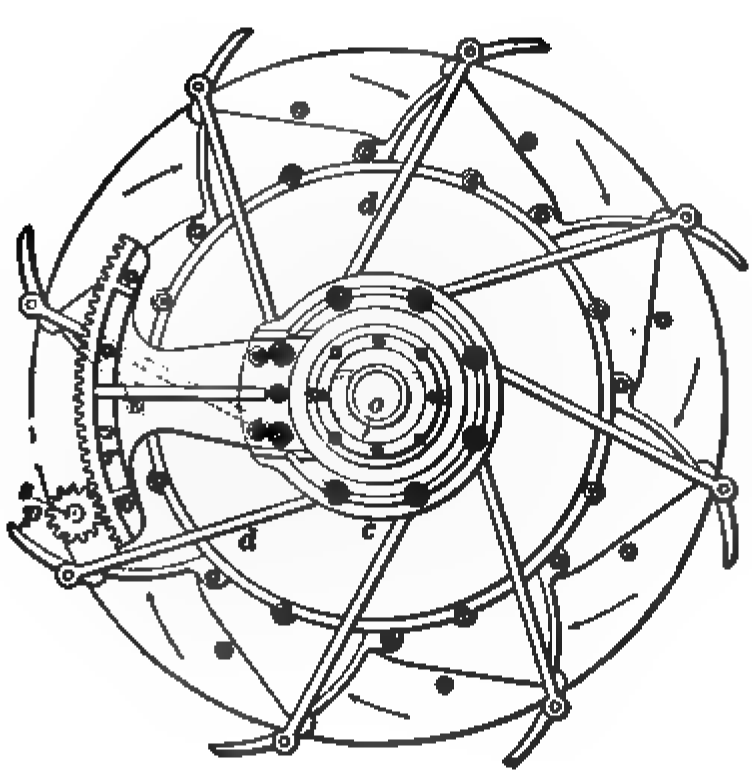


FIG. 38

FIG. 39

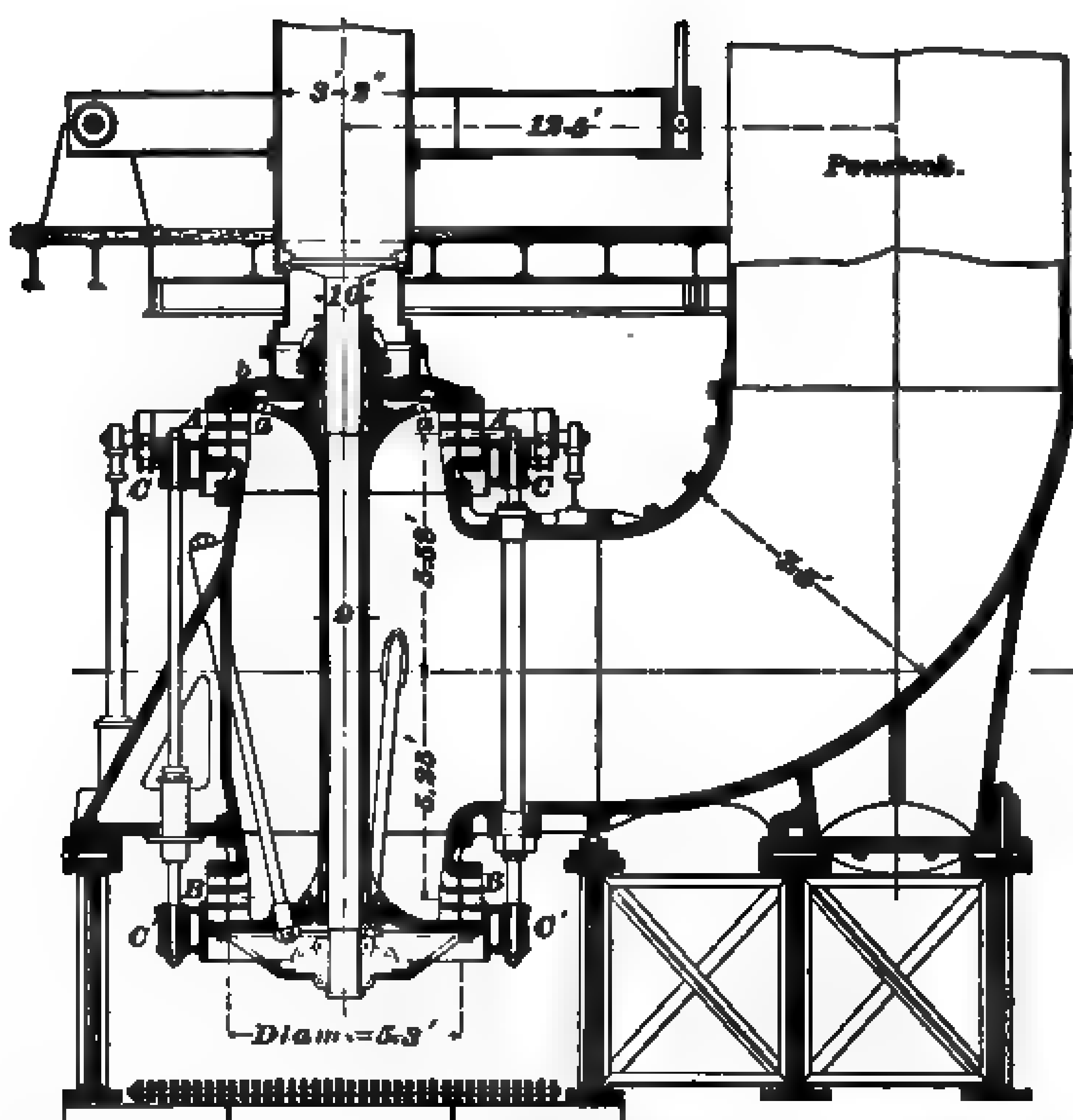


Fig. 40

Fig. 41 shows an enlarged vertical section of the lower wheel and a plan of the lower part of the wheel case with the part above the joint in the case removed; it also shows a partly horizontal section of guides and buckets. The leading dimensions of the wheel are given in the above figures

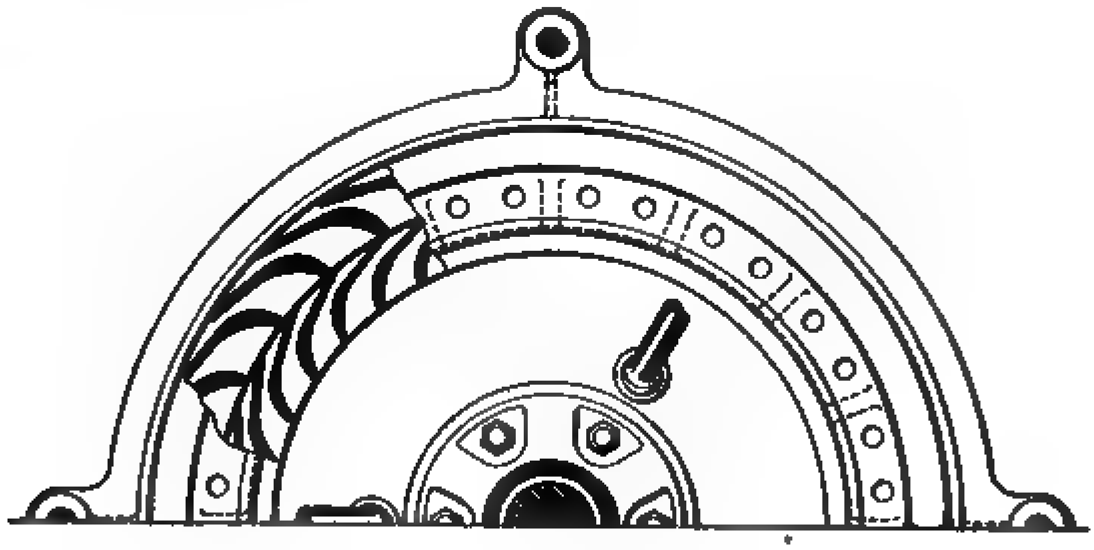


FIG. 41

and in the general plan of its setting. The estimated volume of flow through the wheel at its calculated velocity of 250 revolutions per minute is 430 cubic feet per second. With the mean fall of 136 feet, this gives 6,645 theoretical horsepower, and the rated horsepower will be furnished with an efficiency of 75½ per cent.

HYDRAULICS AND HYDRAULIC MACHINERY

(PART 4)

WATER CONTROL

SLUICES, FLUMES, AND PENSTOCKS

SLUICES

1. The simplest and many times the cheapest form of sluice consists of a canal dug along the side of a hill to lead the water from the dam or weir to the flume that supplies the wheel. In some cases, however, such a canal is subject to considerable loss of water by percolation through the porous soil; and in order to prevent this loss, a sluice of masonry or wood is used. Earthwork canals in porous soil are sometimes lined either with masonry, concrete, or clay to prevent leakage. In order to make the loss of head in the sluice small, its cross-section should be sufficient to make the mean velocity of flow small. In very few cases should this velocity exceed 2 feet per second.

2. Racks and Screens.—Turbines must be protected from leaves, sticks, fish, and similar substances that might clog them by catching between the wheel and guides. This is usually done by means of a rack made of thin bars of

iron placed in the flume just above the penstock. The bars in the rack must be far enough apart to allow the water to flow freely, and the trash that collects must be removed often in order to prevent clogging and consequent loss of head.

FLUMES AND PENSTOCKS

3. Wooden penstocks are used for cheap or temporary work. If carefully constructed on firm foundations, they will give good satisfaction, but great care is necessary in order to secure a firm support for the wheel, so that it will not be moved out of line with its bearings and the gearing. Fig. 1 shows the method of setting a turbine in

FIG. 1

an open wooden penstock with a low fall. In order to secure the advantage of the entire fall, the floor of the penstock must be low enough for the discharge opening of the wheel case to be always submerged in the tail-water. Sufficient space under the flume for the free discharge of the tail-water is a matter of the utmost importance, as is also the area of the cross-section of the tailrace. The velocity

of flow in the tailrace should not exceed 2 feet per second. A penstock suitable for high heads, ranging from 20 to 60 feet, is shown in Fig. 2. The section at (*a*) shows the method of joining the planks at the corners. They must be of uniform width, and each alternate plank of one side is gained to receive the ends of the planks in the adjacent sides.

FIG. 2

The long planks are securely spiked to the corner posts *a*, and small strips *b*, *b*, *b* are nailed in the inner corners. The wheel in this case is set in a cast-iron case that is connected to the penstock by a short cast-iron pipe, and the wheel discharges into a short draft tube.

4. Masonry Flumes. — Well-constructed masonry flumes are sometimes used for moderate falls and furnish a much more substantial support for the wheel than a wooden penstock. Fig. 3 shows a pair of turbines set in a masonry flume. The wheels are carried by iron beams supported by iron columns and the masonry at the sides of the

Fig. 3

tailrace. The two wheels discharge into a common draft tube, and in case of low water, the gates of one wheel may be closed and the other used alone.

5. Steel and Iron Penstocks.—For high heads, penstocks made of riveted steel or iron plates connected to a wheel in an iron casing are most suitable for first-class work. By the use of horizontal wheels and draft tubes, this construction often makes it possible to so place the turbine that the power can be transmitted to the line shaft or machine by a single belt, or, in the case of certain kinds of

machinery that can be so designed as to run at the same speed as the wheel, the driven shaft may be coupled directly to the shaft of the turbine.

Fig. 4 shows a pair of horizontal turbines with a riveted steel-plate penstock, each wheel discharging into a separate

FIG. 4

draft tube. The inlet end of the penstock is provided with a head-gate and protected by a screen to prevent the entrance of any material that might clog or damage the wheels. The wheels are supported on iron beams that span the tailrace. The lower ends of the draft tubes are slightly enlarged, and cone-shaped castings, placed on the bottom of the tailrace immediately beneath them, serve to change the direction of flow gradually from a vertical to a horizontal direction, thus reducing the resistance.

6. An example of a combination of wheels designed to furnish a steady power with a head greatly diminished by high water is given in Fig. 5. Two turbines, one 42 inches and the other 60 inches in diameter, are set in a flume that is fed from a canal in which the level of the water is nearly constant. The discharge is through a curved draft tube

with a branch to each wheel. The smaller wheel is direct-connected to a dynamo on a vertical shaft and is designed to operate the dynamo at the proper speed during ordinary stages of water. At times, however, the level of the tail-water is raised so much by floods that the smaller wheel will

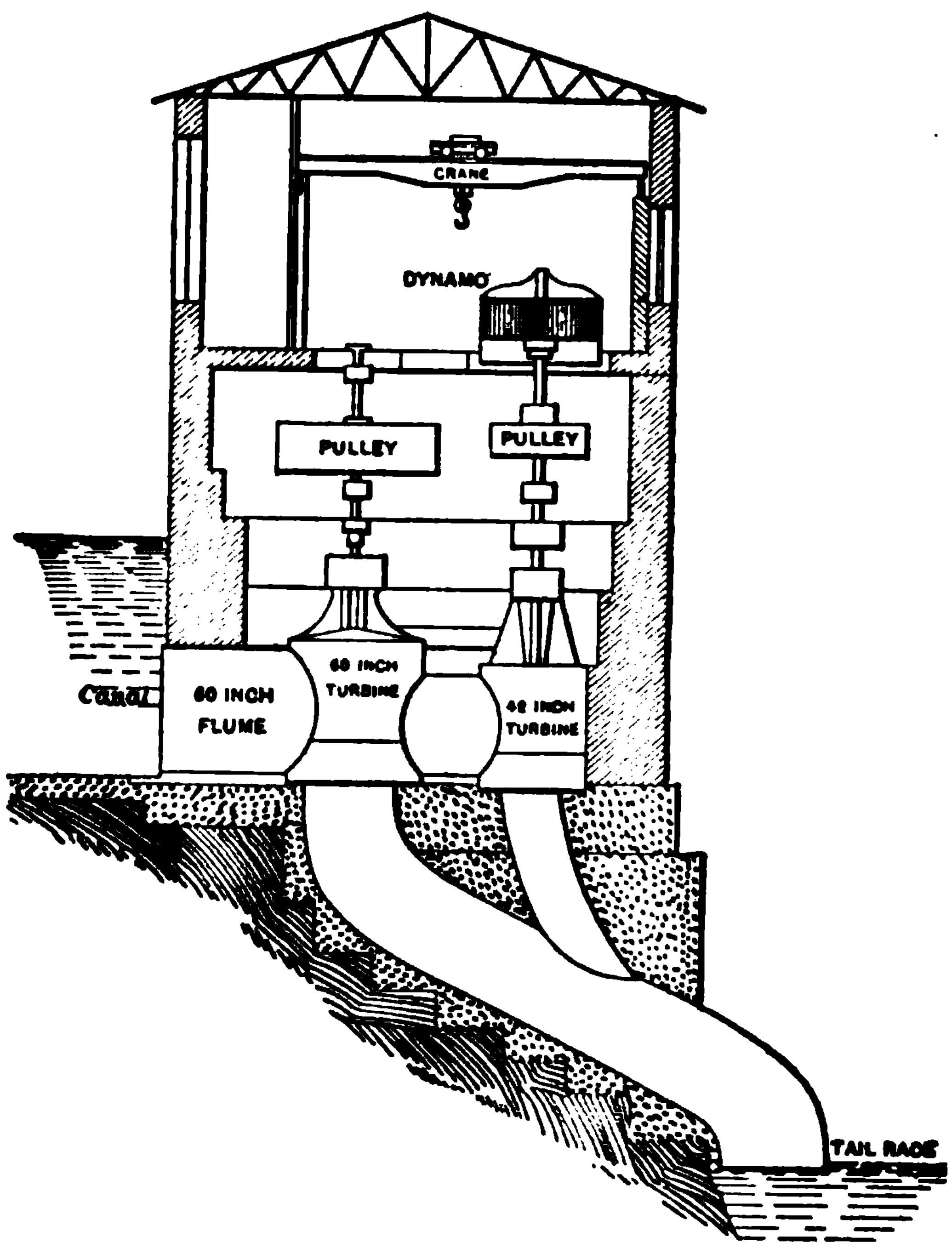


FIG. 5

not furnish sufficient power, and then the larger wheel is connected to the dynamo shaft by a belt on the horizontal pulleys and furnishes the required power under the diminished head by the use of a greater quantity of water.

7. The method of setting one of the 5,000-horsepower turbines at Niagara Falls is shown in Fig. 6. The wheels are located near the bottom of a pit that is about 20 feet wide and will be long enough to accommodate ten wheels when fully completed. Heavy girders built into the masonry at the bottom of the pit support the wheel and the steel

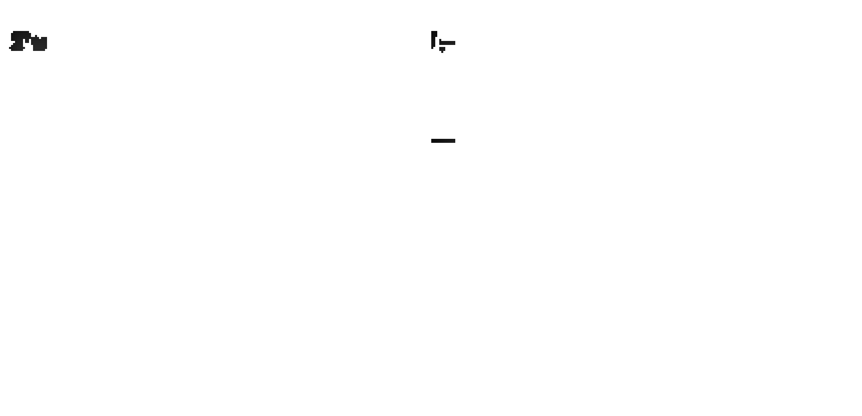


FIG. 6

penstock that brings the water from the supply canal to the wheel case. The shaft, which consists of three sections of steel tubing 38 inches in diameter joined by steel journals *J*, is connected directly to the dynamo at its upper end.

The speed of the wheel is regulated by a governor at the surface, which regulates the gates by means of a system of rods and levers.

HEAD-GATES

8. Head-gates that will close the inlet end of the flume or penstock should always be provided so that the wheel,

FIG. 7

flume, penstock, and sluice may be drained for easy inspection and repairs.

Fig. 7 shows a simple form of head-gate, consisting of a plank gate that slides over the inlet end of a sluice or penstock. The gate is raised or lowered by means of the rack and pinion and a lever that can be inserted in the capstan head *h* of the pinion shaft. A pawl *p* holds the gate wherever desired. Various combinations of screws, worm-gears, and trains of spur wheels are also used for operating head-gates, in place of the simple lever arrangement shown in Fig. 7.

WATERWHEEL GOVERNORS

9. When the work done by waterwheels is variable, and especially when wheels are used to drive electric generators, a governor is required to regulate the speed. Owing to the force required to move the gates of a waterwheel, the common form of centrifugal governor used on steam engines cannot be employed. The system commonly used for regulating turbines is an indirect one, i. e., the common centrifugal governor is made to act on clutches that connect the wheel shaft to the mechanism that operates the gates.

Fig. 8 shows the **Fruen waterwheel governor,** the operation of which is as follows: The bevel pinion *R* is keyed

FIG. 8

to the shaft of the pulley *S*, which is driven by a belt from the waterwheel shaft, and drives the two bevel wheels *D*, *D* in opposite directions. To the upper of these wheels is attached the shaft that carries the governor balls. This shaft is hollow, and through it passes the rod *A*, to which is rigidly attached the friction wheels *B*, *B* and the pinion *C*. The rod *A*, together with the friction wheels *B*, *B* and the pinion *C*, are moved up and down by the rising and falling of the governor balls. When the wheel is running at the proper speed, neither of the friction wheels *B*, *B* is in contact with the bevel wheels *D*, *D*; consequently, they do not

revolve in either direction. Should the speed decrease, the governor balls drop and bring the upper wheels *B* and *D* into contact, so that they move together and with them the pinion *C*. As *C* revolves, it drives the spur wheel *O* and the gate shaft *P*, which opens the gate and admits more water to the wheel. If the speed is too great, the governor balls rise

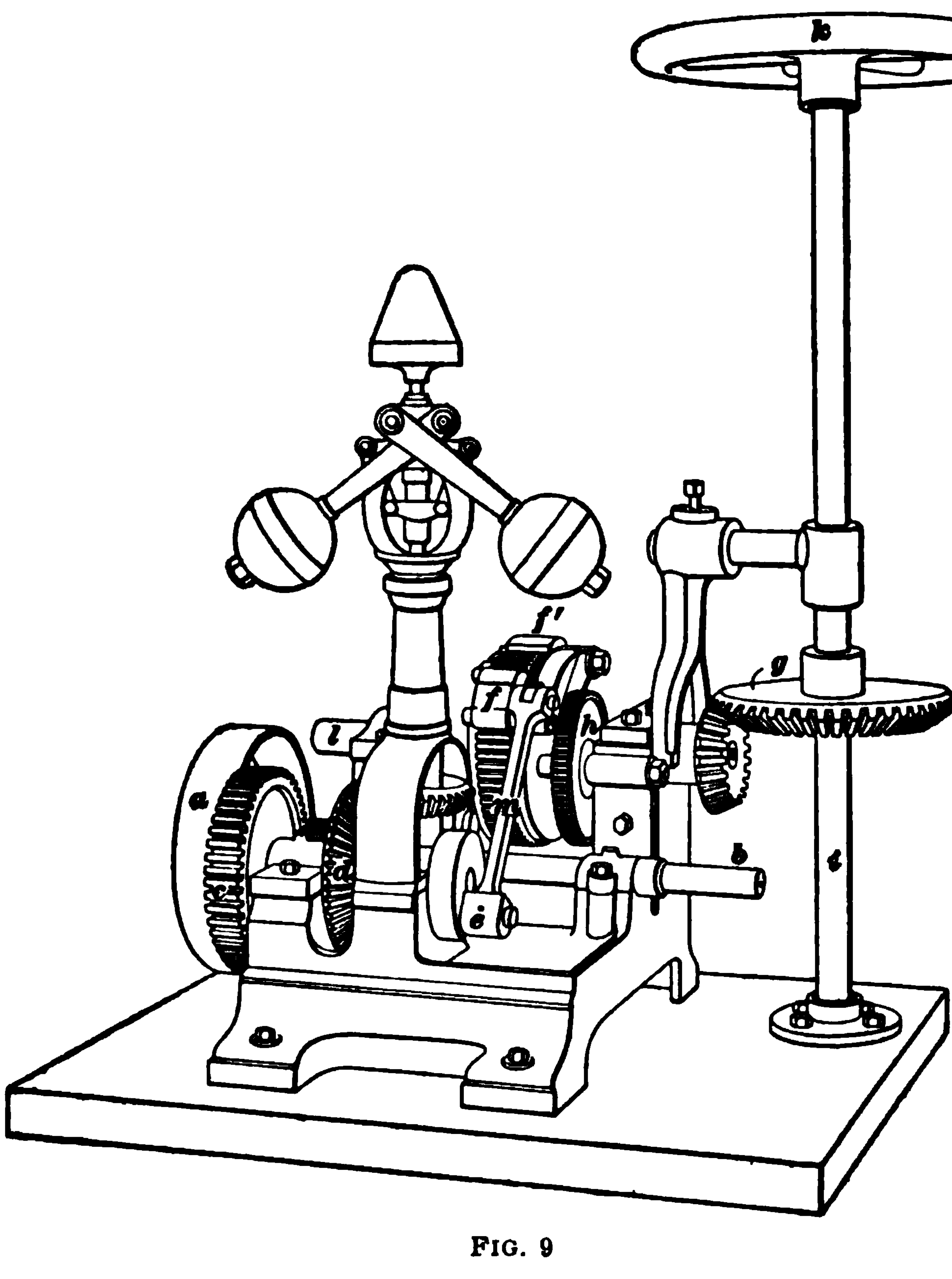


Fig. 9

and bring the lower wheels *B* and *D* into contact; this turns *C* in the opposite direction and closes the gate. The speed can be regulated by varying the compression of the spiral spring below the end of the shaft *A* by means of the hand wheel *G*. The wheel *O* is provided with a clutch *K*, which

acts on the lower cam *H*. If the wheel is overloaded, the spring on *K* allows the clutch to slip on the cam when the gate is fully open, thus preventing breakage. The upper cam *H* is for the purpose of throwing the governor out of gear when stopping or starting the wheel by means of the hand wheel on the shaft *P*. *N* is a nut that is raised or lowered by the screw *L* as *O* revolves, thus showing the amount of opening of the gate by means of the indicator *M*.

10. Fig. 9 shows the **Snow waterwheel governor,** and Fig. 10 is a diagram showing the principles of its

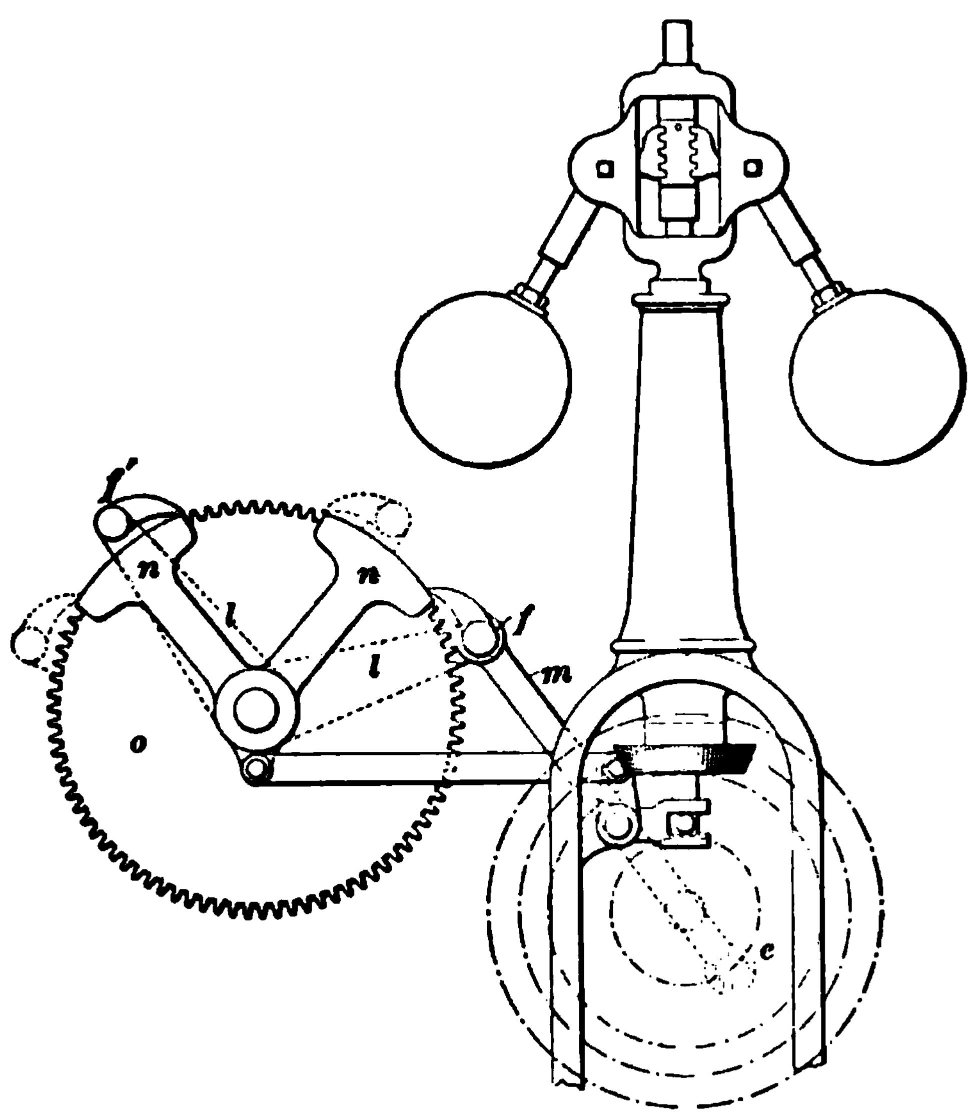


FIG. 10

action. The shaft *b* is driven from the wheel shaft by a belt on the pulley *a*, and drives the spur wheel *c* by a pinion. The shaft to which *c* is keyed carries the bevel wheel *d* and the crank *e*. Two pawls *f*, *f'* on the arms *l* are given a rocking motion by means of the crank *e* and connecting-rod *m*. A cam, formed of two arms *n*, *n*, that is operated by the governor balls, acts on the pawls as follows: When the

wheel is running at the proper speed the cam is held in its central position, as shown in the diagram, and holds both pawls away from the ratchet wheel *o*. If the wheel runs too slow, the governor balls drop and move the cam to the right, thus allowing the pawl *f* to engage the ratchet wheel and turn it to the left. The motion of the ratchet wheel is transmitted to the gate shaft *i* through the bevel gears *g*, and as the ratchet turns to the left the gate is opened, thus admitting more water to the wheel. If the wheel runs too fast, the cam is moved to the left, bringing the pawl *f'* into action; this turns the ratchet to the right and partly closes the gate. The spur wheel *h* acts through a pinion on the ratchet shaft to operate a stop that disengages the ratchet *f* when the gate is fully opened. In order to stop the wheel, the pawl *f* is disengaged by hand, thus leaving the gate shaft free to be turned by the hand wheel *k*.

11. The **Hartford waterwheel governor,** shown in Fig. 11, is an example of a differential governor, i. e., a governor operated by a differential gear. W_1 and W_2 are two wide-faced pulleys driven by the belt that passes over the guide pulleys P_1 and P_2. W_1 is a frustum of a cone, with its diameter at the middle, the same as that of W_2, which is an ordinary pulley. These pulleys give motion to the double bevel wheels *E* and *C*, both of which are loose upon the shaft, through the bevel gears *M* and *R*, respectively. The inner

FIG. 11

parts of *E* and *C* gear with the miter gear *B* on the arm *A*, the latter being keyed to the shaft. The outer part of *C* drives the governor *G* through the gear *K*. The duty of the governor balls is to shift the belt on the cone by means of a fork (not shown) near pulley P_1.

The action of the governor is as follows: When the wheel is running at the proper speed, the belt is at the middle of the cone, and *E* and *C* turn equally in opposite directions; hence, *A* does not move. Should the wheel run too fast, however, the governor balls rise and cause the belt to be shifted towards the small end of the cone, thus increasing the speed of *E*. This, in turn, makes *A*, and hence the shaft, rotate in the direction of *E* and thus operates the gate shaft by means of mechanism not shown. If the wheel should run too slowly, the reverse operation takes place.

The close regulation in speed required by electric-light plants has brought into use several types of governors in which the shifting of the governor balls acts to operate clutches or other shifting mechanism by means of electromagnets. The action of the governors is thus made more prompt than it would be by the action of simple mechanical connections.

HYDRAULIC MACHINERY

INTRODUCTORY

12. Hydraulic machinery is divided into two general classes. The first class comprises all machines for lifting or transporting water included under the general name of *pumps*; the second class comprises those machines in which the energy of water is employed in doing useful work. The second class properly includes waterwheels and turbines, in which the action of the water is largely due to its kinetic energy; but the term *hydraulic machinery* is

more often confined to that special line of machines in which the static pressure of water is made to overcome resistances and thus do useful work.

13. With the exception of the comparatively unimportant devices for transporting water, in which the liquid is confined in a vessel and moves with it, all hydraulic machinery depends for its action on the properties of a fluid. By virtue of these properties, water transmits pressures equally in all directions, and when acted upon by two pressures of different intensities, it changes its form readily and flows freely in any direction against the lesser pressure.

MACHINES FOR LIFTING AND TRANSPORTING WATER

CLASSIFICATION OF PUMPS

14. Pumps may be defined as machines for moving water by producing a change in the pressures that act upon it in such a way that flow takes places in the direction of the lesser pressure. The names applied to the different kinds of pumps are, to a certain extent, an indication of the methods by which this inequality of pressures is produced. Thus, in the *suction pump* the pressure of the atmosphere on the surface of water contained in a pipe is reduced by a process that resembles *suction;* in a *plunger pump* pressure is produced by a special form of piston, commonly called a *plunger;* and in a *centrifugal pump* the pressure is produced by the *centrifugal force* of the water itself when it is given a rapid rotary motion by the action of revolving vanes. Pumps are further classified according to their general form, the power used to drive them, the methods of applying the power, and the special class of work to which they are applied.

DESCRIPTION OF TYPES

15. The Suction Pump.—A section of a suction pump is shown in Fig. 12. Suppose the piston to be at the bottom of the cylinder and to be just on the point of moving upwards in the direction of the arrow.· As the piston rises, it leaves a vacuum behind it; the atmospheric pressure upon the surface of the water in the well causes it to rise in the pipe *P* for the same reason that the mercury rises in the barometer tube. The water rushes up the pipe and lifts the valve *V*, filling the space in the cylinder *B* emptied by the piston. When the piston has reached the end of its stroke, the water entirely fills the space between the bottom of the piston and the bottom of the cylinder and also the pipe *P*. The instant that the piston begins its down stroke the water in the chamber *B* tends to fall back into the well, but its weight forces the valve *V* to its seat and thus prevents any downward flow of the water. The piston now tends to compress the water in the chamber *B*; but this is prevented by the opening of the valves *u*, *u* in the piston. When the piston has reached the end of its downward stroke, the weight of the water above closes the valves *u*, *u*. All the water resting on the top of the piston is then lifted with the piston on its upward stroke and discharged through the spout *A*, the valve *V* again opens, and water fills the space below the piston as before.

FIG. 12

It is evident that the distance between the valve V and the surface of the water in the well must not exceed 34 feet, the highest column of water that the pressure of the atmosphere will sustain, since, otherwise, the water in the pipe will not reach to the height of the valve V. In practice, this distance should not exceed 28 feet or, to obtain the best effects, not more than 22 feet. This is due to the fact that there is a little air left between the bottom of the piston and the bottom of the cylinder; a little air leaks through the valves, which are not perfectly air-tight, and a pressure is needed to raise the valve against its weight, which, of course, acts downwards. There are many varieties of the suction pump, differing principally in the valves and piston, but the principle is the same in all.

16. The Lifting Pump.—A section of a lifting pump is shown in Fig. 13. These pumps are used when water is to be raised to greater heights than can be done with the ordinary suction pump. As will be perceived, it is essentially the same as the pump previously described, except that the spout is fitted with a cock and has a pipe attached to it leading to the point of discharge. If it is desired to discharge the water at the spout, the cock may be opened; otherwise, the cock is closed and the water is lifted by the piston through the pipe P' to the point of discharge, the valve c preventing its falling back into the pump and the valve V preventing the water in the pump falling back into the well.

FIG. 13

In all pumps, the pipe that conducts the water or other liquid to the pump cylinder is called the **suction pipe**; the pipe that conducts the water away from the

pump cylinder is called the **delivery** or **discharge pipe.** In Figs. 13 and 14, P is the suction and P' the delivery pipe. The suction pipe is sometimes called the **inlet pipe.**

17. Force Pumps.—The force pump differs from the lifting pump in several important particulars, but chiefly in the fact that the piston is solid; that is, it has no valves. A section of a suction and force pump is shown in Fig. 14. The water is drawn up the suction pipe P, as before, when the piston rises; but when the piston reverses, the pressure on the water caused by the descent of the piston opens the valve V' and *forces* the water up the delivery pipe P'. When the piston again begins its upward movement, the valve V' is closed by the pressure of the water above it and the valve V is opened by the pressure of the atmosphere on the water below it, as in the previous cases. For an arrangement of this kind, it is not necessary to have a stuffingbox. The water may be forced to almost any desired height.

FIG. 14

18. Double-Acting Pumps.—In the pumps described, the discharge was intermittent; that is, the pump could only discharge when the piston was moving in one direction. In some cases, it is necessary that there should be a continuous discharge; in all cases, it takes more power to run the pump with an intermittent discharge, as a little consideration will show. If the height to which the water is to be raised is considerable, its weight will be very great, and the entire mass must be put in motion during one stroke of the piston.

In order to obtain the advantage of a more continuous discharge, double-acting pumps are used. Fig. 15 shows a part sectional view of such a pump. Two pistons *a* and *b* are used, which are operated by one handle *c*, in the manner shown. The pump has one suction pipe *s* and one discharge pipe *d*. The cylinders *e* and *f* are separated by a partition *g*, so that they cannot communicate with each other above the pistons. In the figure, the handle *c* is moving to

Fig. 15

the right, the piston *a* upwards, and the piston *b* downwards. As the piston *a* moves upwards, it lifts the water above it and causes it to flow through the delivery valve *h* into the discharge pipe *d*. This upward movement of the piston creates a partial vacuum below it in the cylinder *e* and causes the water to rush up the suction pipe *s* into the cylinder, as shown by the arrows. In the cylinder *f*, the downward movement of the piston *b* raises the piston valve *v*,

and the weight of the water on the suction valve *i* keeps it closed. When the handle *c* has completed its movement to the right and begins its return, all the valves on the right-hand side except *v* open and those on the left-hand side except *t* close; water is then discharged into the delivery pipe by the cylinder *f*, and only at the instant of reversal is the flow into the delivery pipe *d* stopped.

19. Air Chambers.—In order to obtain a continuous flow of water in the delivery pipe, with as nearly a uniform velocity as possible, an **air chamber** is usually placed on the delivery pipe of force pumps as near the pump cylinder as the construction of the machine will allow. The air chambers are usually pear-shaped, with the small end connected to the pipe. They are filled with air, which the water compresses during the discharge. During the suction, the air thus compressed expands and acts as an accelerating force upon the moving column of water, a force that diminishes with the expansion of the air and helps to keep the velocity of the moving column more nearly uniform. An air chamber is sometimes placed upon the suction pipe. These not only tend to promote a uniform discharge, but also to equalize the stresses upon the pump and prevent shocks due to the incompressibility of water. They serve the same purpose in pumps that a flywheel does to the steam engine. Unless the pump moves very slowly, an air chamber on the delivery pipe is absolutely necessary.

POWER PUMPS

20. Pumps in which the piston is driven by a crank that receives its motion through a belt or gearing from some outside source of power are usually called **power pumps.** A *single power pump* is one in which but one pump is driven by the shaft. It may be either *single-acting* or *double-acting*.

21. Duplex Pumps.—When two pumps are driven by cranks on a single shaft, the combination is called a **duplex**

pump. The discharge branches from the two pumps are generally combined in such a way that they discharge through a single pipe; and by a proper arrangement of the cranks, the flow through the discharge pipe and the power required to drive the pumps are made nearly constant. If the pumps are single-acting and the cranks are set 180° apart, the discharge from the two pumps will be the same as the discharge from one double-acting pump with the same diameter of piston and length of stroke. Duplex double-acting pumps, with cranks set 90° apart, are much used and give a very steady discharge, since, when one crank is on its dead center and its piston, consequently, is at the end of its stroke and momentarily at rest, the other piston is moving at its maximum velocity and discharging at its maximum rate.

22. Triplex Pumps.—Three pumps driven by cranks on a single shaft form a **triplex pump.** The most common application consists in the use of three single-acting plunger pumps with cranks set 120° apart. With such a combination, at least one of the pumps is always discharging and one taking water from the suction pipe, and the flow is, therefore, continuous and nearly uniform.

FIG. 16

Fig. 16 shows a type of triplex belt-driven power pump much used for feeding boilers or filling elevated tanks in buildings. It consists of three single-acting plunger pumps driven by cranks set at 120° on a single shaft. A tight and a loose pulley provide

the means for starting and stopping the pump, without disturbing the engine or main shaft. The pulley shaft is geared to the crank-shaft by a pinion and spur wheel. I is the suction inlet, D the discharge opening, and C the air chamber.

FIG. 17

Where the supply of power is steady, a belt-driven power pump is very convenient and economical for the purposes for which such pumps can be used, since they get their

power with the same degree of economy as the engine by which they are driven; they are also simple in construction and easily operated. In locations where there is no steam or other power directly available, or where the use of the pump is so intermittent that a steam plant will not be economical, or where the cost of supplying steam is too great, power pumps driven by electric motors may be used to advantage.

23. Fig. 17 shows a horizontal triplex electric pump for mine work. The pump and motor are both fastened to the

FIG. 18

same base or bedplate, a construction that enables them to be set up in any location with only a slight foundation.

24. Small pumps driven by windmills, hot-air engines, gas engines, etc. are much used for supplying water to buildings that have no connection with public water works.

Small, single-acting plunger pumps are most commonly used with these methods of driving, although double-acting pumps are sometimes used.

Where water-power is available, pumps for city water works or for supplying manufacturing establishments are often driven by waterwheels.

25. Fig. 18 shows a duplex double-acting pump driven by a **Pelton wheel.** The wheel shaft carries a pinion that gears into the large spur wheel on the pump crank-shaft. The cranks of the pumps are set at an angle of 90°, an arrangement that distributes the force required to drive the pumps and furnishes a nearly constant discharge. The discharge pipes from the two pumps are joined by a **Y** pipe, to which is attached an air chamber *a*.

Fig. 19

26. Fig. 19 shows a large water-power pumping plant driven by two turbines with horizontal shafts and set in iron flumes. The shafts of the two turbines are in line and are joined by a coupling *A*. Couplings *B*, *B* are also provided, and the pinions may be slid along the shaft, so as to disengage them from the spur wheels. With this

combination, either pump may be driven from either turbine, if it is desirable for any reason to disconnect the other.

Each pump is duplex, with cranks set at 90°, and the discharge pipes are joined with a **Y** branch, to which the air chamber is attached. The discharge from each pair of pumps is joined to the main pipe M by means of a curved **Y** branch, so that the resistance due to sudden changes of direction of flow will be eliminated.

STEAM PUMPS

27. Steam pumps may be divided into two general classes, as follows: *Direct-acting steam pumps*, in which the pump piston is in direct line with and driven by the steam piston, no crank or flywheel being used to regulate the length of the stroke or give motion to the steam valve.

Crank-and-flywheel steam pumps, in which a steam engine, used solely to drive the pump, is provided with a crank and flywheel and some one of the usual methods of operating the steam valves. In this case, the pump may be attached to some reciprocating part of the engine, or it may be driven by a separate crank-shaft geared on to the engine shaft.

DIRECT-ACTING STEAM PUMPS

28. With the possible exception of small hand pumps, used for furnishing water for domestic purposes, direct-acting steam pumps constitute by far the greatest number of all the pumps in use throughout the world. They are made in all sizes, from pumps designed to feed portable boilers or to supply tanks in isolated dwellings, to the largest pumps for city water works; and while they are generally wasteful in their use of steam, their simplicity and light weight, together with the small space they occupy and their low first cost, have been so much in their favor that they have come to be almost universally used in spite of their low efficiency.

The **steam end** of a direct-acting steam pump comprises the steam cylinder, or cylinders, with all accessories, the combination forming a steam engine that may be either simple, compound, or triple expansion. The engine may be run either condensing or non-condensing.

The **water end** comprises the pump cylinder, or cylinders, together with the pistons, valves, etc. The pumps are usually double-acting.

Single direct-acting steam pumps have but one steam end and one water end.

Duplex direct-acting steam pumps have two steam and two water ends. They are usually arranged so that the steam valves of one pump are operated by its mate, in such a way that the strokes of the two pumps alternate with each other. The discharge pipes from the two pumps are combined by a Y branch, so as to form a single main.

29. Single Direct-Acting Steam Pumps.—On account of the conditions under which direct-acting steam pumps work, the distribution of steam in the steam cylinders must be very different from that in the cylinders of the ordinary steam engine. The resistance opposing the motion of the pump plunger is nearly constant throughout the stroke, and there is no flywheel to store up energy during the beginning of the stroke, to be given out again towards the end. As a consequence, the steam pressure on the steam piston must be nearly constant throughout the stroke, in order to furnish the force required. This makes it impossible to use the steam expansively, and, owing partly to the fact that there are no rotating parts, but more especially in order to secure the prompt shifting of the valve near the end of the stroke, a special valve gear is required.

Three of the best-known valve gears used on direct-acting steam pumps are illustrated and their principles of action described.

30. The Knowles Valve Motion.—Fig. 20 shows the steam end of the Knowles steam pump, with the arrangement of the valve gear.

An auxiliary piston p works in the steam chest and drives the main valve v. This auxiliary, or "chest piston," as it is called, is driven backwards and forwards by the pressure of the steam, carrying with it the main valve, which, in turn, gives steam to the steam piston P and operates the pump. The main valve v is a plain slide valve working on a flat seat. The chest piston has a rod R to which is clamped an arm S, the latter being connected to the rocker bar T by a link l. The main piston rod carries an

FIG. 20

arm O, which is provided with a stud, or bolt, on which there is a friction roller. This roller moves back and forth under the curved rocker bar with the motion of the main piston rod and lifts the ends of the bar, thus giving the chest piston a slight rotary motion just at the end of the stroke of the main piston. Each end of the chest piston is provided with a port o, shown in the right-hand end by the partial section, and the solid part of the steam chest has four ports, a, b, and a', b', which open into the space in

which the chest piston moves. *a* and *a′* connect with the live-steam space in the steam chest and serve as steam ports, while *b* and *b′* connect with the exhaust. In the position shown in the figure, the main piston has just reached that point of its stroke where the roller has acted on the rocker bar to rotate the chest piston. This has brought the port *o* (in the right-hand end of the chest piston) into communication with the live steam, admitting the latter to the space at the right of the chest piston. This steam drives the chest piston to the left, which carries the main valve *v* with it, thus exhausting the steam from the right of the main piston and admitting live steam to the left. When the main piston, under the action of this steam, approaches the right end of the cylinder, the roller lifts the right end of the rocker bar, thus rotating the chest piston so as to bring the port *o* in connection with the exhaust port *b* and the port in the opposite end of the chest piston in connection with the steam port *a′*. This drives the chest piston and main valve to the right, allows the steam at the left of the main piston to exhaust, and admits live steam to the right of the main piston again. The chest piston, as it approaches either end of its chamber, covers the exhaust port at that end, thus confining enough of the exhaust steam to form a **cushion** to prevent its striking the end of the steam chest. The main piston also covers the exhaust port before reaching the end of its stroke, as shown in the figure, so that it is cushioned by the exhaust and prevented from striking the cylinder head. Special passages are provided for admitting the steam required to move the piston far enough to uncover the main ports on the return stroke. The arm *O* carries a collar that slides over the chest-piston rod, and in case the steam pressure is not sufficient to move the chest piston, this collar strikes the collars *n*, *n* and thus moves the valve. (One of these collars is just behind the arm *S*.)

31. The **Cameron valve motion,** shown in Fig. 21, possesses the advantage of having no outside gearing. *a* is

the steam cylinder; *c* the piston; *d* the piston rod; *l* the steam chest; *f* the chest piston, the right-hand end of which is shown in section; *g* the main slide valve; *h* the starting

FIG. 21

bar, connected with a handle on the outside; *i*, *i* the reversing valves; *k*, *k* the bonnets over the reversing valve chambers; *e*, *e* are exhaust ports leading from the ends of the steam chest direct to the main exhaust and closed by the reversing valves *i*, *i*.

The action of this valve motion is as follows: The spaces at the ends of the chest piston *f* communicate with the live-steam space by means of small holes, one of which is shown in the right-hand section of *f*. By means of these holes, these spaces and the ports *e*, *e* leading from them are kept filled with live steam as long as the ports are covered by the piston valves *i*, *i*. In the position shown in the figure, the space in the main cylinder to the right of the piston *c* is in communication with the live-steam space in the steam chest; *c* is, therefore, moving to the left. When *c* strikes the stem attached to the valve *i*, it forces *i* to the left and uncovers the left-hand port *e*, thus allowing the steam at the left of *f* to pass out through the exhaust. The steam to the right of *f* then expands and drives *f* and with it the main valve *g* to the left, thus reversing the action of the steam on *c*, which immediately begins to move back towards the

right. Live steam is always acting on the piston *i*, so that as soon as *c* moves to the right this steam pushes *i* back and covers the left port *e* again, after which live steam fills the port and the space connecting with it through the small hole in the end of *f*. When the piston *c* strikes the stem of the right-hand valve *i*, the main valve is again shifted to the right and *c* is started on its stroke to the left. Exhaust steam is confined in the ends of the cylinder to prevent the piston from striking the heads, in the same manner as in the *Knowles* steam pump.

32. The Cataract Steam Pump.—If the load is suddenly thrown off from the ordinary direct-acting steam pump through any cause, as, for example, the bursting of the discharge pipe or the opening of a valve, so as to permit the water to discharge freely under low pressure, the steam is liable to drive the piston to the end of its stroke with so

FIG. 22

much force as to cause serious shocks or even to break some part of the pump. In order to overcome this danger, the *Gordon* steam pump is provided with the arrangement shown in Fig. 22, which is called an **isochronal valve gear.** The main valve is operated by a double chest piston *D D'*, which is actuated by steam controlled by an

auxiliary slide valve F in the small steam chest C. F is provided with a valve stem F', to which two collars e, e' are fastened with setscrews. A slide H, which receives its motion from the main piston rod by means of links I' and I'', the lever I, and the crosshead J, strikes the collars e, e' near the ends of the main piston stroke, thus moving the auxiliary valve F and admitting steam to the chest piston $D D'$, which in its turn operates the main steam valve and reverses the motion of the main piston. In order to prevent the pump running away when the load is suddenly thrown off, the slide H carries a cylinder in which works a piston G fastened to the rod D'' of the chest piston $D D'$. This cylinder, called the **cataract cylinder,** has a cock L that controls a passage joining its two ends, and by means of which the passage may be closed, as desired.

33. The action is as follows: Assume the cataract cylinder to be empty; the piston G will then meet with no resistance and the machine will work as usual. At the end of the stroke, the slide H will strike one of the stops e or e', thus shifting the auxiliary valve F and admitting steam to the piston valve $D D'$, which will move freely through its stroke and thus admit steam to the main piston for the return stroke. Now, if something happens to the water discharge, as, for example, the breaking of a pipe, the load will be removed from the pump and the main piston will be driven suddenly to the end of its stroke and thus be in danger of striking the head with enough force to break it. The object of the cataract cylinder is to overcome this danger. It is filled with liquid, which must be forced from one end to the other by the motion of the piston G. By partly closing the cock L, a resistance is opposed to the passage of the liquid, and the motion of the piston G through its cylinder may be made as slow as desired; consequently, when the main piston moves too rapidly, the motion of the slide H will be transmitted to the piston G, which will shift the main valve so as to shut off the supply of steam to the main piston and thus prevent the pump running too fast.

DUPLEX STEAM PUMPS

34. Fig. 23, which shows an **outside-packed duplex pump** designed for mine work, will show the general arrangement of the valve motion of duplex steam pumps.

FIG. 23

The plunger *A* has completed its stroke in the direction of the arrow, while the plunger *B* is at the end of its stroke in the opposite direction and is just about to begin its return stroke. The steam valve in the steam chest *D* is worked from the piston rod of the opposite pump by means of the levers *F* and *H*, both of which are keyed to a shaft working in the long bearing *G*. The valve in the steam chest *E* is, in like manner, moved by the piston rod on *B*'s side by means of the crank *I* and a crank similar to *H* on the other side. Both pump cylinders discharge into the same delivery pipe *C*. Shoes *K* on the outer ends of both sets of plungers move on the slides *J* and support the ends of the plungers. *L* is the air chamber, and is used so as to make the discharge more uniform.

35. Fig. 24 shows a section of one of the steam cylinders of a Worthington duplex pumping engine. The steam valve *c* is a simple D slide valve operated by the valve stem *d*. As shown in the figure, there are two ports

communicating with each end of the steam cylinder, instead of only one, as is usual in ordinary steam engines. Of these ports, the outer ones *a*, *a* are for the admission of steam, and the inner ones *b*, *b* are for the exhaust. By this arrangement, when the piston approaches the end of its stroke it

FIG. 24

covers the exhaust port and confines sufficient steam in the space between it and the head to provide a cushion, thus preventing the piston striking the head.

The valve has neither inside nor outside lap, consequently the steam is maintained at initial pressure during the full stroke and is not used expansively.

It is usual to allow a certain amount of *lost motion* between the valve stem and valve of a duplex pumping engine, for the purpose of securing a more satisfactory motion of the valve.

36. The action of the duplex direct-acting pump is quite different from that of a double-crank pump with cranks set at right angles to each other. In the duplex direct-acting pump, with the valve motion described above, one piston, as it nears the end of its stroke, shifts the steam valve of the opposite piston and thus starts that on its stroke. The first piston then comes to rest and pauses until its mate has nearly completed its stroke. Just before the second piston reaches the end of its stroke it moves the steam valve of the first and starts it on its return stroke,

while the second gradually comes to rest. The relative motion of the two pistons can be adjusted by means of the lost motion between the valve and its stem, so that when one of the pistons moves at its maximum velocity, the other is at rest, and the second piston gradually begins its stroke as the first begins to come to rest. In this way, the discharge is nearly uniform and the period of rest at the end of each stroke allows the water valves to seat themselves without shock.

COMPOUND DIRECT-ACTING STEAM PUMPS

37. In the direct-acting steam pumps described, no use can be made of the expansive force of the steam, and in order to overcome this difficulty to some extent, many of the larger pumps are made with either compound- or triple-expansion steam cylinders.

38. Fig. 25 shows a common method of arranging the cylinders for a **compound duplex** pumping engine. The

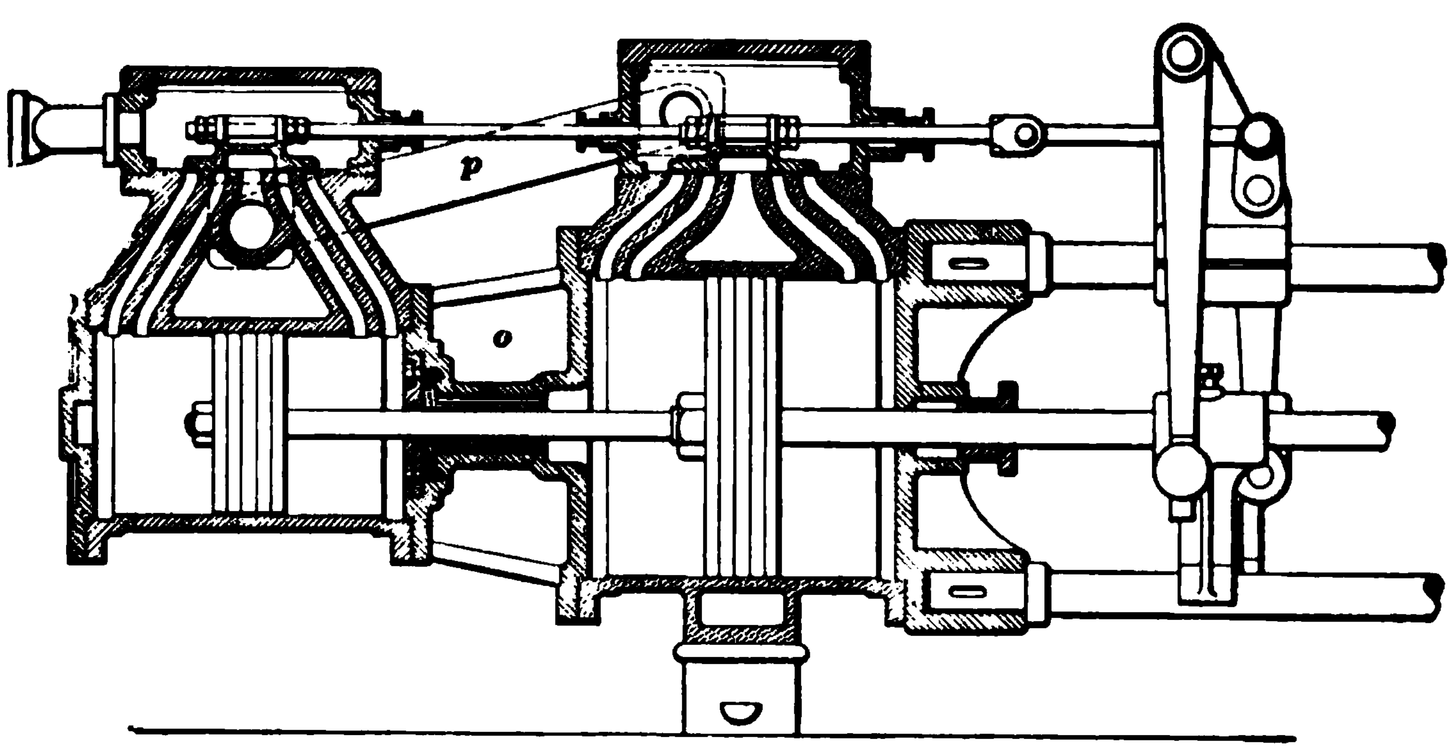


FIG. 25

steam end for each pump is made with two cylinders arranged tandem, the valves for both cylinders being driven from the same valve stem. The high-pressure cylinder is placed outside and connected to the low-pressure cylinder by a cast-iron yoke or spacer *o*, which forms one head for each cylinder.

The high-pressure piston rod passes through a sleeve in this spacer as shown; this sleeve is held in its place by its flange being gripped between the spacer and a plate bolted on to the latter; otherwise the sleeve is free to adjust itself slightly, being a free fit in the body. The exhaust passes directly from the high-pressure cylinder through the pipe *p* to the steam chest of the low-pressure cylinder. Since there is no cut-off in either cylinder, the back pressure on the high-pressure piston is at all times equal to the pressure on the low-pressure piston, neglecting the resistance to the flow of steam through the ports and the pipe *p*.

Since the volume of steam admitted during each stroke is equal to the volume of the high-pressure cylinder, and this steam, when exhausted, just fills the low-pressure cylinder, it is evident that the number of expansions is equal to the ratio of the volume of the low-pressure cylinder to that of the high-pressure cylinder. Also, since the length of stroke is the same for both cylinders, the number of expansions is equal to the ratio of the areas of the low- and the high-pressure piston. The usual number of expansions for small or medium sized pumps ranges from two to three, but for large sizes, four expansions are sometimes used.

FLYWHEEL PUMPING ENGINES

39. We have seen that although direct-acting steam pumps cannot be excelled in simplicity, low first cost, and small expense for repairs, yet they can never be highly economical in their use of steam. In large pumping stations and in many other cases where the cost of fuel is of more importance than the advantages gained from direct-acting pumps, flywheel pumping engines are often used. These are steam engines with cranks and flywheels usually designed for the particular purpose of driving the pump to which they are attached. The steam valves are driven in the ordinary way by means of eccentrics, or some approved automatic valve gear may be used to operate them.

40. By the use of the flywheel, steam may be cut off at the most economical point in the stroke, and the surplus energy imparted to the steam piston during the first part of the stroke will be stored in the flywheel, to be given up towards the end, thus furnishing a nearly uniform driving force for the pump, piston, or plunger.

Fig. 26 shows a section of one side of a *Holley-Gaskill* compound pumping engine. The engine is double, the other

FIG. 26

side being like the one shown in the figure, the two engines having a common flywheel and crank-shaft, with cranks set 90° apart. The high-pressure cylinder is placed directly over the low-pressure, with short passages between them. The connecting-rods from the two cylinders are attached to the opposite ends of a short walking beam B. By this arrangement the pistons move in opposite directions and the exhaust from the high-pressure cylinder passes directly to the low-pressure one. The valves are of the Corliss type, with a releasing gear for regulating the cut-off in the high-pressure cylinder. The connecting-rod that actuates the crank is attached to the upper end of the walking beam and the rod that works the pump plunger P is attached to the cross-head of the low-pressure piston.

VALVES

41. The most important details of a pump of any kind are the valves. They must be so designed and constructed that they will fulfil all the following conditions as thoroughly as possible:

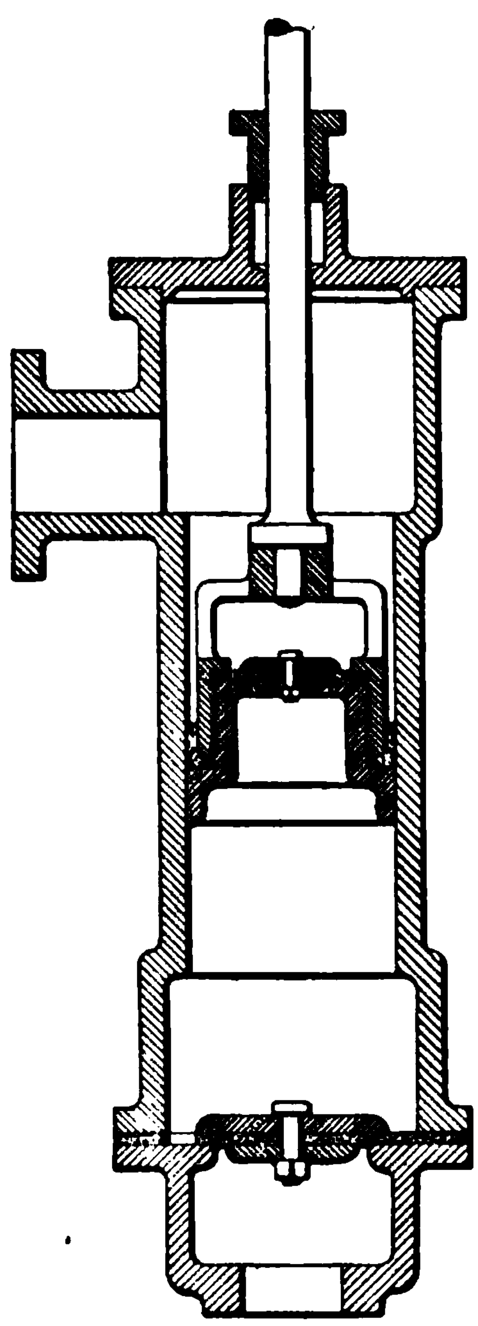


FIG. 27

(*a*) They must open freely under a light pressure.

(*b*) The net area of the passages through the valves should be great enough to limit the velocity of flow through them to 240 feet per minute.

(*c*) The lift of the valves should be small.

(*d*) The passages for the water should be as direct as possible.

(*e*) The valves must close tight under all conditions.

(*f*) The valves and their seats must be durable and of such materials as are not easily affected by the impurities in the water.

(*g*) The valves must return to their seats quickly and without shock as soon as the current through them is stopped.

(*h*) The valves and seats must be easily repaired or removed when worn.

A great variety of valves have been designed with a view of satisfying these requirements, taking into consideration the widely varying conditions under which pumps must work.

For small pumps and moderate lifts, leather **clack valves,** Fig. 27, are often used. They consist simply of a leather disk held at one side and strengthened by a metal plate on top. The leather when wet forms an excellent hinge and a tight valve.

42. For lift pumps working under high pressures, the valves shown in Fig. 28 give good results. The piston

shown at (*a*) has a rubber disk valve working on a gridiron seat. The valve is guided by a central spindle *s* and is held on its seat by a light helical spring that acts on a plate on top of the rubber disk. This piston is very long and has no separate packing. The valve shown at (*b*) is for very

Fig. 28

heavy pressures. It consists of a metal disk guided by a central spindle *s* and held down by a spiral spring in the same manner as the rubber valve. The piston is made with a follower plate, for the purpose of holding a fibrous packing.

43. Fig. 29 shows two valves of a type much used in all classes of pumps for ordinary pressures and service. The valve *v* consists of a vulcanized India-rubber disk that rests on a gun-metal or brass seat *s*. The seat is threaded at *t*, so that it can be screwed into the **deck** of the valve chamber and thus be easily removed. In the design shown at (*a*), the valve is fastened to a spindle *o* by a cap *p*. The spindle is guided by a cage-shaped guard *g* screwed on to the valve seat. The lower end of the spindle is made conical, so as to change the direction of motion of the water gradually and reduce the resistance to flow.

In the design shown at (*b*), the spindle *o* is screwed into the valve seat and carries a guard *g*. A helical spring between this guard and the plate *p* helps to seat the valve quickly.

(*a*) (*b*)

Fig. 29

The size of these valves varies from 2 to 6 inches in diameter, the most common size for ordinary conditions being 3 inches.

44. When used for pumping *hot* water, the disk must be made of a composition that will not be affected by the heat, and for very high pressures, metal disks are used. A cross-section of one such valve disks is shown in Fig. 30.

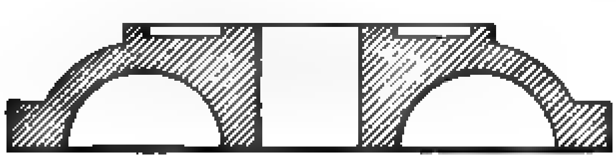


Fig. 30

45. A section of a clack valve is shown in Fig. 31. *A* and *B* are **clacks,** lined with leather on the bottom, so as to make a tighter fit on the seat and thus avoid the necessity of grinding the valve when fitting. *C* is a stop for the valves to strike against so as prevent their opening too far and *E* is the pin on which the clacks are hinged. *D* is a cylindrical casing that forms the valve seat; it may be easily renewed when worn.

Fig. 31

These valves are of the type known as the **butterfly valve,** and are much used for pumps at collieries on account of their cheapness and simplicity of construction.

46. A **single-beat valve** that is suitable for high pressures, up to heads of 500 feet, is shown in Fig. 32. *A* is the valve; *B* is a stem, solid with the valve, which acts as a guide inside the bearing *D*. *C*, *C*, *C*, *C* are rubber rings, kept in position by means of the stem and separated by the washers *E*, *E*, *E*. These rings prevent shock as the valve lifts and also help to close it quickly, thus serving the same purpose as the helical spring in Fig. 29 (*b*).

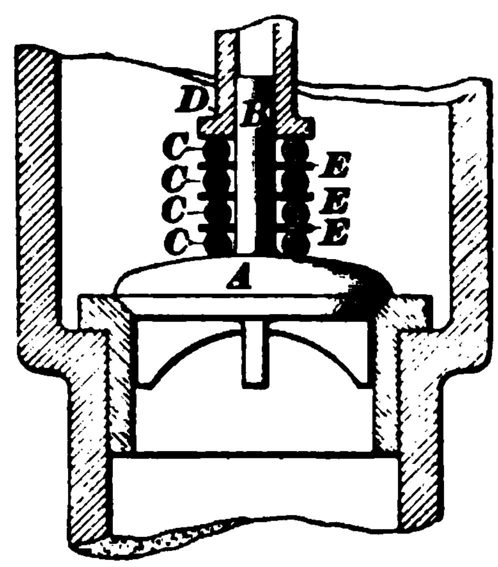


FIG. 32

47. A section of a **Cornish double-beat valve** is shown in Fig. 33. This valve gives excellent results when used in large pumps working under high pressures, and has been applied to pumps working under heads up to 700 feet. It is called a *double-beat* valve because it has two seats and two openings for discharge. *A* is the casing that slides on the vertical stem *B*, its lift being regulated by the nut and washer *E*; when down, it rests on the valve seats *C* and *D*. When the pressure below becomes greater than that above, it raises the casing and the water is discharged through the circular openings at *C* and *D*. The rib around the outside of the casing is for the purpose of strengthening it. The valve seats are conical. The figure shows that one opening discharges the water under the lower edge of the valve and the other through the inside.

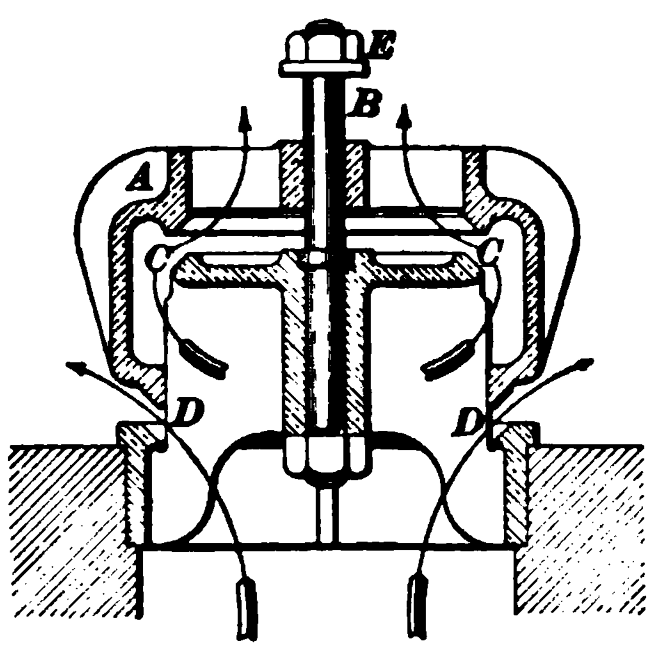


FIG. 33

CHAIN PUMPS

48. The **chain pump** is a device that is much used for raising small quantities of water through moderate distances. It consists of a tube *d* (see Fig. 34), the lower end of which is immersed in the well or reservoir. An endless chain *a a*, to which circular disks *b*, *b* called **buckets** are attached, passes through the tube and over a sprocket wheel *c* above the upper end of the tube. When the sprocket wheel is turned, the buckets, which are made to fit the tube neatly, are drawn up through the tube by the chain and so force the water up ahead of them into the chamber *e*, from the spout of which it flows into the receptacle provided for it. The pump is worked by a belt on the pulley *h*, *g* being the flywheel that equalizes the motion.

ROTARY PUMPS

49. Numerous attempts have been made to replace the reciprocating motion of the piston or plunger, as used in the ordinary pump, by a continuous rotary motion. The results have been unsatisfactory in most cases, owing principally to the difficulty in keeping the moving parts from wearing very rapidly, thus soon producing leakage. Fig. 35 shows one of the oldest and, at the same time, one of the best **rotary pumps.** It consists of a chamber *a*, in which two toothed wheels, or disks, *b*, *b*, revolve in the direction shown by the arrows. The teeth of one wheel fit accurately into the spaces between the teeth of

FIG. 34

its mate; and as the wheels revolve, each tooth acts as a piston that pushes a certain amount of water ahead of it, thus drawing the water from the lower part of the chamber to the upper part, as shown by arrows. It is very important that the flat faces of these wheels, or disks, should be a good fit between the cover and the bottom of the casing or cylinder, and the edges of the teeth also a good fit against the sides of the casing. Most of the rotary pumps that have been at all successful have been modifications of the form just shown, the principal difference being in the number and shape of the teeth on the rotating disks. One of these modifications

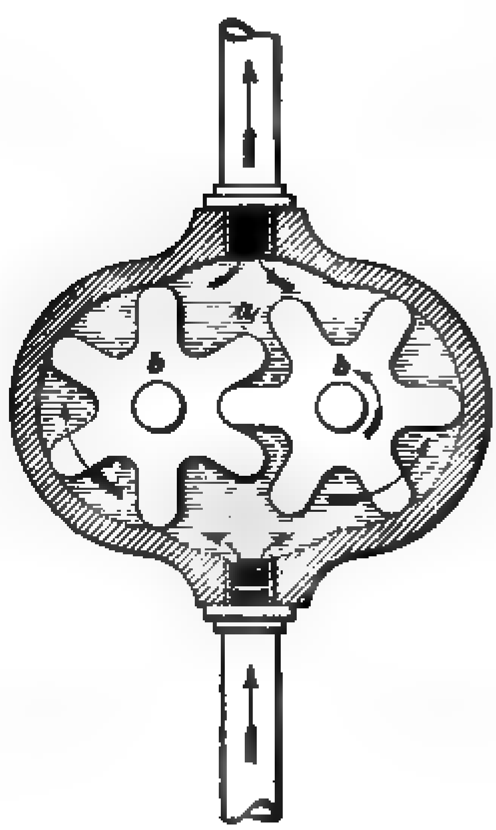


Fig. 35

Fig. 36

is shown in Fig. 36. In this case, the disk *a* has two teeth, or wings, which act as pistons, while its mate *b* has two recesses into which the teeth on *a* fit. The shafts of the two disks are provided with outside gearing that makes their relative motion positive and always keeps them in their proper relative position.

Rotary pumps have never been very efficient, and their principal application has been for fire-engines and similar purposes, where simplicity and light weight are of more importance than economy in the use of power, but since they can lift large quantities of liquid a considerable height

and their parts are readily interchangeable, they are serviceable mill pumps, where acid liquors are to be pumped. They are driven by belts attached to steam, gas, or electrical power generating machines.

CENTRIFUGAL PUMPS

50. Centrifugal pumps depend for their action on the pressure produced by the centrifugul force of a quantity of water rotated rapidly by the vanes of the pump. Fig. 37 is a view of a centrifugal pump with one-half of the casing removed, so as to show the vanes *a*, *a*, *a*, *a* and their location in the pump. The suction pipe connects with the part of the case that has been removed and delivers the water to the vanes at their inner ends, near the hub *S*, the vanes being made sharp at this place, as shown, so as to offer less resistance to the entering water. The vanes revolve in the direction of the arrow, being driven by a shaft through the hub *S*. When the vanes are revolved, the air between them is driven out by centrifugal force, thus forming a partial vacuum. Water is forced in through the suction pipe by the pressure of the atmosphere and fills the space between the vanes. The water, of course, is made to revolve with the vanes, and the action of centrifugal force drives it outwards into the spiral-shaped passage *D D*, which leads it to the discharge pipe.

FIG. 37

The form of the vanes and of the passage *D D* is of great importance in determining the efficiency of a centrifugal pump.

51. Centrifugal pumps are most efficient when working under low heads, and are seldom used for lifts greater than 40 feet. For low heads and large quantities of water they give excellent results, and are especially useful when the water contains grit or other impurities that would destroy the pistons and packing or prevent the closing of the valves of other pumps. Since there are no valves or other restricted passages, centrifugal pumps have been largely used in dredging machines for pumping water containing large quantities of mud, sand, and gravel; and, in fact, anything can be pumped that will be carried through the pump and pipes by a current of water.

In Fig. 38 is shown the shell *a*, an open runner *b*, and a closed runner *c*. The closed runner discharges the entire run

FIG. 38

of the wheel and is considered by the makers to be more efficient than the open runner. It is further claimed that the closed runner wears but little, the wear coming on interchangeable balance rings, while the open runner wears rapidly and requires frequent renewal. Special pumps are sometimes made with only one passageway through the runner, which is large enough at every turn to allow lumps of material as large as the diameter of the pipes fitted to the pumps to pass through.

For millwork where large quantities of liquid or a mixture of liquid and tailings are to be moved, this character of pump should prove exceedingly useful and economical.

General views of a centrifugal pump arranged for belt driving are shown in Fig. 39. *A* is a cast-iron base which

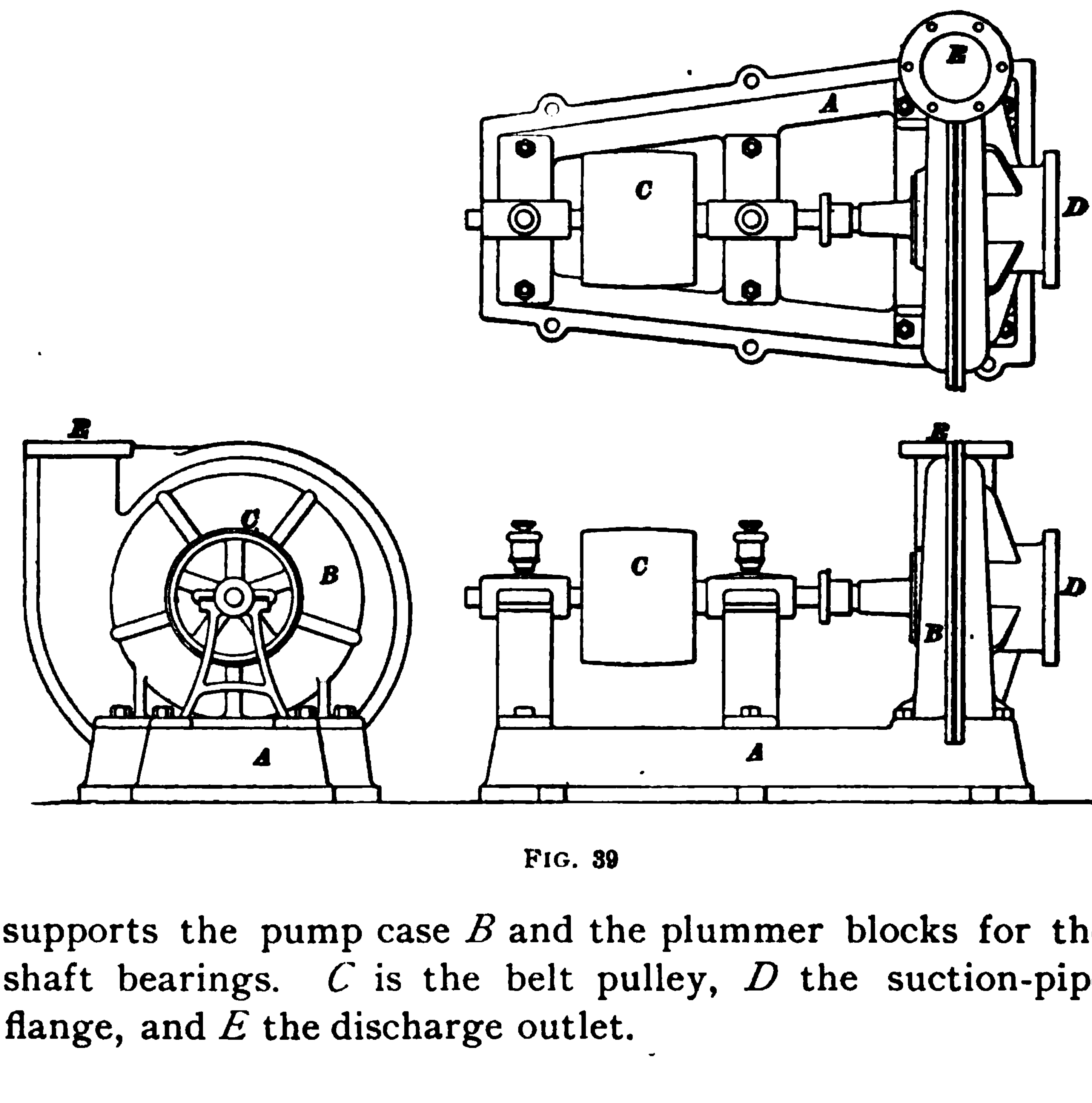


FIG. 39

supports the pump case *B* and the plummer blocks for the shaft bearings. *C* is the belt pulley, *D* the suction-pipe flange, and *E* the discharge outlet.

THE PULSOMETER

52. The **pulsometer** is an interesting and ingenious device for raising water by the direct pressure of steam. Fig. 40 shows a perspective view and Fig. 41 a sectional view of a pulsometer of the latest manufacture. In the sectional view the full lines represent the left-hand half and the dotted lines indicate the position of the discharge valves in the right-hand half of the pulsometer, as shown in Fig. 40.

In the following description the same letters refer to both figures: The steam pipe is connected at E and the suction pipe at S. C is an air chamber, which has no connection with B and A, but communicates with the suction pipe by

FIG. 40

means of the opening I situated below the suction valves F and G. The two latter valves are made of flat rubber and are held to their seats, as shown in the figure, by means of the adjustable spindles R and T, which can be raised or lowered

by means of the nuts f and e. The covers H, H may be removed for the purpose of examining or removing the valves. D is a hard-rubber ball that acts as a valve for admitting steam to the chambers A and B. M and N are

FIG. 41

discharge valves, also made of rubber, situated in the chamber L attached to the other half of the cylinder; their spindles can be raised or lowered by the nuts h and g; K is the discharge pipe.

53. The action of the pulsometer is as follows: Both chambers *A* and *B* (which are divided by the partition *J*) are filled with water to about the height of the water shown in the chamber *B*, Fig. 41. The valve *D* is then opened and steam enters one of the two chambers *A* or *B*. Suppose it enters *B*, the valve *D* being over to the right, as shown. The water in *B* will be forced through the delivery valve *N* into and up through the discharge pipe *K*. This will continue until the water level gets below the edge of the discharge opening *P*. At this point the steam and water mix in the discharge passage and the steam is condensed, creating a partial vacuum in *B*. The pressure in *A* is now greater than in *B*, owing to the vacuum in *B*, and the ball valve *D* is shifted to the left. The steam now enters the chamber *A* and drives the water in it out through the valve *M* into the passage *O* and the discharge pipe *K*. While this is being done, the pressure of the atmosphere forces water up the suction pipe *S*, opens the suction valve *F*, and forces the water into the chamber *B*, filling it. When the suction valve is closed, owing to the reshifting of the ball valve *D* to the other side, the suction water enters the air chamber *I* and is gradually brought to rest by the compression of the air in *C*, thus preventing the shock that would be caused by the sudden stopping of the inflowing water. When the water in *A* has reached the level shown, the steam in *A* is condensed, the ball *D* is shifted to the right, and *B* becomes the driving chamber.

54. In Fig. 40 are shown three small **air valves** *a*, *b*, and *c*. The valve *c* admits air to the air chamber *C*, to replace that which is lost by leakage and by absorption by the water. The valves *a* and *b* admit a small quantity of air to the air chambers *A* and *B*, respectively, just before the suction begins. This impairs the suction somewhat, but is necessary for two reasons: first, it acts as a regulator to govern the amount of water admitted to the chambers; and second, it prevents the steam condensing before the water gets below the edge of the discharge outlet. Suppose

there is a vacuum in *A* owing to the condensation of the steam. The atmospheric pressure forces open the valve *a*, which opens inwards and admits a little air. The incoming water compresses this air and soon closes the valve. When the air has been compressed to such an extent as to balance the force of the inflowing water, the suction valve *G* will close and no more water will enter. Since the same thing occurs in the other chamber, it is evident that the amount of air admitted controls the amount of water admitted during the suction period. The valves *a* and *b* can be adjusted so that the suction valve in either chamber will close at the instant the ball is shifted to the other side, admitting the steam. The air also prevents the steam coming into contact with the water during the forcing process, until the water level has sunk below the edge of the discharge orifice. Air being a poor conductor of heat, the steam does not condense until the mixture of the steam and water has taken place.

The pulsometer is wasteful in its use of steam, but its simplicity and the ease with which it can be set up and run in positions where other pumps would be difficult to manage make it a useful device for many purposes. Since there are no pistons or plungers, it is not badly affected by grit, sand, or any material that will pass through the valves. The limit to which it will raise water by suction is from 20 to 26 feet, and it will discharge water to a height of 100 feet when necessary.

THE HYDRAULIC RAM

55. The **hydraulic ram** is a machine in which the pressure produced by suddenly stopping a column of moving water is used to raise a part of that water to a point above the level of the source of supply. Fig. 42 shows a section of a hydraulic ram as built by a leading manufacturer. *a* is a pipe connecting the ram with the source of supply and is called the **drive pipe.** A valve *b*, which closes the end of the pipe *a*, has a stem that slides freely

through a sleeve *c*. The sleeve *c* is provided with a regulating screw by means of which the stroke of the valve *b* may be regulated. An air chamber *f* is attached to the pipe *a* over an opening closed by a valve *d*, which opens from the pipe towards the air chamber. The action of the ram is as follows: Starting with the valve *b* open, as shown, water will flow from the reservoir through the pipe *a* and out past *b* through the opening at *c*. When the velocity of this water becomes sufficiently rapid, the current flowing past *b* will exert an upward pressure great enough to raise it. This will suddenly stop the current, and the inertia of

FIG 42

the column of water in the pipe *a* will produce sufficient pressure to open the valve *d* and force some of the water into the air chamber *f*. The energy of the moving mass of water is thus absorbed in compressing the air in *f*, and the column is thus brought to rest. The stoppage of the column in *a* is so sudden that there is a slight recoil, which closes *d* and causes a reduction in the pressure sufficient to open *b* again, when the process is repeated. The pressure of the compressed air in *f* forces a nearly constant stream out through the discharge pipe *e*. In order to replenish the air in the air chamber that is absorbed by the water, a **snifting valve** *g* is provided. When the recoil occurs, a

small amount of air is drawn in through *g*, and this air is forced through *d* into the air chamber with the next charge of water.

56. The principal use of these rams is for the purpose of supplying buildings, water tanks, etc. from a source some distance below them. Tests have shown that when they are well made and adjusted their efficiency is a little more than 50 per cent.

Rams may be used when the fall from the supply is no more than 18 inches, but the proportion of the water discharged varies almost directly as the ratio between the fall to the ram and the height to which the water must be raised. With moderate lengths of discharge pipe, the proportion of the water that can be raised is given as follows: One-seventh of the water can be raised to an elevation above the ram 5 times as high as the fall from the supply to the ram; one-fourteenth of the water can be raised to an elevation above the ram 10 times as high as the fall to the ram; and so in the same proportion for other ratios between the fall and the height of discharge.

CALCULATIONS RELATING TO PUMPS

DISPLACEMENT

57. The **displacement** of a pump for a single stroke is the volume of water that would be displaced (that is, driven out of the cylinder) by the piston or plunger during that stroke.

Rule.—*To find the displacement of a pump in cubic feet per minute, when the length of the stroke, diameter of the piston or plunger, and number of strokes per minute are given, multiply the length of the stroke in feet by the number of strokes per minute, and this product by the area of the piston or plunger in square feet. The final product will be the displacement.*

NOTE.—In calculating the displacement of a pump, the number of strokes to be used in this rule is the number of the strokes of the piston or plunger during which water is being discharged. Thus, for a single-acting pump, similar to those shown in Figs. 12 and 13, water is discharged only when the piston moves in one direction; and with the double-acting pump the number of strokes during which discharge occurs is equal to the total number of strokes which the piston makes.

EXAMPLE 1.—A single-acting plunger pump is driven by a crank whose radius is 8 inches and whose number of revolutions is 30 per minute. If the plunger is 6 inches in diameter, what is the displacement in cubic feet per minute?

SOLUTION.—The number of discharging strokes of the plunger is equal to the number of revolutions of the crank, or 30 per minute; the length of the stroke is $\frac{2 \times 8}{12} = 1\frac{1}{3}$ feet; and the area of the piston is $\frac{.7854 \times 6^2}{144} = .196$ square foot. The displacement is, therefore, $1\frac{1}{3} \times 30 \times .196 = 7.84$ cu. ft. per min. Ans.

EXAMPLE 2.—What would be the discharge of a double-acting piston pump driven by the crank in the last example, if the piston has the same diameter as that of the plunger?

SOLUTION.—Since the pump is double-acting, the number of discharging strokes is twice the number of revolutions of the crank; therefore, the displacement is $2 \times 1\frac{1}{3} \times 30 \times .196 = 15.68$ cu. ft. per min. Ans.

DISCHARGE

58. The **theoretical discharge** of a pump is equal to the *displacement*. The **actual discharge** is generally less than the displacement, owing to leakage past the valves and piston and also to the return of water through the valves while they are in the act of closing.

SLIP

59. The difference between the displacement and the actual discharge, expressed as a percentage of the displacement, is called the **slip** of a pump. When the column of water in the suction and discharge pipes of a pump is long and the lift moderate, the energy imparted by the piston

during the discharge stroke may be sufficient to keep the column in motion during all or a part of the return stroke. Under these conditions, the actual discharge will be greater than the displacement, and the slip is said to be *negative.*

Rule.—*To calculate the slip of a pump, find the difference between the displacement and the actual discharge, then divide this difference by the displacement, and the quotient, expressed as a per cent., will be the slip.*

EXAMPLE.—A single-acting plunger pump with a plunger 8 inches in diameter and 36 inches stroke discharges 33.5 cubic feet of water per minute when making 35 discharging strokes. What is the slip?

SOLUTION.—The displacement is $\frac{.7854 \times 8^2}{144} \times 3 \times 35 = 36.652$ cubic feet per minute. The slip, therefore, is $\frac{36.652 - 33.5}{36.652} = .086 = 8.6$ per cent., nearly. Ans.

WORK DONE BY A PUMP

60. The **useful work** in foot-pounds done by a pump is the product of the water raised in pounds multiplied by the vertical distance in feet from the surface of the water in the well or supply reservoir to the outflow end of the discharge pipe.

The **actual work** is always greater than the useful work. Force is required to overcome the friction of the piston or plunger in the cylinder or stuffingbox, and considerable force is also required to overcome the friction of the water in its passage through the pipes and the valves and passages of the pump. Some force must also be expended in giving the water the velocity it has when it leaves the discharge pipe.

The theoretical force required to drive the piston is equal to its area multiplied by the pressure due to a head equal to the vertical distance from the surface of the water in the well to the outlet of the discharge pipe. The actual force can be found by means of a pressure gauge or indicator attached to the pump cylinder, which will give the actual pressure on the piston in pounds per square inch.

According to the principles of hydraulics and the flow of water through pipes, it is evident that the power required to overcome the frictional resistance of the water will be reduced by making the pipes large and direct and the passages through the valves and pump of ample size and as direct as possible, so as to avoid loss from the sudden change of direction of flow.

61. If it is required to find the probable force that will be needed to pump a given quantity of water through a pipe whose size and location are given, first calculate the pressure head required to force the water through the pipe by means of the rules given. To this must be added the head due to the velocity of discharge and the frictional resistances of the pump itself, and also the pressure head due to the elevation to which the water is to be pumped. If a pump is in operation, an indicator diagram taken from the pump cylinder will give the pressure on each square inch of the piston in the same manner as for a steam engine.

RULES FOR CALCULATING SIZES OF PUMPS

62. In laying out pumping plants where the water must be forced through long lines of pipe, as, for example, the pumps for supplying a city water works, it is necessary to know the length, general location, and arrangement of the pipes in the system through which the pump must force the water. If the pumps discharge directly into the mains, it becomes necessary to know the highest point to which the water must be raised, together with the pressure head that must be maintained in the pipes. These data, together with the maximum amount of water the pump will have to discharge, will form a basis on which to calculate the size of the pump cylinders and the pressure at the pump, and from this the power required to drive the pump may be obtained.

The following simple rules and formulas will be found very useful in determining the approximate sizes of pumps for ordinary purposes and the power required to drive them:

To find the pressure in pounds per square inch corresponding to any given head of water:

Rule I.—*Multiply the head in feet by .434; the result is the pressure in pounds per square inch.*

To find the head of water corresponding to a given pressure in pounds per square inch:

Rule II.—*Multiply the given pressure in pounds per square inch by 2.304; the result is the head in feet.*

EXAMPLE 1.—What pressure will a head of 120 feet of water exert?

SOLUTION.—Applying the first rule, $120 \times .434 = 52.08$ lb. per sq. in. Ans.

EXAMPLE 2.—What head of water will exert a pressure of 65 pounds per square inch?

SOLUTION.—Applying the second rule, $65 \times 2.304 = 149.76$ ft. Ans.

63. To find size of plunger or piston necessary to discharge a given number of gallons per minute:

Let G = number of gallons discharged per minute;
S = speed in feet per minute of plunger or piston;
d = diameter of plunger or piston in inches.

Then, the theoretical diameter $d = 4.95\sqrt{\frac{G}{S}}$.

Since there is always more or less slip of the water past the valves and plungers, it is usual to add one-fourth of the required number of gallons to the value of G given in the above formula so as to allow for this slip. Doing so, the formula becomes

$$d = 5.535\sqrt{\frac{G}{S}}. \qquad \textbf{(1.)}$$

Formula **1** should always be used when calculating the size of piston or plunger to discharge a certain number of gallons per minute. The piston speed is the number of feet traveled per minute by the plunger when discharging water; that is, it equals the length of the stroke in feet multiplied by the number of working strokes per minute. If the pump

is double-acting, the number of working strokes is the same as the total number of plunger strokes, both forward and back; if single-acting, half that number.

EXAMPLE.—(*a*) What should be the diameter of a pump plunger required to discharge 130 gallons of water per minute, the speed of the plunger to be 90 feet per minute? If the pump is double-acting and the stroke is twice the diameter, (*b*) how many strokes must it make per minute and (*c*) what is the length of stroke?

SOLUTION.—(*a*) $d = 5.535\sqrt{\frac{G}{S}} = 5.535\sqrt{\frac{130}{90}} = 6.65$ in., nearly, say $6\frac{5}{8}$ in. Ans.

(*b*) Since the stroke is twice the diameter, the stroke $= 6\frac{5}{8} \times 2 = 13\frac{1}{4}$ inches $= \frac{13.25}{12} = 1.104$ ft. Ans.

(*c*) Number of strokes $= 90 \div \frac{13.25}{12} = 81.5$ strokes per. min., nearly. Ans.

NOTE.—This speed is rather high for a pump and should be used only when absolutely necessary.

64. To find the approximate discharge of a pump in gallons per minute, when the diameter and plunger speed are known, use the following formula:

$$G = .03264\, d^2 S. \qquad \textbf{(2.)}$$

The same allowance has been made for slip in this formula as was used in formula **1.** If the theoretical discharge is required, $G = .0408\, d^2 S$.

EXAMPLE 1.—What is the probable discharge of a duplex double-acting mine pump whose plungers are 10 inches in diameter, stroke 24 inches, and which makes 40 strokes per minute?

SOLUTION.—The discharge due to one side of the pump is $G = .03264\, d^2 S = .03264 \times 10^2 \times (2 \times 40) = 261.12$ gallons per minute, since 24 inches = 2 feet and the piston speed $= 2 \times 40$. The total discharge is twice this amount, or $261.12 \times 2 = 522.24$ gal. per min. Ans.

EXAMPLE 2.—In the last example what is the theoretical discharge?

SOLUTION.— $G = .0408\, d^2 S = .0408 \times 10^2 \times (2 \times 40) = 326.4$ gallons per minute; that is, $326.4 \times 2 = 652.8$ gal. per min. for both sides. Ans.

65. To find, approximately, the horsepower required to discharge a certain number of gallons of water per minute

with a given lift, make the necessary substitutions in the following formula, in which H = the horsepower and h = the vertical height in feet between the highest point of the center of the discharge pipe and the level of the surface of the water in the sump or well from which the supply is drawn:

$$H = .00038\,G\,h. \qquad (3.)$$

This is a rough approximation, based on an allowance of 50 per cent. of the theoretical horsepower for friction in the engine, pipes, and pump; hence, the theoretical horsepower is two-thirds of the value given in formula **3.**

EXAMPLE.—How many horsepower should the steam cylinder of a pump that is required to discharge 350 gallons per minute be designed for, the total lift being 320 feet?

SOLUTION.— $H = .00038\,G\,h = .00038 \times 350 \times 320 = 42.56$ H. P. Ans.

In this example the theoretical horsepower is $42.56 \times \frac{2}{3} = 28.37$ horsepower.

66. If it is desired to know the height through which a pump will raise water, when the horsepower of the steam cylinder and discharge of the pump have been determined, use the following formula, in which the letters have the same meaning as in formula **3,**

$$h = \frac{H}{.00038\,G} \qquad (4.)$$

EXAMPLE.—To what height will a 40-horsepower pump force 280 gallons of water per minute?

SOLUTION.— $h = \frac{H}{.00038\,G} = \frac{40}{.00038 \times 280} = 376$ ft., nearly. Ans.

67. To find the size of the steam or air cylinder of a pump, first calculate the horsepower by formula **3,** then proceed as follows: It is customary to design pumps on a basis of 100 feet piston speed per minute, which is a fair allowance and does not bring excessive strain on the pump. If a simple pump is to be used, the mean pressure of the

steam will be the same as the gauge pressure at the pump, since the pressure is carried full stroke. If the pump is also direct-acting, the steam-piston speed will be the same as the pump-piston (or plunger) speed.

Let S = steam-piston speed;
d = diameter of steam cylinder in inches;
r = ratio between length of stroke and diameter of cylinder;
l = length of stroke in feet;
N = number of strokes per minute;
H = horsepower;
P = steam pressure in pounds per square inch.

Then,
$$d = \sqrt[3]{\frac{504{,}201.6 \times H}{rPN}}, \qquad (5.)$$

or
$$d = \sqrt{\frac{42{,}016.8 \times H}{PS}}. \qquad (6.)$$

Having obtained the diameter of the steam piston by either of the above formulas, the stroke can be found by multiplying the diameter by the value of the ratio r. When formula **6** is used, the number of strokes can be found by dividing the piston speed by the length of the stroke in feet.

EXAMPLE.—A pump to be driven by compressed air at a pressure of 45 pounds per square inch is to have a piston speed of 100 feet per minute. If 32 horsepower are required to operate it, what should be (*a*) the size of the air cylinder, and (*b*) the number of strokes?

SOLUTION.—(*a*) Using formula **6**,
$$d = \sqrt{\frac{42{,}016.8 \times H}{PS}} = \sqrt{\frac{42{,}016.8 \times 32}{45 \times 100}} = 17.285,$$
or say $17\frac{1}{4}$ in. Ans.

(*b*) For this case let the stroke be, say, 22 inches, thus making r a little more than $1\frac{1}{4}$. The number of strokes will then be $100 \div \frac{22}{12} = 54\frac{6}{11}$, or say 55. Ans.

APPROXIMATE SIZES OF SUCTION AND DELIVERY PIPES

68. For ordinary work we may allow a velocity of 200 feet per minute in the suction pipe and 400 feet per minute in the delivery pipe. Substituting these values for S in the formula for the theoretical diameter of the plunger and letting d_1 be the diameter of the suction pipe and d_2 the diameter of the delivery pipe, we have

$$d_1 = 4.95\sqrt{\frac{G}{200}}, \text{ or } d_1 = .35\sqrt{G}. \qquad \textbf{(7.)}$$

$$d_2 = 4.95\sqrt{\frac{G}{400}}, \text{ or } d_2 = .25\sqrt{G}. \qquad \textbf{(8.)}$$

The pipes may be made larger than the values calculated by the above formulas, particularly the suction pipe, but it is not good practice to make them any smaller.

EXAMPLE.—What should be the diameters of the suction and delivery pipes of a pump that discharges 225 gallons per minute?

SOLUTION.—Using formula **7**, $d_1 = .35\sqrt{G} = .35\sqrt{225} = 5.25$ inches. Since the pipe sizes above 4 inches in diameter vary by even inches, the suction pipe should be either 5 or 6 inches diameter, preferably the latter.

By formula **8**, $d_2 = .25\sqrt{225} = 3.75$; therefore, make the delivery pipe 4 inches in diameter. Ans.

DUTY OF A PUMP

69. According to the old standard, the **duty** of a pumping engine is *the number of pounds of water raised 1 foot high for each 100 pounds of coal burned in the boiler.*

The duty is calculated by multiplying the number of pounds of water discharged in a given period by 100 and by the total height in feet that the water is lifted, and dividing the product by the number of pounds of coal burned during the same period. Since the discharge is usually given in gallons, the following formula may be used:

$$D = \frac{835.5\, G\, h}{W}, \qquad \textbf{(9.)}$$

in which G = number of gallons discharged in given period;
h = total vertical distance in feet from surface of water in well or other source of supply to point of discharge;
W = number of pounds of coal burned in given period;
D = duty in foot-pounds.

EXAMPLE.—What is the duty of a pump whose discharge is 88,152 gallons in one hour, the height of the lift being 650 feet and the number of pounds of coal burned per hour being 580?

SOLUTION.—Applying formula **9**,

$$D = \frac{835.5\,G\,h}{W} = \frac{835.5 \times 88,152 \times 650}{580} = 82,540,100 \text{ ft.-lb., nearly.} \quad \text{Ans.}$$

70. The above standard for the duty of a pumping engine is not a fair measure of its efficiency, for the following reasons: (*a*) No account is taken of the variation in quality of the different kinds of coal burned. (*b*) The evaporation efficiency of the boilers may be low, thus reducing, through no fault of the pump, the effective work done. (*c*) The friction of the water in the suction and discharge pipes may be so great as to add largely to the actual work done by the pump.

A standard that eliminates the above sources of error is the following, which has been recommended by a committee appointed by the American Society of Mechanical Engineers:

The duty of a pumping engine is equal to the total number of foot-pounds of work actually done by the pump divided by the total number of heat units in the steam used by the pump, including the steam used by the condensers (if any) and boiler feed, and this quotient multiplied by 1,000,000.

71. The number of foot-pounds of work done by the pump is to be found as follows: A pressure gauge is attached to the discharge pipe and a vacuum gauge to the suction pipe, both as near the pump as convenient; then the pressure against which the pump plunger works is equal

to the difference in the pressures shown by these two gauges plus the head due to the difference in level of the points in the pipes to which they are attached; and the number of foot-pounds is equal to the continued product of the net area of the plunger (making allowance for piston rods), the length of the plunger stroke in feet, the number of plunger strokes made during the trial, and the pressure against which the pump plungers work, as shown by the gauges.

72. The number of heat units furnished to the pump is the number of British thermal units (B. T. U.) in the steam from the boilers and is to be determined by an evaporation test of the boilers. Let

A = net area of plunger in square inches;
P = pressure in pounds per square inch indicated by gauge on the discharge pipe;
p = pressure in pounds per square inch corresponding to vacuum indicated by gauge on suction pipe.
S = pressure in pounds per square inch corresponding to difference in level between two gauges;
L = average length of stroke of pump plunger in feet;
N = total number of single strokes of plunger made during trial;
H = total number of heat units consumed by engine during trial;
W = total number of foot-pounds of work done by pump during trial;
D = duty.

Then,
$$W = A\,(P \pm p + S)\,L N, \qquad \textbf{(10.)}$$

and
$$D = \frac{W}{H} \times 1{,}000{,}000$$

$$= \frac{A\,(P \pm p + S)\,L N}{H} \times 1{,}000{,}000. \qquad \textbf{(11.)}$$

EXAMPLE.—A crank-and-flywheel pump has two double-acting water plungers, each 20 inches in diameter and 36 inches stroke. Each

plunger has a piston rod 3 inches in diameter extending through one pump-cylinder head.

During a 10-hour duty trial the total heat in the steam supplied to the engine was 35,752,340 B. T. U. and the engine made 9,527 revolutions. If the average pressure indicated by a gauge on the discharge pipe was $95\frac{1}{2}$ pounds, the average vacuum indicated by a gauge on the suction pipe $8\frac{1}{4}$ inches, and the difference in level between the centers of the vacuum and the pressure gauge 8 feet, what was the duty of the pump?

SOLUTION.—The area of a plunger 20 inches in diameter is 314.16 square inches and the cross-sectional area of a rod 3 inches in diameter is 7.07 square inches. Since the rod extends through only one end of the pump cylinder, the average effective area of the two ends of each plunger is $314.16 - \frac{7.07}{2} = 310.63$ square inches.

The pressure corresponding to a vacuum of $8\frac{1}{4}$ inches is $p = 8.25 \div 2.037 = 4.05$ pounds per square inch, and the pressure corresponding to a difference in level of 8 feet is $S = 8 \times .434 = 3.47$ pounds per square inch.

Since there are two double-acting plungers, the total number of plunger strokes corresponding to 9,527 revolutions is $N = 9{,}527 \times 4 = 38{,}108$.

From formula **11**, the duty is found to be

$$D = \frac{A\,(P + p + S) \times L \times N}{H} \times 1{,}000{,}000$$

$$= \frac{310.63\,(95.5 + 4.05 + 3.47) \times 3 \times 38{,}108}{35{,}752{,}340} \times 1{,}000{,}000 \doteq 102{,}327{,}650. \text{ Ans.}$$

A SERIES

OF

QUESTIONS AND EXAMPLES

RELATING TO THE SUBJECTS

TREATED OF IN THIS VOLUME.

It will be noticed that the various Examination Questions that follow have been divided into sections to which have been assigned the same section numbers as the Instruction Papers to which they refer. No attempt should be made to answer any of the questions or to solve any of the examples until the Instruction Paper having the same section number as the section in which the questions or examples occur has been carefully studied.

ARITHMETIC.

(PART 1.)

EXAMINATION QUESTIONS.

(1) What is a number?

(2) What is the difference between a concrete number and an abstract number?

(3) Write each of the following numbers in words:

(*a*) 980; (*b*) 605; (*c*) 28,284; (*d*) 9,006,042; (*e*) 850,317,002; (*f*) 700,004.

(4) Represent in figures the following expressions:

(*a*) Seven thousand six hundred; (*b*) eighty-one thousand four hundred two; (*c*) five million four thousand seven; (*d*) one hundred eight million ten thousand one; (*e*) eighteen million six; (*f*) thirty thousand ten.

(5) $709 + 8,304,725 + 391 + 100,302 + 300 + 909 =$ what?

Ans. 8,407,336.

(6) Find the difference between the following:

(*a*) 50,962 and 3,338; (*b*) 10,001 and 15,339.

Ans. $\begin{cases} (a) & 47,624. \\ (b) & 5,338. \end{cases}$

(7) (*a*) $70,968 - 32,975 = ?$ (*b*) $100,000 - 98,735 = ?$

Ans. $\begin{cases} (a) & 37,993. \\ (b) & 1,265. \end{cases}$

(8) A man paid $125,000 for three shipments of ore. For the first he gave $44,675; for the second $26,380. What did he pay for the third shipment? Ans. $53,945.

(9) The greater of two numbers is 1,004 and their difference is 49; what is their sum? Ans. 1,959.

(10) From 5,962 + 8,471 + 9,023 take 3,874 + 2,039.

Ans. 17,543.

(11) Find the products of the following:

(*a*) 526,387 × 7; (*b*) 700,298 × 17; (*c*) 217 × 103 × 67.

Ans. $\begin{cases} (a)\ 3{,}684{,}709. \\ (b)\ 11{,}905{,}066. \\ (c)\ 1{,}497{,}517. \end{cases}$

(12) An engine and a boiler in a manufactory are worth \$3,246. The building is worth three times as much plus \$1,200, and the tools are worth twice as much as the building plus \$1,875. (*a*) What is the value of the building and tools? (*b*) What is the value of the whole plant?

Ans. $\begin{cases} (a)\ \$34{,}689. \\ (b)\ \$37{,}935. \end{cases}$

(13) Solve the following by cancelation:

(*a*) $\dfrac{72 \times 48 \times 28 \times 5}{96 \times 15 \times 7 \times 6} = ?$ (*b*) $\dfrac{80 \times 60 \times 50 \times 16 \times 14}{70 \times 50 \times 24 \times 20} = ?$

Ans. $\begin{cases} (a)\ 8. \\ (b)\ 32. \end{cases}$

(14) If a millman earns \$1,500 a year and spends \$968 a year, how long will it take him to save enough to purchase 28 acres of land at \$133 an acre? Ans. 7 years.

(15) A jaw crusher worked up 365 tons of ore in one week, and three times as much, lacking 246 tons, the next week; how many tons did it crush the second week?

Ans. 849 tons.

(16) What is the quotient of:

(*a*) 589,824 ÷ 576? (*b*) 369,730,620 ÷ 43,911? (*c*) 2,527,525 ÷ 505? (*d*) 4,961,794,302 ÷ 1,234?

Ans. $\begin{cases} (a)\ 1{,}024. \\ (b)\ 8{,}420. \\ (c)\ 5{,}005. \\ (d)\ 4{,}020{,}903. \end{cases}$

(17) A man paid \$444 for a horse, wagon, and harness. If the horse cost \$264 and the wagon \$153, how much did the harness cost? Ans. \$27.

(18) What is the product of:

(*a*) 1,024 × 576? (*b*) 5,005 × 505? (*c*) 43,911 × 8,420?

Ans. (*a*) 589,824. (*b*) 2,527,525. (*c*) 369,730,620.

(19) If a man receives 30 cents an hour for his wages, how much will he earn in a year, working 10 hours a day and averaging 25 days per month? Ans. \$900.

ARITHMETIC.

(PART 2.)

EXAMINATION QUESTIONS.

(1) What is a fraction?

(2) What does the denominator show?

(3) What does the numerator show?

(4) Is $\frac{13}{8}$ a proper or an improper fraction, and why?

(5) Write three mixed numbers.

(6) Reduce the following fractions to their lowest terms: $\frac{4}{8}$, $\frac{4}{16}$, $\frac{8}{32}$, $\frac{32}{64}$. Ans. $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{2}$.

(7) Reduce 6 to an improper fraction whose denominator is 4. Ans. $\frac{24}{4}$.

(8) Reduce $7\frac{7}{8}$, $13\frac{5}{16}$, and $10\frac{3}{4}$ to improper fractions.

Ans. $\frac{63}{8}$, $\frac{213}{16}$, $\frac{43}{4}$.

(9) Solve the following:

(*a*) $35 \div \frac{5}{16}$; (*b*) $\frac{9}{16} \div 3$; (*c*) $\frac{17}{2} \div 9$; (*d*) $\frac{113}{64} \div \frac{7}{16}$; (*e*) $15\frac{3}{4} \div 4\frac{3}{8}$.

Ans. $\begin{cases} (a)\ 112. \\ (b)\ \frac{3}{16}. \\ (c)\ \frac{17}{18}. \\ (d)\ 4\frac{1}{28}. \\ (e)\ 3\frac{3}{5}. \end{cases}$

(10) $\frac{1}{8} + \frac{2}{8} + \frac{5}{8} = ?$ Ans. 1.

(11) $42 + 31\frac{5}{8} + 9\frac{7}{16} = ?$ Ans. $83\frac{1}{16}$.

(12) An iron screen is divided into four sections; the first contains $29\frac{3}{4}$ square inches; the second, $50\frac{5}{8}$ square inches;

the third, 41 square inches; and the fourth, $69\frac{3}{16}$ square inches. How many square inches are in the screen?

Ans. $190\frac{9}{16}$ sq. in.

(13) Find the value of each of the following:

$$(a)\ \frac{7}{\frac{3}{16}};\quad (b)\ \frac{\frac{15}{32}}{\frac{5}{8}};\quad (c)\ \frac{\frac{4+3}{2+6}}{5}.$$

Ans. $\begin{cases} (a)\ 37\frac{1}{3}. \\ (b)\ \frac{3}{4}. \\ (c)\ \frac{7}{40}. \end{cases}$

(14) The numerator of a fraction is 28, and the value of the fraction is $\frac{7}{8}$; what is the denominator? Ans. 32.

(15) What is the difference between (*a*) $\frac{7}{8}$ and $\frac{7}{16}$? (*b*) 13 and $7\frac{7}{16}$? (*c*) $312\frac{9}{16}$ and $229\frac{5}{32}$?

Ans. $\begin{cases} (a)\ \frac{7}{16}. \\ (b)\ 5\frac{9}{16}. \\ (c)\ 83\frac{13}{32}. \end{cases}$

(16) If a mill stamps $85\frac{5}{12}$ tons in one day, $78\frac{9}{15}$ tons in another day, and $125\frac{17}{35}$ tons in another day, how many tons did it stamp in the three days? Ans. $289\frac{211}{420}$ tons.

(17) From $573\frac{4}{5}$ tons take $216\frac{5}{8}$ tons. Ans. $357\frac{7}{40}$ tons.

(18) Multiply $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{7}{11}$ of $\frac{19}{20}$ of 11 by $\frac{7}{8}$ of $\frac{5}{6}$ of 45.

Ans. $109\frac{13}{128}$.

(19) How many times is $\frac{2}{3}$ contained in $\frac{3}{4}$ of 16?

Ans. 18 times.

(20) Bought $211\frac{1}{4}$ pounds of old lead for $1\frac{7}{8}$ cents per pound. Sold a part of it for $2\frac{1}{2}$ cents per pound, receiving for it the same amount as I paid for the whole. How many pounds did I have left? Ans. $52\frac{13}{16}$ lb.

ARITHMETIC.

(PART 3.)

EXAMINATION QUESTIONS.

(1) Write out in words the following numbers: .08, .131, .0001, .000027, .0108, and 93.0101.

(2) State the difference between a fraction and a decimal.

(3) State how to reduce a fraction to a decimal.

(4) Reduce the following fractions to equivalent decimals: $\frac{1}{2}$, $\frac{7}{8}$, $\frac{5}{32}$, $\frac{65}{100}$, and $\frac{125}{1000}$.

$$\text{Ans.} \begin{cases} .5. \\ .875. \\ .15625. \\ .65. \\ .125. \end{cases}$$

(5) Solve the following:

(*a*) $\dfrac{32.5 + .29 + 1.5}{4.7 + 9}$; (*b*) $\dfrac{1.283 \times \overline{8 + 5}}{2.63}$;

(*c*) $\dfrac{\overline{589 + 27} \times \overline{163 - 8}}{25 + 39}$; (*d*) $\dfrac{\overline{40.6 + 7.1} \times (3.029 - 1.874)}{6.27 + 8.53 - 8.01}$.

$$\text{Ans.} \begin{cases} (a) \ 2.5029. \\ (b) \ 6.3418. \\ (c) \ 1{,}491.875. \\ (d) \ 8.1139. \end{cases}$$

(6) How many inches in .875 of a foot? Ans. $10\frac{1}{2}$ in.

(7) What decimal part of a foot is $\frac{3}{16}$ of an inch?

Ans. .015625.

(8) A cubic inch of water weighs .03617 of a pound. What is the weight of a body of water whose volume is 1,500 cubic inches ? Ans. 54.255 lb.

(9) A cubic foot of ore weighs 176 pounds and a cubic foot of water 62.5 pounds. How many times heavier is the ore than the water ? Ans. 2.816 times.

(10) Divide 17,892 by 231 and carry the result to four decimal places. Ans. 77.4545+.

(11) Express approximately (*a*) .7928 in 64ths; (*b*) .1416 in 32ds; (*c*) .47915 in 16ths.

$$\text{Ans.}\begin{cases}(a)\ \frac{51}{64}.\\(b)\ \frac{5}{32}.\\(c)\ \frac{8}{16}.\end{cases}$$

(12) What is the sum of .125, .7, .089, .4005, .9, and .000027 ? Ans. 2.214527.

(13) Solve the following:

(*a*) $(\frac{7}{16} - .13) \times \overline{.625 + \frac{5}{8}}$; (*b*) $(\frac{19}{32} \times .21) - (.02 \times \frac{3}{16})$; (*c*) $(\frac{13}{4} + .013 - 2.17) \times \overline{13\frac{1}{4} - 7\frac{5}{16}}$.

$$\text{Ans.}\begin{cases}(a)\ .384375.\\(b)\ .1209375.\\(c)\ 6.4896875.\end{cases}$$

(14) Find the value of the following expression:

$$\frac{1.25 \times 20 \times 3}{\dfrac{87 + (11 \times 8)}{459 + 32}}$$

Ans. $210\frac{3}{7}$.

(15) The cost of building a reverberatory roasting furnace 60 feet long was $2,632. What was its cost per running foot ? Ans. $43.87, nearly.

(16) If limestone weighs 2.4 times as much as water, and a cubic foot of water weighs 62.5 pounds, what is the weight of a cubic foot of limestone ? Ans. 150 lb.

(17) If a cubic inch of water weighs .03617 pound, and a cubic inch of mercury weighs .49175 pound, how many cubic inches of water will balance the weight of one cubic inch of mercury ? Ans. 13.5955 cu. in.

(18) A cubic foot of water contains 7.48 gallons. What is the capacity in cubic feet of a tank holding 2,000 gallons of water ? Ans. 267.4 cu. ft., nearly.

(19) Raw ore dried and crushed fine will average in weight about 92.5 pounds per cubic foot. How many tons of 2,000 pounds will there be in a tank having a capacity of 267.3 cubic feet? Ans. 12.36 tons.

(20) Pulverized ore shrinks in bulk about .18 when wet. What space will 267.3 cubic feet occupy after being wet? Ans. 219.19 cu. ft.

ARITHMETIC.

(PART 4.)

EXAMINATION QUESTIONS.

(1) What per cent. of 50 is 2? Ans. 4%.

(2) A farmer gained 15% on his farm by selling it for $5,500. What did it cost him? Ans. $4,782.61.

(3) A man receives a salary of $950. He pays 24% of it for board, 12½% of it for clothing, and 17% of it for other expenses. How much does he save in a year?

Ans. $441.75.

(4) If 37½ per cent. of a number is 961.38, what is the number? Ans. 2,563.68.

(5) What sum diminished by 35% of itself equals $4,810? Ans. $7,400.

(6) The distance between two stations on a certain railroad is 16.5 miles, which is 12½% of the entire length of the road. What is the length of the road? Ans. 132 miles.

(7) Reduce 28 rd. 4 yd. 2 ft. 10 in. to inches.

Ans. 5,722 in.

(8) Reduce 5,722 in. to higher denominations.

Ans. 28 rd. 4 yd. 2 ft. 10 in.

(9) How many pounds, ounces, pennyweights, and grains are contained in 13,750 gr.?

Ans. 2 lb. 4 oz. 12 pwt. 22 gr.

(10) What is the sum of 3 gal. 3 qt. 1 pt. 3 gi.; 6 gal. 1 pt. 2 gi.; 4 gal. 1 gi.; 8 qt. 5 pt.? Ans. 16 gal. 3 qt. 2 gi.

(11) What is the sum of 11° 16′ 12″; 13° 19′ 30″; 20° 25″; 26′ 29″; 10° 17′ 11″? Ans. 55° 19′ 47″.

(12) What is the sum of 130 rd. 5 yd 1 ft. 6 in.; 215 rd. 2 ft. 8 in.; 304 rd. 4 yd. 11 in.?

Ans. 2 mi. 10 rd. 5 yd. 7 in.

(13) What is the sum of 21 A. 67 sq. ch. 3 sq. rd. 21 sq. li.; 28 A. 78 sq. ch. 2 sq. rd. 23 sq. li.; 47 A. 6 sq. ch. 2 sq. rd. 18 sq. li.; 56 A. 59 sq. ch. 2 sq. rd. 16 sq. li.; 25 A. 38 sq. ch. 3 sq. rd. 23 sq. li.; 46 A. 75 sq. ch. 2 sq. rd. 21 sq. li.?

Ans. 255 A. 3 sq. ch. 14 sq. rd. 122 sq. li.

(14) From 20 rd. 2 yd. 2 ft. 9 in. take 300 ft.

Ans. 2 rd. 1 yd. 2 ft. 9 in.

(15) A note was given August 5, 1890, and was paid June 3, 1892. What length of time did it run?

Ans. 1 yr. 9 mo. 28 da.

(16) If 1 iron rail is 17 ft. 3 in. long, how long would 51 rails be if placed end to end? Ans. 53 rd. $1\frac{1}{2}$ yd. 9 in.

(17) Multiply 3 lb. 10 oz. 13 pwt. 12 gr. by 1.5.

Ans. 5 lb. 10 oz. 6 gr.

(18) Multiply 7 T. 15 cwt. 10.5 lb. by 1.7.

Ans. 13 T. 3 cwt. 67.85 lb.

(19) Divide 282 bu. 3 pk. 1 qt. 1 pt. by 12.

Ans. 23 bu. 2 pk. 2 qt. $\frac{1}{4}$ pt.

(20) If 16 square miles are equally divided into 62 farms, how much land will each contain?

Ans. 165 A. 25 sq. rd. 24 sq. yd. 3 sq. ft. 80+ sq. in.

ARITHMETIC.

(PART 5.)

EXAMINATION QUESTIONS.

(1) What is the square of 108 ? Ans. 11,664.

(2) Find the fifth power of 9. Ans. 59,049.

(3) What is the value of $.0133^3$? Ans. .000002352637.

(4) In what respect does evolution differ from involution ?

(5) Extract the square root of 90. Ans. 9.4868+.

(6) Find the value of $(3\frac{3}{4})^3$. Ans. $52\frac{47}{64}$, or 52.734375.

(7) What is the cube root of 92,416 ? Ans. 45.212−.

(8) Find the value of $\sqrt{502,681}$. Ans. 709.

(9) What is the value of $\sqrt[3]{\frac{27}{64}}$? Ans. $\frac{3}{4}$.

(10) From the cube of 4 take the cube root of 8. Ans. 62.

(11) What is the value of $\sqrt[3]{\frac{3}{8}}$. Ans. .72112+.

(12) Extract the square root of .3364. Ans. .58.

(13) Find the square root of 3.1416. Ans. 1.7725−.

(14) What number multiplied by itself equals 114.9184 ? Ans. 10.72.

(15) Extract the square root of 3,486,784. Ans. 1,867.3−.

(16) Find the square root of .00041209. Ans. .0203.

(17) Find the fifth root of 4,558.4. Ans. 5.3922−.

(18) Extract the fifth root of .127. Ans. .66185+.

(19) $\sqrt[5]{72.415} = ?$ Ans. 2.3549−.

ARITHMETIC.

(PART 6.)

EXAMINATION QUESTIONS.

Find the value of x in the following:

(1) $11.7 : 13 :: 20 : x$. Ans. 22.22+.

(2) (*a*) $20 + 7 : 10 + 8 :: 3 : x$; (*b*) $12^2 : 100^2 :: 4 : x$.

Ans. $\begin{cases} (a) & 2. \\ (b) & 277.7+. \end{cases}$

(3) (*a*) $\frac{4}{x} = \frac{7}{21}$; (*b*) $\frac{x}{24} = \frac{8}{16}$; (*c*) $\frac{2}{10} = \frac{x}{100}$; (*d*) $\frac{15}{45} = \frac{60}{x}$; (*e*) $\frac{10}{150} = \frac{x}{600}$.

Ans. $\begin{cases} (a) & x = 12. \\ (b) & x = 12. \\ (c) & x = 20. \\ (d) & x = 180. \\ (e) & x = 40. \end{cases}$

(4) $45 : 60 :: x : 24$. Ans. 18.

(5) $x : 35 :: 4 : 7$. Ans. 20.

(6) $\sqrt[3]{1,000} : \sqrt[3]{1,331} = 27 : x$. Ans. 29.7.

(7) $64 : 81 = 21^2 : x^2$. Ans. 23.625.

(8) $7 + 8 : 7 = 30 : x$. Ans. 14.

(9) If a column of mercury 27.63 in. high weighs .76 of a pound, what will be the weight of a column of mercury having the same diameter, 29.4 inches high? Ans. .808+ lb.

(10) If 5 men by working 8 hours a day can do a certain amount of work, how many men by working 10 hours a day can do the same work? Ans. 4 men.

(11) If a man travel 540 miles in 20 days of 10 hours each, how many hours a day must he travel to cover 630 miles in 25 days? Ans. $9\frac{1}{3}$ hr.

(12) Five grams of lead ore were taken for assay and 2.5 grams of lead were obtained in a button. What proportion of lead is there in the ore? Ans. .5.

(13) In a zinc assay, 3.042 gm. of zinc ore were taken and .0492 gm. of zinc obtained. What proportion of the ore is zinc? Ans. .2134 part zinc.

(14) Ferrous oxide is composed of 1 part iron and 1 part oxygen. The atomic weight of the one part iron is 56, of the one part oxygen 16. What proportion of the total weight is oxygen? Ans. 22.22 parts.

(15) Carbon monoxide, one of the gases in furnace reduction of ores, is composed of 1 part carbon and 1 part oxygen. The total atomic weight of the gas is 28; the atomic weight of the one part oxygen is 16, of the one part carbon 12. What proportion is carbon and what oxygen?

Ans. $\begin{cases} \text{42.86 parts carbon.} \\ \text{57.14 parts oxygen.} \end{cases}$

MENSURATION AND USE OF LETTERS IN FORMULAS.

EXAMINATION QUESTIONS.

$A = 5 \qquad h = 200$

$B = 10 \qquad x = 12$

$i = 3.5 \qquad D = 120$

Work out the solutions to the following formulas, using the above values for the letters:

(1) $C = \frac{D - x}{B + i}$. Ans. $C = 8$.

(2) $Q = \frac{A h + D}{2 x + 6} + D$. Ans. $Q = 157\frac{1}{3}$.

(3) $v = \sqrt{\frac{A D}{i B + 1.5}}$. Ans. $v = 4.05+$.

(4) $g = \frac{(B - A)^2 - \sqrt{h + 2 B + A}}{A^2 - (1 + D)}$. Ans. $g = 2\frac{1}{2}$.

(5) If one of the angles formed by one straight line meeting another straight line equals 152° 3′, what is the other angle equal to? Ans. 27° 57′.

(6) Draw an obtuse angle, a right angle, and an acute angle. State the name of each angle by using letters to designate them.

(7) Draw a rhombus and then draw a rectangle having the same area.

(8) A sheet of zinc measures $11\frac{1}{2}$ inches by $2\frac{1}{2}$ feet. How many square inches does it contain? Ans. 345 sq. in.

(9) How many boards 16 feet long and 5 inches wide would be required to lay a floor measuring 15 ft. × 24 ft.?

Ans. 54 boards.

(10) The accompanying figure shows the floor plan of an electric-light station. From the dimensions given, calculate the number of square feet of unoccupied floor space.

Ans. 2,059.08 sq. ft.

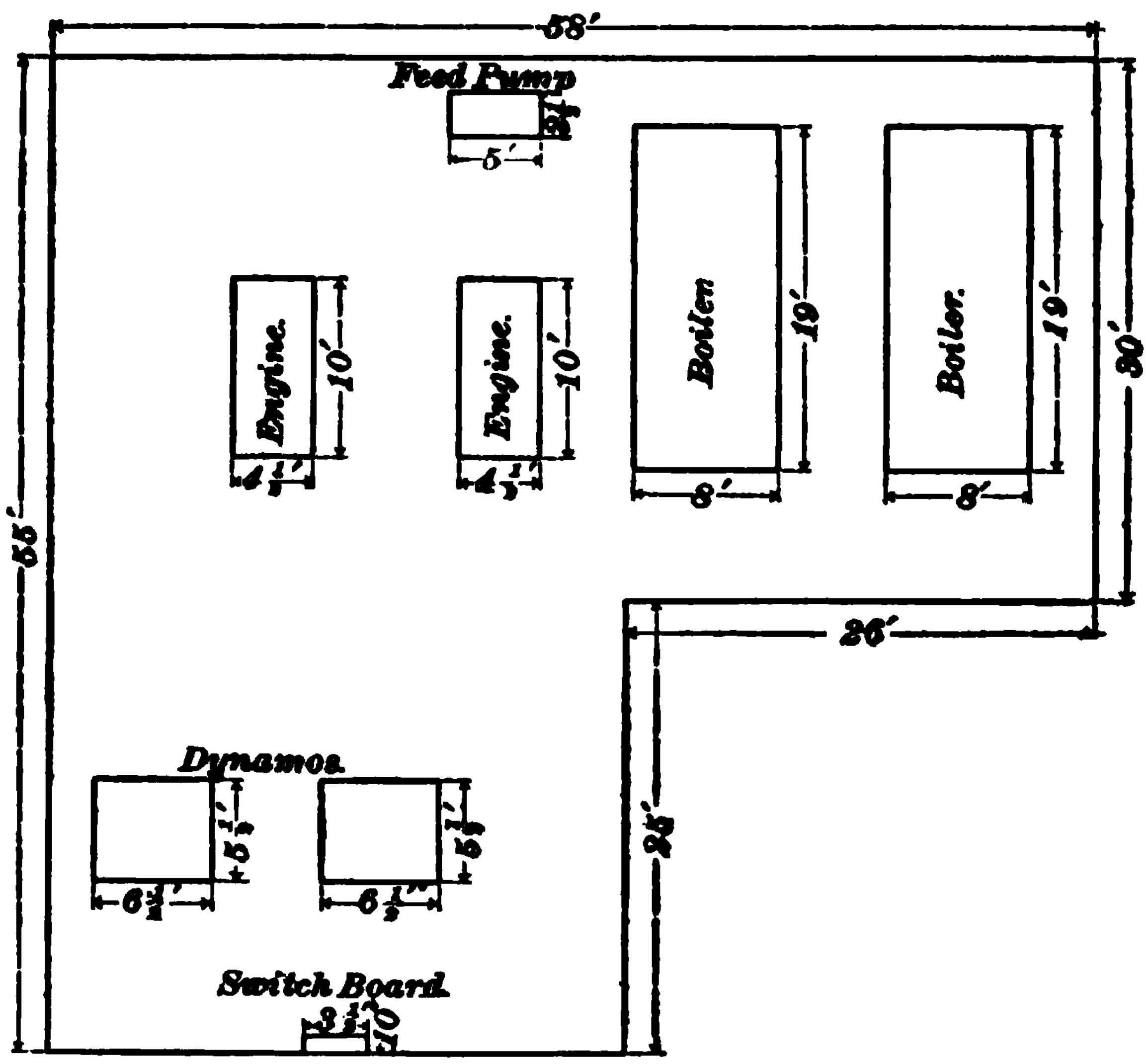


FIG. I.

(11) A triangle has three equal angles; what is it called?

(12) If a triangle has two equal angles, what kind of a triangle is it?

(13) In a triangle $A\ B\ C$, angle $A = 23°$ and $B = 32° \ 32'$; what does angle C equal? Ans. $C = 124° \ 28'$.

(14) In the figure, if $AD = 10$ inches, $AB = 24$ inches, and $BC = 13\frac{1}{2}$ inches, how long is DE, DE being parallel to BC? Ans. $DE = 5.625$ in.

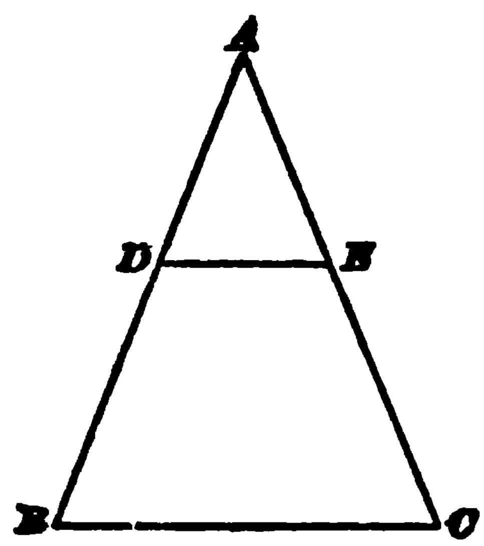


FIG. II.

(15) An engine room is 52 feet long and 39 feet wide. How many feet is it from one corner to a diagonally opposite one, measured in a straight line? Ans. 65 ft.

(16) It is required to make a miter-box in which to cut molding to fit around an octagon post. At what angle with the side of the box should the saw run? Ans. $67\frac{1}{2}°$.

(17) If the distance between two opposite corners of a hexagonal nut is 2 inches, what is the distance between two opposite sides? Ans. 1.732+ in.

(18) In the accompanying figure, if the distance BI is 6 inches and HK 18 inches, what is the diameter of the circle? Ans. 19.5 in.

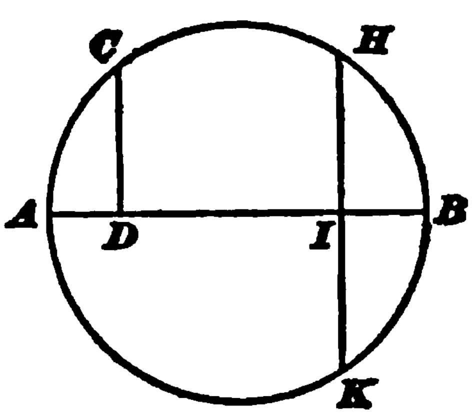


FIG. III.

(19) How many revolutions will a 72-inch locomotive driver make in going 1 mile? Ans. 280.112 revolutions.

(20) A pipe has an internal diameter of 6.06 inches; what is the area of a circle having this diameter? Ans. 28.8427 sq. in.

(21) How long must the arc of a circle be to contain 12°, supposing the radius of the circle to be 6 inches? Ans. 1.25664 in.

(22) What is the area of the sector of a circle 15 inches in diameter, the angle between the two radii forming the sector being $12\frac{1}{2}°$? Ans. 6.1359 sq. in.

(23) (*a*) What would be the length of the side of a square metal plate having an area of 103.8691 square inches?

(*b*) What would be the diameter of a round plate having this area? (*c*) How much shorter is the circumference of the round plate than the perimeter of the square plate?

Ans. $\begin{cases} (a) & 10.1916 \text{ in.} \\ (b) & 11\frac{1}{2} \text{ in.} \\ (c) & 4.638 \text{ in.} \end{cases}$

(24) Find the area in square feet of the entire surface of a hexagonal column 12 feet long, each edge of the ends of the column being 4 inches long. Ans. 24.5774 sq. ft.

(25) Find the cubical contents of the above column in cubic inches. Ans. 5,985.9648 cu. in.

(26) Compute the weight per foot of an iron boiler tube 4 inches outside diameter and 3.73 inches inside diameter, the weight of the iron being taken at .28 pound per cubic inch. Ans. $5\frac{1}{2}$ lb.

(27) The dimensions of a return-tubular boiler are as follows: Diameter, 60 inches; length between heads, 16 feet; outside diameter of tubes, $3\frac{1}{2}$ inches; number of tubes, 64; distance of mean water-line from top of boiler, 18 inches. (*a*) Compute the steam space of the boiler in cubic feet. (*b*) Determine the number of gallons of water that will be required to fill the boiler up to the mean water level.

Ans. $\begin{cases} (a) & 79.2 \text{ cu. ft.} \\ (b) & 1{,}246 \text{ gal., nearly.} \end{cases}$

(28) The length of the circumference of the base of a cone is 18.8496 inches and its slant height is 10 inches. Find the area of the entire surface of the cone.

Ans. 122.5224 sq. in.

(29) If the altitude of the above cone were 9 inches, what would be its volume? Ans. 84.8232 cu. in.

(30) A square vat is 11 feet deep, 15 feet square at the top, and 12 feet square at the bottom. How many gallons will it hold? Ans. 15,058.29 gal.

(31) How many pails of water would be required to fill the vat, the pail having the following dimensions: Depth,

11 inches; diameter at the top, 12 inches; diameter at the bottom, 9 inches? Ans. 3,627.28.

(32) Find (*a*) the area of the surface, and (*b*) the cubical contents of a ball $22\frac{1}{2}$ inches in diameter.

Ans. $\begin{cases} (a) & 1{,}590.435 \text{ sq. in.} \\ (b) & 5{,}964.1313 \text{ cu. in.} \end{cases}$

(33) (*a*) What is the volume and area of a cylindrical ring whose outside diameter is 16 inches and inside diameter 13 inches? (*b*) If made of cast iron, what is its weight? Take the weight of 1 cubic inch of cast iron as .261 pound.

Ans. Weight = 21 lb.

ELEMENTARY ALGEBRA

AND

TRIGONOMETRIC FUNCTIONS.

EXAMINATION QUESTIONS.

(1) How may the signs of all the terms of the denominator of a fraction be changed from $+$ to $-$ or from $-$ to $+$ without altering the value of the fraction?

(2) Divide $3a^3 + 2 - 4a^5 + 7a + 2a^6 - 5a^4 + 10a^2$ by $a^3 - 1 - a^2 - 2a$. Ans. $2a^3 - 2a^2 - 3a - 2$.

(3) Multiply: (*a*) $2 + 4a - 5a^2 - 6a^3$ by $7a^3$; (*b*) $4x^2 - 4y^2 + 6z^2$ by $3x^2y$; (*c*) $3b + 5c - 2d$ by $6a$.

$$\text{Ans.}\begin{cases} (a)\ 14a^3 + 28a^4 - 35a^5 - 42a^6. \\ (b)\ 12x^4y - 12x^2y^3 + 18x^2yz^2. \\ (c)\ 18ab + 30ac - 12ad. \end{cases}$$

(4) A man performed a journey of 48 miles in a certain number of hours; but if he had traveled 4 miles more each hour, he would have performed the journey in 6 hours. How many miles did he travel per hour? Ans. 4 mi.

(5) Translate the following algebraic expressions into ordinary language: $\sqrt{\frac{a+b+c}{n}} + \sqrt{a} + \frac{b+c}{n} + \sqrt{a+b} + \frac{c}{n} + (a+b)\,c + a + bc$.

(6) Find the products of:

(*a*) $\frac{9m^2n^2}{8p^3q^3}$, $\frac{5p^2q}{2xy}$, and $\frac{24x^2y^2}{90mn}$;

(*b*) $3ax + 4$ and $\frac{a^2}{9a^3x^2 + 24a^2x + 16a}$.

$$\text{Ans.}\begin{cases} (a)\ \frac{3mnxy}{4pq^2}. \\ (b)\ \frac{a}{3ax+4}. \end{cases}$$

(7) A vessel containing some water was filled by pouring in 42 more gallons; there was then 7 times as much water in the vessel as at first. How much did the vessel hold?

Ans. 49 gal.

(8) Reduce $\dfrac{c(a+b)+cd}{(a+b)c}$ to its simplest form.

(9) Simplify $\dfrac{3+2x}{2-x} - \dfrac{2-3x}{2+x} + \dfrac{16x-x^2}{x^2-4}$. Ans. $\dfrac{1}{x+2}$.

(10) From $a^4 - b^4$ take $5a^3b - 7a^2b^2 + 5ab^3$, and from the result take $3a^4 - 4a^3b + 6a^2b^2 + 5ab^3 - 3b^4$.

Ans. $-2a^4 - a^3b + a^2b^2 - 10ab^3 + 2b^4$.

(11) (*a*) Give an illustration, not contained in the section on *Algebra*, that will explain the difference between positive and negative quantities. (*b*) In what respects are addition and subtraction different in algebra from addition and subtraction in arithmetic?

(12) (*a*) What is the value of a^0? (*b*) What does $a^3 \div a^{-1}$ equal?

(13) Change the fraction $-\dfrac{c-(a-b)}{c+(a+b)}$ so that the sign before the dividing line will be $+$.

(14) (*a*) What is the reciprocal of $\frac{24}{45}$? (*b*) Of what number is 700 the reciprocal?

(15) (*a*) Explain in your own words the difference between a coefficient and an exponent. (*b*) How are coefficients and exponents treated in multiplication, and how in division? (*c*) What is the law of signs in multiplication?

(16) Solve the equations:

(*a*) $\dfrac{9x+20}{36} = \dfrac{4(x-3)}{5x-4} + \dfrac{x}{4}$; (*b*) $ax - \dfrac{3a-bx}{2} = \dfrac{1}{2}$;

(*c*) $am - b - \dfrac{ax}{b} + \dfrac{x}{m} = 0$.

$$\text{Ans.} \begin{cases} (a)\ x = 8. \\ (b)\ x = \dfrac{3a+1}{2a+b}. \\ (c)\ x = bm. \end{cases}$$

(17) State how you would read the following expressions: (*a*) $a^2x^2 + 2a^3b^5 - (a+b)$; (*b*) $\sqrt[3]{x} + y(a-n^2)^{\frac{2}{3}}$;

(c) $(m+n)(m-n)^2\left(m-\frac{n}{2}\right)$.

(18) (a) Write a monomial; a binomial; a polynomial. (b) In the expression $a + 2ab - b^2$, why cannot the indicated addition and subtraction be performed ? (c) What operation is indicated between the quantities in $4ac^2d$?

(19) Resolve into their factors: (a) $45x^7y^{10} - 90x^6y^7 - 360x^4y^n$; (b) $a^2b^2 + 2abcd + c^2d^2$; (c) $(a+b)^2 - (c-d)^2$.

Ans. (c) $(a+b+c-d)(a+b-c+d)$.

(20) Remove the symbols of aggregation from the following:

(a) $2a - \{3b + [4c - 4a - (2a + 2b)] + [3a - \overline{b + c}]\}$;

(b) $7a - \{3a - [(2a - 5a) + 4a]\}$;

(c) $a - \{2b + [3c - 3a - (a + b)] + [2a - (b + c)]\}$.

$$\text{Ans.}\begin{cases}(a)\ 5a - 3c.\\ (b)\ 5a.\\ (c)\ 3a - 2c.\end{cases}$$

(21) (a) Arrange $a^3b^2 + 2abc + 3 - 7a^2b^3 + 6a^4b^4$ according to the decreasing powers of a; (b) according to the increasing powers of b. (c) With $a^2 + 1 + 2a^3 + ax$ arranged according to the *increasing* powers of a, should the 1 be placed first or last, and why ?

(22) (a) Express with radical signs: $x^{\frac{5}{2}}$; $3x^{\frac{1}{2}}y^{-\frac{3}{4}}$; $3x^{\frac{1}{4}}y^{-\frac{5}{3}}z^{\frac{1}{3}}$.

(b) Clear $a^{-1}b^{\frac{1}{2}} + \dfrac{c^{-2}}{a+b} + (m-n)^{-1} - \dfrac{a^2b^{-2}c}{c^{-3}}$ of negative exponents. (c) Express with fractional exponents: $\sqrt[7]{x^6}$; $\sqrt[3]{x^{-4}}$; $(\sqrt[4]{b^6x^2})^3$.

(23) Divide:

(a) $\dfrac{2ax + x^2}{a^3 - x^3}$ by $\dfrac{x}{a-x}$; (b) $\dfrac{6m^2n^2 - 3n}{4m^4n^2 - 4m^2n + 1}$ by $\dfrac{3n}{4m^4n^2 - 1}$;

(c) $9 + \dfrac{5y^2}{x^2 - y^2}$ by $3 + \dfrac{5y}{x-y}$.

$$\text{Ans.}\begin{cases}(a)\ \dfrac{2a + x}{a^2 + ax + x^2}.\\ (b)\ 2m^2n + 1.\\ (c)\ \dfrac{3x - 2y}{x + y}.\end{cases}$$

(24) Perform the indicated additions:

(a) $\dfrac{x}{x-y}+\dfrac{x-y}{y-x}$; (b) $\dfrac{x^2}{x^2-1}+\dfrac{x}{x+1}-\dfrac{x}{1-x}$;

(c) $\dfrac{3a-4b}{7}-\dfrac{2a-b+c}{3}+\dfrac{13a-4c}{12}$.

$$\text{Ans.}\begin{cases}(a)\ \dfrac{y}{x-y}.\\ (b)\ \dfrac{3x^2}{x^2-1}.\\ (c)\ \dfrac{71a-20b-56c}{84}.\end{cases}$$

(25) Factor the following: (a) $9x^4+12x^2y^2+4y^4$; (b) $49a^4-154a^2b^2+121b^4$; (c) $64x^2y^2+64xy+16$.

(26) Divide: (a) $3x^2+x+9x^3-1$ by $3x-1$; (b) $a^3-2ab^2+b^3$ by $a-b$; (c) $7x^3+58x-24x^2-21$ by $7x-3$.

$$\text{Ans.}\begin{cases}(a)\ 3x^2+2x+1.\\ (b)\ a^2+ab-b^2.\\ (c)\ x^2-3x+7.\end{cases}$$

(27) What is the ratio of (a) x^4-1 to x^2+1? (b) $x^4-2x^2y^2+y^4$ to $x-y$? Ans. (b) $(x^2-y^2)(x+y)$.

(28) Solve the following:

(a) $\dfrac{10x+3}{3}-\dfrac{6x-7}{2}=10(x-1)$;

(b) $(a^2+x)^2=x^2+4a^2+a^4$; (c) $\dfrac{x-1}{x-2}-\dfrac{x+1}{x+2}=\dfrac{3}{x^2-4}$.

$$\text{Ans.}\begin{cases}(a)\ x=1\tfrac{1}{2}.\\ (b)\ x=2.\\ (c)\ x=1\tfrac{1}{2}.\end{cases}$$

(29) Divide: (a) $35m^3y+28m^2y^2-14my^3$ by $-7my$; (b) $4a^4-3a^5b-a^6b^2$ by a^4; (c) $4x^3-8x^5+12x^7-16x^9$ by $4x^2$.

$$\text{Ans.}\begin{cases}(a)\ -5m^2-4my+2y^2.\\ (b)\ 4-3ab-a^2b^2.\\ (c)\ x-2x^3+3x^5-4x^7.\end{cases}$$

(30) A post has $\frac{1}{5}$ of its length in the earth, $\frac{3}{7}$ in the water, and 13 feet in the air. What is its length?

Ans. 35 ft.

(31) In the composition of a quantity of gunpowder, the niter was 10 pounds more than $\frac{2}{3}$ of the whole, the sulphur was $4\frac{1}{2}$ pounds less than $\frac{1}{6}$ of the whole, and the charcoal was 2 pounds less than $\frac{1}{7}$ of the niter. What was the amount of gunpowder and of each of the ingredients?

Ans. $\begin{cases} \text{Gunpowder, 69 lb.} \\ \text{Niter, 56 lb.} \\ \text{Sulphur, 7 lb.} \\ \text{Charcoal, 6 lb.} \end{cases}$

(32) Find the values of the following:

(a) $\sqrt[3]{-125x^3y^6z^9}$; (b) $\sqrt[4]{10{,}000a^{16}b^{20}c^8}$; (c) $\sqrt[5]{243m^{15}n^{20}}$; (d) $\sqrt[5]{-\frac{x^5y^{10}z^{15}}{a^{20}b^{15}c^{10}d^5}}$.

Ans. $\begin{cases} (a) \ -5xy^2z^3. \\ (b) \ \pm 10a^4b^5c^2. \\ (c) \ 3m^3n^4. \\ (d) \ -\frac{xy^2z^3}{a^4b^3c^2d}. \end{cases}$

(33) Find the sine, cosine, and tangent of 17° 27′ 37″.

(34) Sine=.27038, cosine=.27038, and tangent=2.27038; find the corresponding angles.

(35) In a right triangle $A\ B\ C$, the hypotenuse $A\ B = 17.69$ feet, and the side $A\ C = 9$ feet 9 inches; find the other three parts.

Ans. $\begin{cases} 56° \ 33' \ 15''. \\ 33° \ 26' \ 45''. \\ \text{14 ft. 9 in.} \end{cases}$

(36) In a right triangle $A\ B\ C$, right-angled at C, side $A\ C = 17.5$, side $B\ C = 21.3$; find the other three parts.

Ans. $\begin{cases} 39° \ 24' \ 23''. \\ 50° \ 35' \ 37''. \\ \text{27.57, nearly.} \end{cases}$

(37) In a right triangle $A\ B\ C$, angle $A = 65° \ 13' \ 29''$, hypotenuse $A\ B = 5\frac{1}{2}$ yards; find the other three parts. Give sides in feet and inches.

Ans. $\begin{cases} 24° \ 46' \ 31''. \\ \text{14 ft. } 11\frac{3}{4} \text{ in.} \\ \text{6 ft. 11 in., nearly.} \end{cases}$

(38) Add 159° 27′ 34.6″, 25° 16′ 8.7″, and 3° 48′ 53″.

(39) In a triangle $A\ B\ C$, side $A\ B = 70$ feet, side $B\ C = 42$ feet, and angle $A = 36°\ 10'$. Find the angles B and C and the side $A\ C$.

$$\text{Ans.} \begin{cases} \text{Angle } B = 64°\ 14'. \\ \text{Angle } C = 79°\ 36'. \\ \text{Side } A\ C = 64.1 \text{ ft.} \end{cases}$$

(40) (*a*) What is the supplement of 72° 11′ 36″? (*b*) What is the complement of 22° 34′ 17″?

$$\text{Ans.} \begin{cases} (a)\ 107°\ 48'\ 24''. \\ (b)\ 67°\ 25'\ 43''. \end{cases}$$

MECHANICS.

(PART 1.)

EXAMINATION QUESTIONS.

(1) (*a*) What is a molecule? (*b*) What is an atom?

(2) The number of teeth in a spur-gear is 50 and the pitch is $1\frac{1}{2}$ inches; (*a*) what is the pitch diameter? (*b*) What is the outside diameter?

Ans. $\begin{cases} (a) \; 23.87''. \\ (b) \; 24.77''. \end{cases}$

(3) What pressure can be exerted by a force of 24 pounds on a half-inch screw which has 13 threads per inch, the distance from the center of the screw to the point on the handle where the force is applied being 11 inches?

Ans. 21,563.94 lb.

(4) A ball weighing 5 pounds revolves in a circle whose radius is 32 inches, at the rate of 350 R. P. M.; what is the pull on the support caused by the ball? Ans. $555\frac{1}{3}$ lb.

(5) A body weighing 2 pounds has a velocity of 600 feet per second; what is its kinetic energy? Ans. 11,194 ft.-lb.

(6) What should be the width of a double leather belt to transmit 150 horsepower, when the belt has a velocity of 3,000 feet per minute, and has 7 feet of its length in contact with the smaller pulley, whose diameter is 63 inches? Give width to nearest half inch. Ans. 29.5 in.

(7) (*a*) What are the three states of matter? Name (*b*) some of the general properties of matter; (*c*) some of the specific properties.

(8) What is meant by *center of gravity?*

(9) (*a*) Why is crowning usually given to the face of a pulley? (*b*) Why should high-speed pulleys be balanced?

(10) At what speed must the engine run when the diameter of the band-wheel is 13 feet and of the main pulley 91 inches, if the speed of the main shaft is to be 108 R. P. M.?

Ans. 63 R. P. M.

(11) What do you understand by *specific gravity?*

(12) What should be the width of a single leather belt to transmit $2\frac{1}{2}$ horsepower when the belt has a velocity of 2,000 feet per minute? The diameter of the smaller pulley is 14 inches, and the belt has 18 inches of its length in contact with it. Ans. $1\frac{1}{8}$ inch.

(13) What is meant (*a*) by inertia? (*b*) by weight? (*c*) How is weight measured?

(14) The speed of a certain belt is 3,000 feet per minute; if it drives a 48-inch pulley, how long will it take the pulley to make 100 revolutions? Ans. 25.13 sec., nearly.

(15) Find the point of suspension of a rectangular cast-iron lever 4 feet 6 inches long, 2 inches deep, and $\frac{3}{4}$ inch thick, having weights 47 and 71 pounds hung from each end, in order that there may be equilibrium. Take the weight of a cubic inch of cast iron as .261 pound.

$$\text{Ans.} \begin{cases} \text{Short arm} = 22.343 \text{ in.} \\ \text{Long arm} = 31.657 \text{ in.} \end{cases}$$

(16) A cubic foot of a certain kind of wood weighs 51 pounds; what is its specific gravity? Ans. .816.

(17) What is (*a*) motion? (*b*) velocity? (*c*) rest? (*d*) Can a body be in motion with respect to one object and at rest with respect to another? Explain fully.

(18) (*a*) What is force? (*b*) Name several kinds of forces.

(19) Find by measurement the center of gravity of a triangle whose sides are 4 inches, 5 inches, and 6 inches long. Ans. $1\frac{3}{32}$ inches from 6-inch side.

(20) What horsepower can be safely transmitted by a gear whose pitch is 1.57 inches, pitch diameter is 30 inches, and which makes 100 revolutions per minute?

Ans. 19.36 H. P.

(21) (*a*) What is uniform motion? (*b*) What is variable motion? (*c*) If a body moves 10 feet the first second, 12 feet the second second, 15 feet the third second, etc., is its motion uniform or variable, and why?

(22) In a train of gears used to raise a weight of 6,000 pounds in a manner similar to that shown in Fig. 612, the diameters of the drivers and belt pulley are 18 inches, 12 inches, 15 inches, and 12 inches, and of the pinions and drum, 6 inches, 5 inches, 8 inches, and 3 inches. What force must be applied to the belt to raise the weight, if 20% of the total force is lost through friction? Ans. $138\frac{8}{9}$ lb.

(23) The pitch diameter of a gear is 24.16 inches, and the number of teeth is 38; what is the pitch?

Ans. 1.9974 in.

(24) It is required to raise a load of 1,890 pounds by means of a block and tackle which has four fixed and four movable pulleys; what force is required to be applied to the free end of the rope? Ans. $236\frac{1}{4}$ lb.

(25) A piece of lead is $\frac{1}{2}$ inch in diameter and 10 inches long; how much does it weigh? Ans. 12.91 oz.

(26) It is required to raise a weight of 1,500 pounds by means of a lever like that shown in Fig. 596. The length of the lever is 4 feet, and the distance from the fulcrum to the weight is 4 inches; what force will it be necessary to apply? Ans. $136\frac{4}{11}$ lb.

(27) Had the lever in the above example been like that shown in Fig. 597, what force would have been required?

Ans. 125 lb.

(28) What is (*a*) a spur-gear? (*b*) a miter-gear? (*c*) a bevel-gear?

(29) The length of an inclined plane is 400 feet and the height is 45 feet. What force acting parallel to the

plane will be required to pull up the plane a weight of 4,000 pounds? Ans. 450 lb.

(30) The diameters of two pulleys are 14 inches and 18 inches, and the distance between their centers is 14 feet; what must be the length of a belt to drive these pulleys? Ans. 32 ft. 4 in.

(31) What is (*a*) a rack? (*b*) a worm-wheel? (*c*) a worm?

(32) (*a*) What distinguishes epicycloidal teeth from the involute teeth? (*b*) Name two advantages which the latter possess over the former.

(33) An inclined plane has a length of 1,200 feet and a height of 125 feet. It is required to pull a load of 50,000 pounds up this plane. A block and tackle having 6 fixed and 6 movable pulleys is stationed at the top of the plane, and the weight end of the rope is attached to the load. If the rope which connects the block to the load is parallel to the plane, what force will it be necessary to exert on the free end of the rope to pull up the load, no allowance being made for friction? Ans. 434 lb.

(34) What do you understand (*a*) by centrifugal force? (*b*) by centripetal force?

(35) Why is it difficult to jump from a rowboat?

(36) A compound lever, similar to the one shown in Fig. 602, is required to lift a weight of 1,250 pounds. The lengths of the power-arms $P F$ are 30 inches, 20 inches, 10 inches, and 15 inches, respectively, and the lengths of the weight-arms $W F$ are 6 inches, 5 inches, 4 inches, and 7 inches; what force will be required? Ans. $11\frac{2}{3}$ lb.

(37) How is the diameter of a gear measured?

(38) How much work can be done by 20 cubic feet of water falling from a height of 50 feet? Ans. 62,500 ft.-lb.

(39) It is required to raise a weight of 18,000 pounds by means of a screw having 3 threads per inch; if the length of

the handle is 15 inches, and there is a loss of 10,000 pounds, due to friction, etc., what force will it be necessary to apply to the handle? Ans. 99 lb., nearly.

(40) If the distance between the center line of the handle and the axis of the drum shown in Fig. 604 is $14\frac{1}{2}$ inches, and the diameter of the drum is 5 inches, what load will a force of 30 pounds exerted on the handle raise?

Ans. 174 lb.

MECHANICS.

(PART 2.)

EXAMINATION QUESTIONS.

(1) What is meant by the expression, *the resultant of several forces?*

(2) If in Fig. 631 the tension in the rope is $3\frac{3}{4}$ tons, and the angle at *d* between the directions of the two parts of the rope is 30°, what is the total load on the shaft of the head-wheel?

(3) What do you understand (*a*) by tensile strength of a material? (*b*) by working stress?

(4) A close-link wrought-iron chain is made from $\frac{5}{8}$-inch iron; what is the greatest safe load that it will carry?

Ans. 1,687.5 lb.

(5) What is the allowable working load for a steel-wire rope $5\frac{1}{4}$ inches in circumference? Ans. 27,562.5 lb.

(6) If a line 5 inches long represents a force of 20 pounds, (*a*) how long must the line be to represent a force of 1 pound? (*b*) of $6\frac{1}{4}$ lb?

(7) What is (*a*) cold-rolled shafting? (*b*) bright shafting? (*c*) black shafting?

(8) Find the resultant of the forces acting in Fig. I—all acting towards the same point?

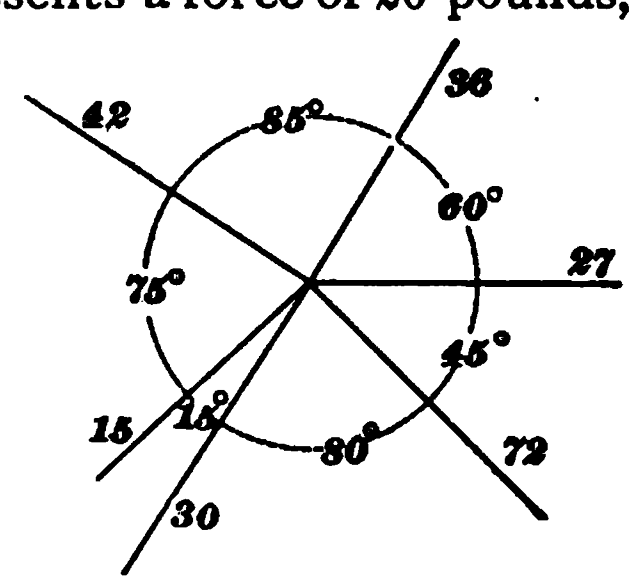


FIG. I.

(9) What load can be safely sustained by a round wooden pillar, 8 inches in diameter and 10 feet long, having both ends flat? Ans. $13\frac{1}{8}$ tons.

(10) What are the components of a force?

(11) What should be the least diameter of a wrought-iron bolt that is to resist a sudden pull of 12,000 pounds? Ans. 1.74+ in.

(12) A white-pine beam supported at both ends has a rectangular cross-section 8 inches wide by 10 inches deep; if the beam is 28 feet long, what total uniform load will it support in safety? Ans. $6{,}857\frac{1}{7}$ lb.

(13) What horsepower can a 10-inch wrought-iron crank-shaft transmit when running at 200 revolutions per minute? Ans. $2{,}857\frac{1}{7}$ H. P.

(14) A force of 87 pounds acts at an angle of 23° to the horizontal; what are its horizontal and vertical components? Find, first, graphically, by the method of triangle of forces, and, second, by trigonometry. Ans. $\begin{cases} 33.994 \text{ lb.} \\ 80.084 \text{ lb.} \end{cases}$

(15) What is the greatest safe load that may be applied to a stud-link wrought-iron chain, if the diameter of the iron from which the link is made is $\frac{1}{2}$ inch? Ans. 4,500 lb.

(16) It is desired to haul loads up to 14,000 pounds by means of an iron-wire rope; what should be its circumference? Ans. 4.83 in., nearly.

(17) Two forces act upon a body at a common point—one with a force of 75 pounds, and the other with a force of 40 pounds; if the angle between them is 60°, and both forces act towards the body, what is the value of the resultant? Solve by the method of triangle of forces and parallelogram of forces, and mark the direction of the resultant. Ans. 101+ lb.

(18) In the last example, if one force (the one of 75 pounds) acts away from the body, and the other towards it, what is the resultant? Solve by the method of triangle of forces

and parallelogram of forces, and mark the direction of the resultant. Ans. 65 lb.

(19) If two forces, of 27 pounds and 46 pounds, respectively, act in exactly opposite directions upon a body, what is the resultant?

(20) A bar of steel having a cross-section of $1\frac{3}{4}$ inches by 3 inches is subjected to a tensile stress; if the stress is suddenly applied, what is the greatest load that it will safely carry? Ans. 31,500 lb.

(21) In laying out an engine-plane, it was found necessary to lead the rope around two guiding-sheaves, as shown

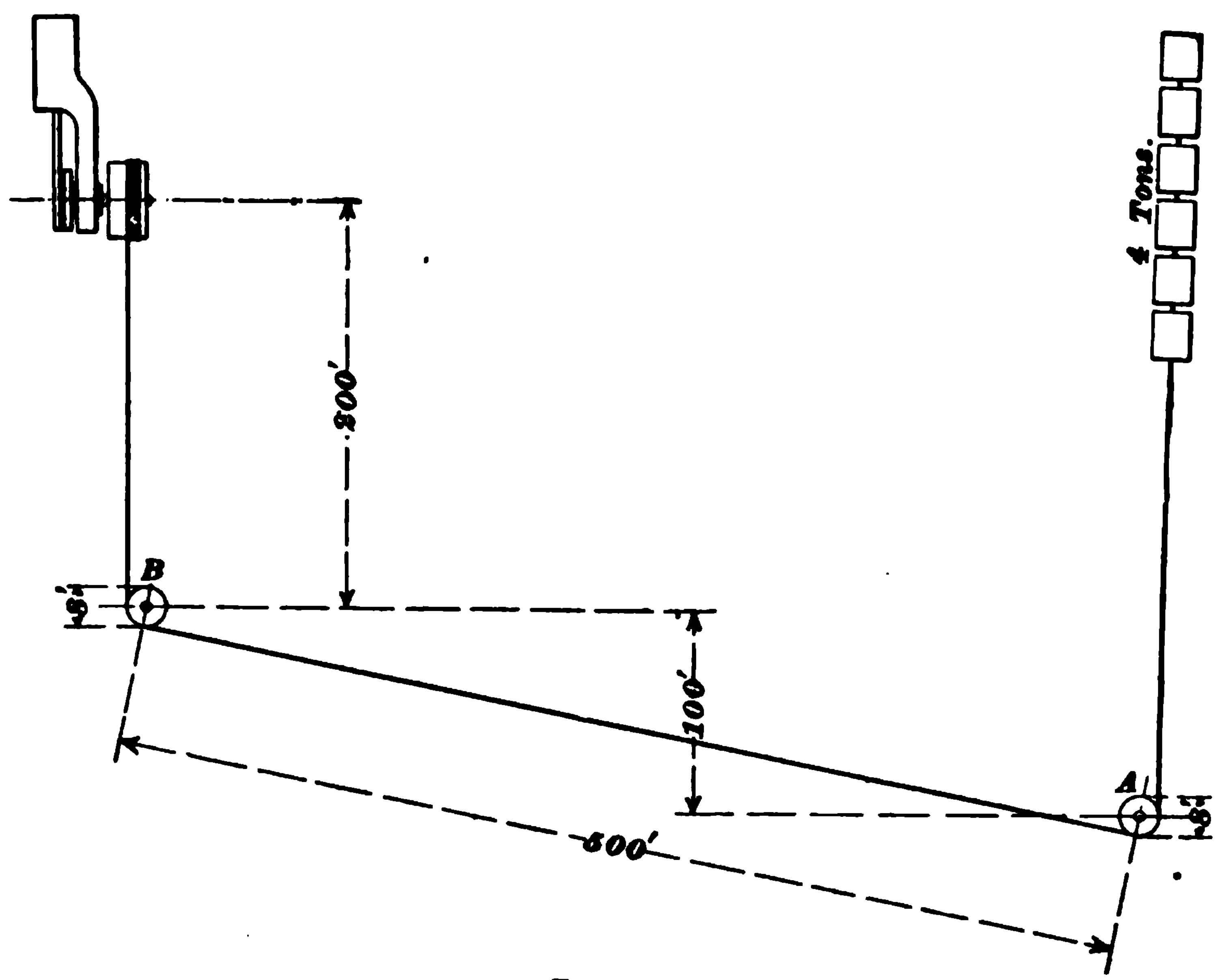


FIG. II.

in Fig. II. The portion of the rope between the car and the sheave A is parallel to the portion of the rope led from the engine to the sheave B. The locations of the sheaves are found from the dimensions given. The resistance due to the cars and coal—that is, the tension in the rope—is

4 tons. What is the greatest pressure on the shaft of each sheave? Solve graphically by means of the parallelogram of forces.

Ans. { Pressure on sheave A, 12,400 lb., nearly.
Pressure on sheave B, 10,125 lb., nearly.

(22) What is (*a*) a stress? (*b*) a strain? (*c*) a unit stress?

(23) A steel-wire rope is used to haul cars up an inclined plane; the greatest stress in the rope is 8,000 pounds; what should its circumference be? Ans. 2.83 in.

(24) What uniform load can be safely sustained by a steel beam 20 feet long, 2 inches wide, and 6 inches deep?

Ans. 4,608 lb.

(25) What is (*a*) elasticity? (*b*) elastic limit? (*c*) What is meant by set?

(26) What is the allowable working load for an iron-wire rope 6 inches in circumference? Ans. 21,600 lb.

(27) What force is required to shear a wrought-iron strip 4 feet long and $\frac{1}{2}$ inch thick? Ans. 960,000 lb.

(28) A 7-inch wrought-iron crank-shaft is to transmit 200 horsepower; how many revolutions per minute must it make? Ans. 40.8 rev., nearly.

(29) An iron-wire rope 4 inches in circumference is used for hoisting; what is the greatest load that the rope will sustain with safety? Ans. 9,600 lb.

(30) A cast-iron rectangular cantilever beam, having a cross-section of $1\frac{1}{2}$ inches wide by $2\frac{1}{2}$ inches deep, is 4 feet 8 inches long; how great a weight will the beam sustain at its end? Ans. 201 lb., nearly.

(31) What horsepower will a $2\frac{7}{16}$-inch steel shaft transmit when running at 120 revolutions per minute, there being pulleys between bearings? Ans. 20,445 H. P.

(32) What safe steady load can be sustained by a $1\frac{1}{8}$-inch round wrought-iron bar, the load producing a tensile stress?

Ans. 21,205.2 lb.

(33) What load will a hollow cast-iron pillar support with safety, if the pillar is 20 feet long, outside diameter 14 inches, inside diameter $11\frac{1}{2}$ inches, and both ends are fixed?

Ans. 219.24 tons.

(34) What force is required to punch a hole $1\frac{1}{2}$ inches in diameter through a $\frac{3}{4}$-inch steel plate? Ans. 212,058 lb.

(35) A weight of 325 pounds rests upon a smooth inclined plane, as shown in Fig. 636. If the angle of the plane is 15°, (*a*) what is the perpendicular pressure against it? (*b*) What force would it be necessary to exert parallel to the plane to keep it from sliding downwards, there being no friction? Solve by trigonometry, and also graphically by the method of the triangle of forces.

Ans. $\begin{cases} (a) \ 313.93 \text{ lb.} \\ (b) \ 84.12 \text{ lb.} \end{cases}$

HYDRAULICS AND HYDRAULIC MACHINERY

(PART 1)

EXAMINATION QUESTIONS

(1) If the area of the piston *e* in Fig. 2 is 8.25 square inches and the force of 150 pounds is applied to it, what forces must be applied to the other pistons to keep them in equilibrium, assuming *a* to be 20; *b*, 7; *c*, 1; *d*, 6; and *f*, 4 square inches area, respectively ?

(2) A well 4 feet in diameter and 20 feet deep is filled with water. (*a*) What is the pressure on a strip of the wall 2 inches wide, the center of which is 1.5 feet from the bottom ? (*b*) What is the upward pressure 1.5 feet from the bottom ?

Ans. { (*a*) 2,409 lb. per sq. in.
(*b*) 7.99 lb. per sq. in.

(3) Define (*a*) hydraulics; (*b*) hydrostatics.

(4) What is the rule for the velocity of efflux ?

(5) What is the diameter of a contracted vein at its smallest section ?

(6) What effect has the hydraulic grade line upon the head and consequently the flow of water in pipes ?

(7) The water level in a city reservoir is 250 feet above a certain street hydrant; what pressure per square inch should there be at this hydrant ? Ans. 108 lb. per sq. in.

(8) In example 7, will the water from the hydrant spurt 250 feet high, and if not, what causes prevent ?

(9) A stream of water issuing from a hose nozzle has a velocity of 175 feet per second; what must be the head to give it this velocity? Ans. 476 ft.

(10) There is an actual discharge from a vessel of .1435 cubic foot per second; what should be the theoretical discharge? Ans. 14 cu. ft. per min.

(11) How were coefficients for smooth and rough pipes determined?

(12) What is water hammering and how may it be prevented?

(13) What advantages have flumes over ditches in leading water to the place it is to be utilized?

(14) When water flows from a pressure box into a pipe what occurs, and how is collapse prevented?

HYDRAULICS AND HYDRAULIC MACHINERY

(PART 2)

EXAMINATION QUESTIONS

(1) How will the hydraulic engineer proceed to ascertain whether he can obtain a sufficient supply of water for a mill?

(2) Name the resistances to the flow of water through conduits.

(3) What considerations limit the choice of the form of cross-section for a ditch?

(4) What limits the velocity in a canal, as far as depth enters into the subject?

(5) What objections are raised to the use of earthen canals or ditches?

(6) From what sources may the engineer locating a mill in the mountains expect to draw water?

(7) What steps are to be taken before locating a dam?

(8) Why are center cores desirable in all dam foundations not constructed on solid bed rock?

(9) What rate of slope should be given earthen dams?

(10) Describe a spillway, its object and proportions.

(11) Why are wing dams constructed?

(12) A trapezoidal wall 9 feet at the base, 4 feet at the top, and 18 feet high has a density of 130 pounds. (*a*) Determine its moment of resistance. (*b*) What is its factor of safety ?

(13) Given the height 40 feet, density 140 pounds, top of the wall 8 feet, factor of safety 2.5; what must be the width of the base of a wall to resist overturning ?

(14) Define a miner's inch.

(15) Why is a greater width of base required to guard against sliding rather than against the overturning of a masonry dam ?

(16) Name the different methods of stream measurement.

HYDRAULICS AND HYDRAULIC MACHINERY

(PART 3)

EXAMINATION QUESTIONS

(1) What is the theoretical work a weight of water can perform ?

(2) How may the energy of a given weight of water be expressed ?

(3) When a jet strikes a surface at right angles, will it produce work ?

(4) When is the pressure of a jet of water greatest ?

(5) What is the theoretical energy in a jet of water whose area is 3.1416 square inches and whose head on the orifice is 75 feet with the coefficient of velocity .98 ?

(6) In what three ways may the energy of falling water be made to perform work ?

(7) What is the efficiency of a waterwheel that delivers 40 horsepower when using 1,600 pounds of water per second with a head of 30 feet ?

(8) How many horsepower will be furnished by a turbine using 1,600 pounds of water per second with a head of 30 feet, when the efficiency is 60 per cent. ?

(9) In designing an overshot wheel, what are the chief factors to be considered in order to obtain the highest efficiency ?

(10) Compute the principal dimensions of an overshot waterwheel to utilize 15 cubic feet of water per second with a total head of 15 feet.

(11) What good points have breast wheels?

(12) Compute the principal dimensions of a breast wheel to utilize 15 cubic feet of water per second with a total head of $7\frac{1}{2}$ feet.

(13) When is the best efficiency of an impulse wheel to be obtained?

(14) What should be the diameter of an impulse wheel that is directly connected to the pulley wheel on a stamp-mill shaft making 90 revolutions per minute, if the pressure head is 100 feet and the velocity of the jet is .98?

(15) What head tends to produce flow when draft tubes are used?

(16) How does it make a difference whether the turbine is above or below the level of discharge in a tailrace?

HYDRAULICS AND HYDRAULIC MACHINERY

(PART 4)

EXAMINATION QUESTIONS

(1) How is the water which flows through a penstock regulated so as to keep the power as steady as possible?

(2) What two classes of hydraulic machinery are there in general use?

(3) What three kinds of pumps may be used about a mill?

(4) (*a*) How does a lift pump differ from a force pump? (*b*) To what height will each raise water in the suction pipe?

(5) (*a*) What is a double-acting pump? (*b*) What is a duplex pump?

(6) What is the action of the air in the air chambers that are placed on some pumping engines?

(7) How do power pumps differ in principle from direct-acting pumps?

(8) What is a double-acting duplex power pump?

(9) How are the steam valves thrown so as to reverse the piston in the Knowles and Cameron pumps?

(10) How are the steam valves thrown on direct-acting duplex steam pumps?

(11) Describe a compound direct-acting steam pump.

(12) What are the conditions a good pump valve must fulfil ?

(13) What kind of valves are to be used when pumping cold, hot, and acid solutions ?

(14) State the difference between the actions of rotary and centrifugal pumps.

(15) Can the centrifugal pump be used to advantage in handling water containing slimes and sand ?

(16) In what particulars do useful and actual work differ in pumping engines ?

(17) Why does the theoretical and actual discharge from pumping engines differ ?

(18) What pressure in pounds per square inch will be on the water end of a steam pump when the column pipe is 8 inches in diameter and 112 feet high ?

(19) What head of water will exert a pressure of 48.6 pounds per square inch on the plunger in the water end of a pump ?

(20) To what height will a 60-horsepower pump force 500 gallons of water per minute ?

A KEY

TO ALL THE

QUESTIONS AND EXAMPLES

CONTAINED IN THE

EXAMINATION QUESTIONS

INCLUDED IN THIS VOLUME.

The Keys that follow have been divided into sections corresponding to the Examination Questions to which they refer, and have been given corresponding section numbers. The answers and solutions have been numbered to correspond with the questions. When the answer to a question involves a repetition of statements given in the Instruction Paper, the reader has been referred to a numbered article, the reading of which will enable him to answer the question himself.

To be of the greatest benefit, the Keys should be used sparingly. They should be used much in the same manner as a pupil would go to a teacher for instruction with regard to answering some example he was unable to solve. If used in this manner, the Keys will be of great help and assistance to the student, and will be a source of encouragement to him in studying the various papers composing the Course.

ARITHMETIC.

(PART 1.)

(1) See Art. **3.**

(2) See Arts. **5** and **6.**

(3) (*a*) 980 = Nine hundred eighty.

(*b*) 605 = Six hundred five.

(*c*) 28,284 = Twenty-eight thousand two hundred eighty-four.

(*d*) 9,006,042 = Nine million six thousand and forty-two.

(*e*) 850,317,002 = Eight hundred fifty million three hundred seventeen thousand and two.

(*f*) 700,004 = Seven hundred thousand and four.

(4) (*a*) Seven thousand six hundred = 7,600.

(*b*) Eighty-one thousand four hundred two = 81,402.

(*c*) Five million four thousand and seven = 5,004,007.

(*d*) One hundred and eight million ten thousand and one = 108,010,001.

(*e*) Eighteen million and six = 18,000,006.

(*f*) Thirty thousand and ten = 30,010.

(5)

$$\begin{array}{r} 709 \\ 8304725 \\ 391 \\ 100302 \\ 300 \\ 909 \\ \hline 8407336 \end{array} \text{ Ans.}$$

(6) (*a*) In subtracting whole numbers, place the subtrahend or smaller number under the minuend or larger number, so that the right-hand figures stand directly under each other. Begin at the right to subtract. We cannot subtract 8 units from 2 units, so we take 1 ten from the 6 tens and add it to the 2 units. As 1 *ten* = 10 *units*, we have 10 units + 2 units = 12 units. Then, 8 units from 12 units leaves 4 units. We took 1 ten from 6 tens, so only 5 tens remain. 3 tens from 5 tens leaves 2 tens. In the hundreds column we have 3 hundreds from 9 hundreds leaves 6 hundreds. We cannot subtract 3 thousands from 0 thousands, so we take 1 ten-thousand from 5 ten-thousands and add it to the 0 thousands. 1 *ten-thousand* = 10 *thousands*, and ten thousands + 0 thousands = 10 thousands. Subtracting, we have 3 thousands from 10 thousands leaves 7 thousands. We took 1 ten-thousand from 5 ten-thousands and have 4 ten-thousands remaining. Since there are no ten-thousands in the subtrahend, the 4 in the ten-thousands column in the minuend is brought down into the same column in the remainder, because 0 from 4 leaves 4.

$$\begin{array}{r} 50962 \\ 3338 \\ \hline 47624 \end{array} \text{ Ans.}$$

(*b*)
$$\begin{array}{r} 15339 \\ 10001 \\ \hline 5338 \end{array} \text{ Ans.}$$

(7) (*a*)
$$\begin{array}{r} 70968 \\ 32975 \\ \hline 37993 \end{array} \text{ Ans.}$$

(*b*)
$$\begin{array}{r} 100000 \\ 98735 \\ \hline 1265 \end{array} \text{ Ans.}$$

(8)
$$\begin{array}{rl} \$44675 = & \text{amount paid for first shipment.} \\ 26380 = & \text{amount paid for second shipment.} \\ \hline \$71055 = & \text{amount paid for both shipments.} \\ \$125000 = & \text{amount paid for three shipments.} \\ 71055 = & \text{amount paid for two shipments.} \\ \hline \$53945 = & \text{amount paid for third shipment. Ans.} \end{array}$$

(9) We have given the minuend or greater number (1,004) and the difference or remainder (49). Placing these

in the usual form of subtraction, we have $\begin{array}{r}1004\\ \underline{—}\\ 49\end{array}$ in which the dash (——) represents the number sought. This number is evidently *less* than 1,004 by the difference 49; hence, 1,004 − 49 = 955, the smaller number. For the sum of the two numbers we then have

$$\begin{array}{rl} 1004 & \textit{larger} \\ 955 & \textit{smaller} \\ \hline 1959 & \textit{sum.} \quad \text{Ans.} \end{array}$$

Or, this problem may be solved as follows: If the greater of two numbers is 1,004, and the difference between them is 49, then it is evident that the smaller number must be equal to the difference between the greater number (1,004) and the difference (49); or 1,004 − 49 = 955, the smaller number. Since the greater number equals 1,004 and the smaller number equals 955, their sum equals 1,004 + 955 = 1,959 sum. Ans.

(10) The numbers connected by the plus (+) sign must first be added. Performing these operations, we have

$$\begin{array}{rl} 5962 & \\ 8471 & \\ 9023 & \\ \hline 23456 & \textit{sum.} \end{array} \qquad \begin{array}{rl} 3874 & \\ 2039 & \\ \hline 5913 & \textit{sum.} \end{array}$$

Subtracting the smaller number (5,913) from the greater (23,456) we have

$$\begin{array}{rl} 23456 & \\ 5913 & \\ \hline 17543 & \textit{difference.} \quad \text{Ans.} \end{array}$$

(11) (*a*) In the multiplication of whole numbers, place the multiplier under the multiplicand, and multiply each term of the multiplicand by each term of the multiplier, writing the right-hand figure of each product obtained under the term of the multiplier which produces it.

7 × 7 units = 49 units, or 4 tens and 9 units. We write the 9 units and reserve the 4 tens. 7 times 8 tens = 56 tens;

56 tens + 4 tens reserved = 60 tens, or 6 hundreds and 0 tens. Write the 0 tens and reserve the 6 hundreds. 7 × 3 hundreds = 21 hundreds; 21 + 6 hundreds reserved = 27 hundreds, or 2 thousands and 7 hundreds. Write the 7 hundreds and reserve the 2 thousands. 7 × 6 thousands = 42 thousands; 42 + 2 thousands reserved = 44 thousands or 4 ten-thousands and 4 thousands. Write the 4 thousands and reserve the 4 ten-thousands. 7 × 2 ten-thousands = 14 ten-thousands; 14 + 4 ten-thousands reserved = 18 ten-thousands, or 1 hundred-thousand and 8 ten-thousands. Write the 8 ten-thousands and reserve the 1 hundred-thousand. 7 × 5 hundred-thousands = 35 hundred-thousands; 35 + 1 hundred-thousand reserved = 36 hundred-thousands. Since there are no more figures in the multiplicand to be multiplied, we write the 36 hundred-thousands in the product. This completes the multiplication.

$$\begin{array}{r} 526387 \\ 7 \\ \hline 3684709 \end{array} \text{ Ans.}$$

A simpler explanation of the same problem is the following:

7 times 7 = 49; write the 9 and reserve the 4. 7 times 8 = 56; 56 + 4 reserved = 60; write the 0 and reserve the 6. 7 times 3 = 21; 21 + 6 reserved = 27; write the 7 and reserve the 2. 7 × 6 = 42; 42 + 2 reserved = 44; write the 4 and reserve 4. 7 × 2 = 14; 14 + 4 reserved = 18; write the 8 and reserve the 1. 7 × 5 = 35; 35 + 1 reserved = 36; write the 36.

(*b*) In this case the multiplier is 17 *units*, or 1 *ten* and 7 *units*, so that the product is obtained by adding two partial products, namely, 7 × 700,298 and 10 × 700,298. The actual operation is performed as follows:

$$(b)\quad \begin{array}{r} 700298 \\ 17 \\ \hline 4902086 \\ 700298 \\ \hline 11905066 \end{array} \text{ Ans.}$$

7 times 8 = 56; write the 6 and reserve the 5. 7 times 9 = 63; 63 + 5 reserved = 68; write the 8 and reserve the 6. 7 times 2 = 14; 14 + 6 reserved = 20; write the 0 and reserve the 2. 7 times 0 = 0; 0 + 2 reserved = 2; write

the 2. 7 times 0 = 0; 0 + 0 reserved = 0; write the 0. 7 times 7 = 49; 49 + 0 reserved = 49; write the 49.

To multiply by the 1 ten we say 1 times 700298 = 700298, and write 700298 under the first partial product, as shown, with the right-hand figure 8 under the multiplier 1. Add the two partial products; their sum equals the entire product.

(*c*) Multiply any two of the numbers together and multiply their product by the third number.

$$\begin{array}{r} 217 \\ 103 \\ \hline 651 \\ 2170 \\ \hline 22351 \\ 67 \\ \hline 156457 \\ 134106 \\ \hline 1497517 \end{array} \text{ Ans.}$$

(**12**) If an engine and a boiler are worth \$3,246, and the building is worth three times as much plus \$1,200, then the building is worth

$$\begin{array}{rr} & \$3246 \\ & 3 \\ \hline & 9738 \\ \text{plus} & 1200 \\ \hline & \$10938 \end{array} = \text{value of building.}$$

If the tools are worth twice as much as the building plus \$1,875, then the tools are worth

$$\begin{array}{rr} & \$10938 \\ & 2 \\ \hline & 21876 \\ \text{plus} & 1875 \\ \hline & \$23751 \end{array} = \text{value of tools.}$$

$$\begin{array}{lr} \text{Value of building} = & \$10938 \\ \text{Value of tools} \quad\;\; = & 23751 \\ \hline & \$34689 \end{array} = \text{value of the building and tools. } (a) \text{ Ans.}$$

Value of engine and boiler = \$ 3246

Value of building and tools = 34689

\$37935 = value of the whole plant. (*b*) Ans.

(13) (*a*) $(72 \times 48 \times 28 \times 5) \div (96 \times 15 \times 7 \times 6)$.

Placing the numerator over the denominator, the problem becomes

$$\frac{72 \times 48 \times 28 \times 5}{96 \times 15 \times 7 \times 6} = ?$$

The 5 in the *dividend* and 15 in the *divisor* are both divisible by 5, since 5 divided by 5 equals 1, and 15 divided by 5 equals 3. Cross off the 5 and write the 1 over it; also, cross off the 15 and write the three under it. Thus,

$$\frac{72 \times 48 \times 28 \times \overset{1}{\cancel{5}}}{96 \times \underset{3}{\cancel{15}} \times 7 \times 6} =$$

The 5 and 15 are not to be considered any longer, and, in fact, may be erased entirely and the 1 and 3 placed in their stead, and treated as if the 5 and 15 never existed. Thus,

$$\frac{72 \times 48 \times 28 \times 1}{96 \times 3 \times 7 \times 6} =$$

72 in the dividend and 96 in the divisor are divisible by 12, since 72 divided by 12 equals 6, and 96 divided by 12 equals 8. Cross off the 72 and write the 6 over it; also, cross off the 96 and write the 8 under it. Thus,

$$\frac{\overset{6}{\cancel{72}} \times 48 \times 28 \times 1}{\underset{8}{\cancel{96}} \times 3 \times 7 \times 6} =$$

The 72 and 96 are not to be considered any longer, and, in fact, may be erased entirely and the 6 and 8 placed in

their stead, and treated as if the 72 and 96 never existed. Thus,

$$\frac{6 \times 48 \times 28 \times 1}{8 \times 3 \times 7 \times 6} =$$

Again, 28 in the dividend and 7 in the divisor are divisible by 7, since 28 divided by 7 equals 4, and 7 divided by 7 equals 1. Cross off the 28 and write the 4 over it; also, cross off the 7 and write the 1 under it. Thus,

$$\frac{6 \times 48 \times \overset{4}{\cancel{28}} \times 1}{8 \times 3 \times \underset{1}{\cancel{7}} \times 6} =$$

The 28 and 7 are not to be considered any longer, and, in fact, may be erased entirely and the 4 and 1 placed in their stead, and treated as if the 28 and 7 never existed. Thus,

$$\frac{6 \times 48 \times 4 \times 1}{8 \times 3 \times 1 \times 6} =$$

Again, 48 in the dividend and 6 in the divisor are divisible by 6, since 48 divided by 6 equals 8, and 6 divided by 6 equals 1. Cross off the 48 and write the 8 over it; also, cross off the 6 and write the 1 under it. Thus,

$$\frac{6 \times \overset{8}{\cancel{48}} \times 4 \times 1}{8 \times 3 \times 1 \times \underset{1}{\cancel{6}}} =$$

The 48 and 6 are not to be considered any longer, and, in fact, may be erased entirely and the 8 and 1 placed in their stead, and treated as if the 48 and 6 never existed. Thus,

$$\frac{6 \times 8 \times 4 \times 1}{8 \times 3 \times 1 \times 1} =$$

Again, 6 in the dividend and 3 in the divisor are divisible by 3, since 6 divided by 3 equals 2, and 3 divided by 3

equals 1. Cross off the 6 and write the 2 over it; also, cross off the 3 and write the 1 under it. Thus,

$$\frac{\overset{2}{\cancel{6}} \times 8 \times 4 \times 1}{8 \times \underset{1}{\cancel{3}} \times 1 \times 1} =$$

The 6 and 3 are not to be considered any longer, and, in fact, may be erased entirely and the 2 and 1 placed in their stead, and treated as if the 6 and 3 never existed. Thus,

$$\frac{2 \times 8 \times 4 \times 1}{8 \times 1 \times 1 \times 1} =$$

Canceling the 8 in the dividend and the 8 in the divisor, the result is

$$\frac{2 \times \overset{1}{\cancel{8}} \times 4 \times 1}{\underset{1}{\cancel{8}} \times 1 \times 1 \times 1} = \frac{2 \times 1 \times 4 \times 1}{1 \times 1 \times 1 \times 1}.$$

Since there are no two remaining numbers (one in the dividend and one in the divisor) divisible by any number except 1, without a remainder, it is impossible to cancel further.

Multiply all the uncanceled numbers in the dividend together and divide their product by the product of all the uncanceled numbers in the divisor. The result will be the quotient. The product of all the uncanceled numbers in the dividend equals $2 \times 1 \times 4 \times 1 = 8$; the product of all the uncanceled numbers in the divisor equals $1 \times 1 \times 1 \times 1 = 1$.

Hence, $$\frac{2 \times 1 \times 4 \times 1}{1 \times 1 \times 1 \times 1} = \frac{8}{1} = 8. \quad \text{Ans.}$$

Or, $$\frac{\overset{\overset{2}{\cancel{6}}}{\cancel{72}} \times \overset{\cancel{8}}{\cancel{48}} \times \overset{4}{\cancel{28}} \times \overset{1}{\cancel{5}}}{\underset{\underset{1}{\cancel{8}}}{\cancel{96}} \times \underset{\underset{1}{\cancel{3}}}{\cancel{15}} \times \cancel{7} \times \underset{1}{\cancel{6}}} = \frac{8}{1} = 8. \quad \text{Ans.}$$

(*b*) $(80 \times 60 \times 50 \times 16 \times 14) \div (70 \times 50 \times 24 \times 20)$.

Placing the numerator over the denominator, the problem becomes

$$\frac{80 \times 60 \times 50 \times 16 \times 14}{70 \times 50 \times 24 \times 20} = ?$$

The remainder of the work will be readily apparent.

$$\text{Hence, } \frac{\overset{4}{\cancel{80}} \times \overset{\overset{2}{\cancel{6}}}{\cancel{60}} \times \overset{\overset{1}{\cancel{5}}}{\cancel{50}} \times \overset{2}{\cancel{16}} \times \overset{2}{\cancel{14}}}{\underset{\underset{1}{\cancel{7}}}{\cancel{70}} \times \underset{\underset{1}{\cancel{5}}}{\cancel{50}} \times \underset{\underset{1}{\cancel{3}}}{\cancel{24}} \times \underset{1}{\cancel{20}}} = \frac{4 \times 2 \times 1 \times 2 \times 2}{1 \times 1 \times 1 \times 1} = \frac{32}{1} = 32. \text{ Ans.}$$

(**14**) If a millman earns \$1,500 a year and his expenses are \$968 per year, then he would save \$1500 − \$968, or \$532 per year.

$$\begin{array}{r} \$1500 \\ 968 \\ \hline \$532 \end{array}$$

If he saves \$532 in 1 year, to save \$3,724 it would take as many years as \$532 is contained times in \$3,724, or 7 years.

$$\begin{array}{l} 532\,)\,3724\,(\,7 \text{ years. Ans.} \\ 3724 \\ \hline \end{array}$$

(**15**) If the jaw crusher worked up 365 tons in one week and three times as many lacking 246 tons the next week, then it crushed $(3 \times 365 \text{ tons}) - 246$ tons, or 849 tons the second week. Thus,

$$\begin{array}{rr} & 365 \\ & 3 \\ \hline & 1095 \\ & 246 \\ \hline \textit{difference} & 849 \text{ tons. Ans.} \end{array}$$

(**16**) (*a*)

$$\begin{array}{l} 576\,)\,589824\,(\,1024 \text{ Ans.} \\ 576 \\ \overline{1382} \\ 1152 \\ \overline{2304} \\ 2304 \end{array}$$

(*b*)

```
43911)369730620(8420  Ans.
      351288
      ------
       184426
       175644
       ------
         87822
         87822
         -----
             0
```

(*c*)

```
505)2527525(5005  Ans.
    2525
    ----
       2525
       2525
       ----
```

(*d*)

```
1234)4961794302(4020903  Ans.
     4936
     ----
       2579
       2468
       ----
        11143
        11106
        -----
            3702
            3702
            ----
```

(**17**) The harness evidently cost the difference between $444 and the amount that he paid for the horse and wagon.

Since $264 + $153 = $417, the amount paid for the horse and wagon, $444 − $417 = $27, the cost of the harness.

```
$264              $444
  153                417
 ----               ----
$417                $27  Ans.
```

(**18**) (*a*)

```
   1024
    576
   ----
   6144
  7168
 5120
 ------
 589824  Ans.
```

(*b*)

$$\begin{array}{r@{\quad}l} 5005 & \\ 505 & \\ \hline 25025 & \\ 250250 & \\ \hline 2527525 & \text{Ans.} \end{array}$$

(*c*)

$$\begin{array}{r@{\quad}l} 43911 & \\ 8420 & \\ \hline 878220 & \\ 175644 & \\ 351288 & \\ \hline 369730620 & \text{Ans.} \end{array}$$

(19) Since there are 12 months in a year, the number of days the man works is $25 \times 12 = 300$ days. As he works 10 hours each day, the number of hours that he works in one year is $300 \times 10 = 3{,}000$ hours. Hence, he receives for his work $3{,}000 \times 30 = 90{,}000$ cents, or $90{,}000 \div 100 = \$900$. Ans.

ARITHMETIC.

(PART 2.)

(1) See Art. **71.**

(2) See Art. **73.**

(3) See Art. **73.**

(4) $\frac{13}{8}$ is an improper fraction, since its numerator 13 is greater than its denominator 8.

(5) $4\frac{1}{2}$; $14\frac{3}{10}$; $85\frac{4}{19}$.

(6) To reduce a fraction to its lowest terms means to change its form without changing its value. In order to do this, we must divide both numerator and denominator by the same number, until we can no longer find any number (except 1) which will divide both of these terms without a remainder.

To reduce the fraction $\frac{4}{8}$ to its lowest terms, we divide both numerator and denominator by 4, and obtain as a result the fraction $\frac{1}{2}$. Thus, $\frac{4 \div 4}{8 \div 4} = \frac{1}{2}$; similarly, $\frac{4 \div 4}{16 \div 4} = \frac{1}{4}$; $\frac{8 \div 4}{32 \div 4} = \frac{2 \div 2}{8 \div 2} = \frac{1}{4}$; $\frac{32 \div 8}{64 \div 8} = \frac{4 \div 4}{8 \div 4} = \frac{1}{2}$. Ans.

(7) When the denominator of any number is not expressed, it is understood to be 1; so that $\frac{6}{1}$ is the same as $6 \div 1$, or 6. To reduce $\frac{6}{1}$ to an improper fraction whose

denominator is 4, we must multiply both numerator and denominator by some number which will make the denominator of 6 equal to 4. Since this denominator is 1, by multiplying both terms of $\frac{6}{1}$ by 4 we shall have $\frac{6 \times 4}{1 \times 4} = \frac{24}{4}$, which has the *same value* as 6, but has a *different form.* Ans.

(**8**) In order to reduce a mixed number to an improper fraction, we must multiply the whole number by the denominator of the fraction and add the numerator of the fraction to that product. This result is the numerator of the improper fraction, of which the denominator is the denominator of the fractional part of the mixed number.

$7\frac{7}{8}$ means the same as $7 + \frac{7}{8}$. In 1 there are $\frac{8}{8}$, hence in 7 there are $7 \times \frac{8}{8} = \frac{56}{8}$; $\frac{56}{8}$ plus the $\frac{7}{8}$ of the mixed number $= \frac{56}{8} + \frac{7}{8} = \frac{63}{8}$, which is the required improper fraction.

$$13\frac{5}{16} = \frac{(13 \times 16) + 5}{16} = \frac{213}{16};\ 10\frac{3}{4} = \frac{(10 \times 4) + 3}{4} = \frac{43}{4}.$$

(**9**) In division of fractions, invert the divisor (or, in other words, turn it upside down) and proceed as in multiplication.

(*a*) $35 \div \frac{5}{16} = \frac{35}{1} \times \frac{16}{5} = \frac{35 \times 16}{1 \times 5} = \frac{560}{5} = 112.$ Ans.

(*b*) $\frac{9}{16} \div 3 = \frac{9}{16} \div \frac{3}{1} = \frac{9}{16} \times \frac{1}{3} = \frac{9 \times 1}{16 \times 3} = \frac{9}{48} = \frac{3}{16}.$ Ans.

(*c*) $\frac{17}{2} \div 9 = \frac{17}{2} \div \frac{9}{1} = \frac{17}{2} \times \frac{1}{9} = \frac{17 \times 1}{2 \times 9} = \frac{17}{18}.$ Ans.

(*d*) $\frac{113}{64} \div \frac{7}{16} = \frac{113}{64} \times \frac{16}{7} = \frac{113 \times 16}{64 \times 7} = \frac{1,808}{448} = \frac{452}{112}$

$= \frac{113}{28}$) 1 1 3 ($4\frac{1}{28}$. Ans.

1 1 2

1

(*e*) $15\frac{3}{4} \div 4\frac{3}{8} = ?$ Before proceeding with the division, reduce both of the mixed numbers to improper fractions. Thus, $15\frac{3}{4} = \frac{(15 \times 4) + 3}{4} = \frac{60 + 3}{4} = \frac{63}{4}$, and $4\frac{3}{8} = \frac{(4 \times 8) + 3}{8}$ $= \frac{32 + 3}{8} = \frac{35}{8}$. The problem is now $\frac{63}{4} \div \frac{35}{8} = ?$ As before, invert the divisor and multiply; $\frac{63}{4} \div \frac{35}{8} = \frac{63}{4} \times \frac{8}{35} = \frac{63 \times 8}{4 \times 35}$ $= \frac{504}{140} = \frac{252}{70} = \frac{126}{35} = \frac{18}{5}$.

$$\begin{array}{r}
18 \\
\overline{5}\,)\,1\,8\,(\,3\frac{3}{5}. \quad \text{Ans.} \\
1\,5 \\
\hline
3
\end{array}$$

(10) $\frac{1}{8} + \frac{2}{8} + \frac{5}{8} = \frac{1 + 2 + 5}{8} = \frac{8}{8} = 1.$ Ans.

When the denominators of the fractions to be added are alike, we know that the units are divided into the same number of parts (in this case eighths); we therefore add the numerators of the fractions to find the number of parts (eighths) taken or considered, thereby obtaining $\frac{8}{8}$ or 1 as the sum.

(11) When mixed numbers and whole numbers are to be added, add the fractional parts of the mixed numbers separately, and if the resulting fraction is an improper fraction, reduce it to a whole or mixed number. Next, add all the whole numbers, including the one obtained from the addition of the fractional parts, and annex to their sum the fraction of the mixed number obtained from reducing the improper fraction.

$42 + 31\frac{5}{8} + 9\frac{7}{16} = ?$ Reducing $\frac{5}{8}$ to a fraction having a denominator of 16, we have $\frac{5 \times 2}{8 \times 2} = \frac{10}{16}$. Adding the two

fractional parts of the mixed numbers, we have $\frac{10}{16} + \frac{7}{16}$ $= \frac{10+7}{16} = \frac{17}{16} = 1\frac{1}{16}$.

The problem now becomes $42 + 31 + 9 + 1\frac{1}{16} = ?$

$$\begin{array}{r} 42 \\ 31 \\ 9 \\ 1\frac{1}{16} \\ \hline 83\frac{1}{16} \end{array} \text{ Ans.}$$

Adding all the whole numbers and the number obtained from adding the fractional parts of the mixed numbers, we obtain $83\frac{1}{16}$ as their sum.

(12) $29\frac{3}{4} + 50\frac{5}{8} + 41 + 69\frac{3}{16} = ?$ $\frac{3}{4} = \frac{3 \times 4}{4 \times 4} = \frac{12}{16}$.

$\frac{5}{8} = \frac{5 \times 2}{8 \times 2} = \frac{10}{16}$. $\frac{12}{16} + \frac{10}{16} + \frac{3}{16} = \frac{12 + 10 + 3}{16} = \frac{25}{16} = 1\frac{9}{16}$.

The problem now becomes $29 + 50 + 41 + 69 + 1\frac{9}{16} = ?$

$$\begin{array}{rl} 29 & \text{square inches.} \\ 50 & \text{square inches.} \\ 41 & \text{square inches.} \\ 69 & \text{square inches.} \\ 1\frac{9}{16} & \text{square inches.} \\ \hline 190\frac{9}{16} & \text{square inches.} \quad \text{Ans.} \end{array}$$

(13) (*a*) $\dfrac{7}{\frac{3}{16}} = 7 \div \frac{3}{16} = 7 \times \frac{16}{3} = \frac{7 \times 16}{3} = \frac{112}{3} = 37\frac{1}{3}$. Ans.

The heavy line between 7 and $\frac{3}{16}$ means that 7 is to be divided by $\frac{3}{16}$.

(*b*) $\dfrac{\frac{15}{32}}{\frac{5}{8}} = \frac{15}{32} \div \frac{5}{8} = \frac{15}{32} \times \frac{8}{5} = \frac{\overset{3}{\cancel{15}} \times \cancel{8}}{\underset{4}{\cancel{32}} \times \cancel{5}} = \frac{3}{4}$. Ans.

(*c*) $\dfrac{\frac{4+3}{2+6}}{5} = \dfrac{\frac{7}{8}}{5} = \frac{7}{8 \times 5} = \frac{7}{40}$. (See Art. **131.**) Ans.

(14) $\frac{7}{8}$ = value of the fraction, and 28 = the numerator. We find that 4 multiplied by 7 = 28, so multiplying 8, the denominator of the fraction, by 4, we have 32 for the required denominator, and $\frac{28}{32} = \frac{7}{8}$. Hence, 32 is the required denominator. Ans.

(15) (*a*) $\frac{7}{8} - \frac{7}{16} = ?$ When the denominators of fractions are not alike, it is evident that the units are divided into unequal parts; therefore, before subtracting, reduce the fractions to fractions having a common denominator. Then subtract the numerators and place the remainder over the common denominator.

$$\frac{7 \times 2}{8 \times 2} = \frac{14}{16}. \quad \frac{14}{16} - \frac{7}{16} = \frac{14 - 7}{16} = \frac{7}{16}. \quad \text{Ans.}$$

(*b*) $13 - 7\frac{7}{16} = ?$ This problem may be solved in two ways:

First. $13 = 12\frac{16}{16}$, since $\frac{16}{16} = 1$, and $12\frac{16}{16} = 12 + \frac{16}{16} = 12 + 1 = 13$.

We can now subtract the whole numbers separately and the fractions separately, and obtain $12 - 7 = 5$ and $\frac{16}{16} - \frac{7}{16} = \frac{16 - 7}{16} = \frac{9}{16}$. $5 + \frac{9}{16} = 5\frac{9}{16}$. Ans.

$$\begin{array}{r} 12\frac{16}{16} \\ 7\frac{7}{16} \\ \hline 5\frac{9}{16} \end{array}$$

Second. By reducing both numbers to improper fractions having a denominator of 16.

$$13 = \frac{13}{1} = \frac{13 \times 16}{1 \times 16} = \frac{208}{16}. \quad 7\frac{7}{16} = \frac{(7 \times 16) + 7}{16} = \frac{112 + 7}{16} = \frac{119}{16}.$$

Subtracting, we have $\frac{208}{16} - \frac{119}{16} = \frac{208 - 119}{16} = \frac{89}{16}$ and $\frac{89}{16} = 16\,)\,89\,(\,5\frac{9}{16}$ the same result that was obtained by the first method.

$$\begin{array}{r} 16\,)\,89\,(\,5\frac{9}{16} \\ 80 \\ \hline 9 \\ \hline 16 \end{array}$$

(*c*) $312\frac{9}{16} - 229\frac{5}{32} = ?$ We first reduce the fractions of the two mixed numbers

to fractions having a common denominator. Doing this we have $\frac{9}{16} = \frac{9 \times 2}{16 \times 2} = \frac{18}{32}$. We can now subtract the whole numbers and fractions separately, and have $312 - 229 = 83$ and $\frac{18}{32} - \frac{5}{32} = \frac{18-5}{32} = \frac{13}{32}$.

$$\begin{array}{r} 312\frac{18}{32} \\ 229\frac{5}{32} \\ \hline 83\frac{13}{32} \end{array} \quad 83 + \frac{13}{32} = 83\frac{13}{32}. \text{ Ans.}$$

(16) The mill stamps $85\frac{5}{12} + 78\frac{9}{15} + 125\frac{17}{35}$ tons.

Adding the fractions separately in this case,

$$\frac{5}{12} + \frac{9}{15} + \frac{17}{35} = \frac{5}{12} + \frac{3}{5} + \frac{17}{35} = \frac{175 + 252 + 204}{420} = \frac{631}{420} = 1\frac{211}{420}.$$

Adding the whole numbers and the mixed number representing the sum of the fractions, the sum is $289\frac{211}{420}$ tons. Ans.

$$\begin{array}{r} 85 \\ 78 \\ 125 \\ 1\frac{211}{420} \\ \hline 289\frac{211}{420} \end{array}$$

To find the least common denominator, we have

$$\begin{array}{r|l} 5 & 12, \ 5, \ 35 \\ \hline 7 & 12, \ 1, \ 7 \\ \hline & 12, \ 1, \ 1, \end{array} \text{ or } 5 \times 7 \times 12 = 420.$$

(17)
$$\begin{array}{rl} 573\frac{4}{5} & \text{tons.} \\ 216\frac{5}{8} & \text{tons.} \\ \hline \textit{difference } 357\frac{7}{40} & \text{tons. Ans.} \end{array} \qquad \begin{array}{l} \frac{4}{5} = \frac{32}{40} \\ \frac{5}{8} = \frac{25}{40} \\ \hline \frac{7}{40} = \textit{difference.} \end{array}$$

(18) Referring to Arts. **114** and **119**,

$\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{7}{11}$ of $\frac{19}{20}$ of 11 multiplied by $\frac{7}{8}$ of $\frac{5}{6}$ of 45 is equal to

$$\frac{\cancel{2}\times\cancel{3}\times 7\times 19\times\cancel{11}\times 7\times 5\times\overset{\overset{3}{\cancel{15}}}{\cancel{45}}}{\cancel{3}\times 4\times\cancel{11}\times\underset{4}{\cancel{20}}\times 1\times 8\times\underset{\cancel{3}}{\cancel{6}}\times 1}=\frac{7\times 19\times 7\times 5\times 3}{4\times 4\times 8}=\frac{13{,}965}{128}$$

$= 109\frac{13}{128}$. Ans.

(19) $\frac{3}{4}$ of $16=\frac{3}{\cancel{4}}\times\frac{\overset{4}{\cancel{16}}}{1}=12.$ $12\div\frac{2}{3}=\frac{\overset{6}{\cancel{12}}}{1}\times\frac{3}{\cancel{2}}=18.$ Ans.

(20) $211\frac{1}{4}\times 1\frac{7}{8}=\frac{845}{4}\times\frac{15}{8}$, reducing the mixed numbers to improper fractions. $\frac{845}{4}\times\frac{15}{8}=\frac{12{,}675}{32}$ cents = amount paid for the lead. The number of pounds sold is evidently $\frac{12{,}675}{32}\div 2\frac{1}{2}=\frac{\overset{2{,}535}{\cancel{12{,}675}}}{\underset{16}{\cancel{32}}}\times\frac{\cancel{2}}{\cancel{5}}=\frac{2{,}535}{16}=158\frac{7}{16}$ pounds. The amount remaining is $211\frac{1}{4}-158\frac{7}{16}=\frac{845}{4}-\frac{2{,}535}{16}=\frac{3{,}380}{16}-\frac{2{,}535}{16}=\frac{845}{16}=52\frac{13}{16}$ pounds. Ans.

ARITHMETIC.

(PART 3)

(1)

tenths.	hundredths.	
.0	8	= *Eight* **hundredths.**

tenths.	hundredths.	thousandths.	
.1	3	1	= *One hundred thirty-one* **thousandths**

tenths.	hundredths.	thousandths.	ten-thousandths.	
.0	0	0	1	= *One* **ten-thousandth.**

tenths.	hundredths.	thousandths.	ten thousandths.	hundred-thousandths.	millionths.	
.0	0	0	0	2	7	= *Twenty-seven* **millionths.**

tenths.	hundredths.	thousandths.	ten-thousandths.	
.0	1	0	8	= *One hundred eight* **ten-thousandths.**

	tenths.	hundredths.	thousandths.	ten-thousandths.	
93.	0	1	0	1	= Ninety-three, and *one hundred one* **ten-thousandths.**

In reading decimals, read the number just as you would if there were no ciphers before it. Then count from the decimal point towards the right, beginning with tenths, to as many places as there are figures, and the *name* of the last figure must be annexed to the previous reading of the figures to give the decimal reading. Thus, in the first example above, the simple reading of the figure is *eight*, and the name of its position in the decimal scale is **hundredths,** so that the decimal reading is *eight* **hundredths.** Similarly, the figures in the fourth example are ordinarily read *twenty-seven;* the name of the position of the figure 7 in the decimal scale is **millionths,** giving, therefore, the decimal reading as *twenty-seven* **millionths.**

If there should be a whole number before the decimal point, read it as you would read any whole number, and read the decimal as you would if the whole number were not there; or, read the whole number and then say, "and" so many hundredths, thousandths, or whatever it may be, as "ninety-three, *and* one hundred one ten-thousandths."

(2) A fraction is one or more of the equal parts of a unit, and is expressed by a numerator and a denominator, while a decimal fraction is a number of *tenths*, *hundredths*, *thousandths*, etc., of a unit, and is expressed by placing a period (.), called a decimal point, to the left of the figures of the number, and omitting the denominator.

(3) See Art. **165.**

(4) To reduce the fraction $\frac{1}{2}$ to a decimal, we annex one cipher to the numerator, which makes it 1.0. Dividing 1.0, the numerator, by 2, the denominator, gives a quotient of .5, the decimal point being placed before the *one* figure

of the quotient, or .5, since only *one* cipher was annexed to the numerator. Ans.

$$\frac{7}{8} \quad 8\,)\,\underline{7.000} \quad .875 \quad \text{Ans.}$$

$$\frac{5}{32} \quad 32\,)\,5.00000\,(.15625 \quad \text{Ans.}$$

```
32
---
180
160
---
 200
 192
 ---
   80
   64
   ---
   160
   160
   ---
```

$$\frac{125}{1000} = .125. \quad \text{Ans.}$$

Since $.65 = \frac{65}{100}$, then, $\frac{65}{100}$ must equal .65. Or, when the denominator is 10, 100, 1,000, etc., point off as many places in the numerator as there are ciphers in the denominator. Doing so, $\frac{65}{100} = .65$. Ans.

(**5**) (*a*) This example, written in the form of a fraction, means that the numerator $(32.5 + .29 + 1.5)$ is to be divided by the denominator $(4.7 + 9)$. The operation is as follows:

$$\frac{32.5 + .29 + 1.5}{4.7 + 9} = ?$$

```
          32.5
       +    .29
       +   1.5
       ----------
 13.7 ) 34.29000 ( 2.5029  Ans.
        274
        ---
 4.7     689
+9.0     685
----     ---
13.7       400
           274
           ---
           1260
           1233
           ----
             27
```

Since there are 5 decimal places in the dividend and 1 in the divisor, there are 5 − 1 or 4 places to be pointed off in the quotient. The fifth figure of the decimal is evidently less than 5.

(*b*) Here again the problem is to divide the numerator, which is $(1.283 \times \overline{8 + 5})$, by the denominator, which is 2.63. The operation is as follows:

$$\frac{1.283 \times \overline{8+5}}{2.63} = ? \quad \overline{8+5} = 13.$$

$$\begin{array}{r}
1.283 \\
\times \quad 13 \\
\hline
3849 \\
1283 \\
\end{array}$$

$$\begin{array}{l}
2.63)\,16.679000\,(6.3418 \quad \text{Ans.} \\
1578 \\
\overline{899} \\
789 \\
\overline{1100} \\
1052 \\
\overline{480}
\end{array}
\qquad
\begin{array}{r}
480 \\
263 \\
\hline
2170 \\
2104 \\
\hline
66
\end{array}$$

(c) $\dfrac{\overline{589+27} \times \overline{163-8}}{25+39} = ?$

$$\begin{array}{r}
163 \\
-\quad 8 \\
\hline
155 \\
\times 616 \\
\hline
930 \\
155 \\
930
\end{array}$$

$$\begin{array}{r}
589 \\
+\ 27 \\
\hline
616
\end{array}$$

$$\begin{array}{l}
64)\,95480.000\,(1491.875 \quad \text{Ans.} \\
64 \\
\overline{314} \\
256 \\
\overline{588} \\
576 \\
\overline{120} \\
64 \\
\overline{560} \\
512 \\
\overline{480} \\
448 \\
\overline{320} \\
320 \\
\overline{}
\end{array}$$

$$\begin{array}{r}
25 \\
+\ 39 \\
\hline
64
\end{array}$$

There are three decimal places in the quotient, since three ciphers were annexed to the dividend.

(*d*) $\dfrac{\overline{40.6 + 7.1} \times (3.029 - 1.874)}{6.27 + 8.53 - 8.01} = ?$

$$\begin{array}{r} 40.6 \\ + \quad 7.1 \\ \hline 47.7 \end{array} \qquad \begin{array}{r} 3.029 \\ -1.874 \\ \hline 1.155 \\ \times \quad 47.7 \\ \hline 8085 \\ 8085 \\ 4620 \end{array}$$

$$\begin{array}{r} 6.27 \\ + \ 8.53 \\ \hline 14.80 \\ - \ 8.01 \\ \hline 6.79 \end{array}$$

$$6.79 \overline{)\, 55.093500} \,(8.1139 \quad \text{Ans.}$$

$$\begin{array}{l} 5432 \\ \hline 773 \\ 679 \\ \hline 945 \\ 679 \\ \hline 2660 \\ 2037 \\ \hline 6230 \\ 6111 \\ \hline 119 \end{array}$$

6 decimal places in the dividend — 2 decimal places in the divisor = 4 decimal places to be pointed off in the quotient.

(6) $.875 = \dfrac{875}{1{,}000} = \dfrac{175}{200} = \dfrac{7}{8}$ of a foot.

1 foot = 12 inches.

$$\frac{7}{8} \text{ of 1 foot} = \frac{7}{\cancel{8}_{2}} \times \frac{\cancel{12}^{3}}{1} = \frac{21}{2} = 10\tfrac{1}{2} \text{ inches.} \quad \text{Ans.}$$

(7) 12 inches = 1 foot.

$$\frac{3}{16} \text{ of an inch} = \frac{3}{16} \div 12 = \frac{\cancel{3}}{16} \times \frac{1}{\cancel{12}_{4}} = \frac{1}{64} \text{ of a foot.}$$

Point off 6 decimal places in the quotient, since we annexed six ciphers to the dividend, the divisor containing

no decimal places; hence, 6 − 0 = 6 places to be pointed off.

```
 1
---
64 ) 1.000000 ( .015625   Ans.
        64
        ---
        360
        320
        ---
         400
         384
         ---
          160
          128
          ---
           320
           320
           ---
```

(8) If 1 cubic inch of water weighs .03617 of a pound, the weight of 1,500 cubic inches will be .03617 × 1,500 = 54.255 lb.

```
     .03617 lb.
       1500
    -------
    1808500
    3617
    -------
    54.25500 lb.   Ans.
```

(9) Since a cubic foot of ore weighs 176 pounds and a cubic foot of water weighs 62.5 pounds, then the ore is as many times heavier than the water as 62.5 is contained times in 176, or 2.816 times.

```
62.5 ) 176.000 ( 2.816
       1250
       ----
        5100
        5000
        ----
         1000
          625
         ----
          3750
          3750
          ----
```

(10)

$$
\begin{array}{r}
231\,)\,17892.00000\,(\,77.45454 \\
\underline{1617} \\
1722 \\
\underline{1617} \\
1050 \\
\underline{924} \\
1260 \\
\underline{1155} \\
1050 \\
\underline{924} \\
1260 \\
\underline{1155} \\
1050
\end{array}
$$

77.45454, or 77.4545 to four decimal places. Ans.

(11) See Art. **174.** Applying rule in Art. **175,**

(*a*) $.7928 \times \frac{64}{64} = \frac{50.7392}{64} = \frac{51}{64}.$ Ans.

(*b*) $.1416 \times \frac{32}{32} = \frac{4.5312}{32} = \frac{5}{32}.$ Ans.

(*c*) $.47915 \times \frac{16}{16} = \frac{7.6664}{16} = \frac{8}{16} = \frac{1}{2}.$ Ans.

(12) In addition of decimals the decimal points must be placed directly under one another, so that tenths will come under tenths, hundredths under hundredths, thousandths under thousandths, etc. The addition is then performed as in whole numbers, the decimal point of the sum being placed directly under the decimal points above.

$$
\begin{array}{l}
.125 \\
.7 \\
.089 \\
.4005 \\
.9 \\
\underline{.000027} \\
2.214527
\end{array}
$$

Ans.

(13) (*a*) $\left(\frac{7}{16} - .13\right) \times \overline{.625 + \frac{5}{8}} = ?$

First perform the operation indicated by the parenthesis.

$$\frac{7}{16} = 16)\,7.0000\,(.4375$$

```
16)7.0000(.4375
   64
   --
    60
    48
    --
    120
    112
    ---
      80
      80
      --
```

We point off four decimal places, since we annexed four ciphers.

```
              .4375
              .13
              -----
Subtracting, we obtain .3075
```

The vinculum has the same meaning as the parenthesis; hence, we perform the operation indicated by it. We point off three decimal places, since three ciphers were annexed to the 5.

$$\frac{5}{8} = 8)\,\underline{5.000}\quad .625$$

Adding the terms included by the vinculum, we obtain

```
 .625
 .625
 -----
1.250
```

The final operation is to perform the work indicated by the sign between the parenthesis and the vinculum, thus,

```
  .3075
   1.25
  ------
  15375
  6150
 3075
 -------
.384375  Ans.
```

(b) $\left(\frac{19}{32} \times .21\right) - \left(.02 \times \frac{3}{16}\right) = ?$

$$.21 = \frac{21}{100}. \quad \frac{19}{32} \times \frac{21}{100} = \frac{399}{3200}. \quad .02 = \frac{2}{100}. \quad \frac{2}{100} \times \frac{3}{16} = \frac{6}{1600} = \frac{3}{800}.$$

$$\frac{3}{800} = \frac{3 \times 4}{800 \times 4} = \frac{12}{3200}. \qquad \frac{399}{3200} - \frac{12}{3200} = \frac{399 - 12}{3200} = \frac{387}{3200}.$$

Reducing $\frac{387}{3200}$ to a decimal, we obtain

$$
\begin{array}{l}
\frac{387}{3200}) \; 387.0000000 \; (.1209375 \quad \text{Ans.} \\
\quad 3200 \\
\hline
\quad\;\; 6700 \\
\quad\;\; 6400 \\
\hline
\quad\;\;\; 30000 \\
\quad\;\;\; 28800 \\
\hline
\quad\;\;\;\; 12000 \\
\quad\;\;\;\;\; 9600 \\
\hline
\quad\;\;\;\;\; 24000 \\
\quad\;\;\;\;\; 22400 \\
\hline
\quad\;\;\;\;\;\; 16000 \\
\quad\;\;\;\;\;\; 16000 \\
\hline
\end{array}
$$

Point off seven decimal places, since seven ciphers were annexed to the dividend.

(*c*) $\left(\frac{13}{4} + .013 - 2.17\right) \times \overline{13\frac{1}{4} - 7\frac{5}{16}} = ?$

$\frac{13}{4} = \frac{13}{4)\,13.00} \atop \quad\;\; 3.25$

$$
\frac{13}{4} = 4)\,\underline{13.00} \\ \qquad\quad 3.25
$$

Point off two decimal places, since two ciphers were annexed to the dividend.

$$
\begin{array}{r}
3.25 \\
+\;.013 \\
\hline
3.263 \\
-2.17 \\
\hline
1.093
\end{array}
$$

$\frac{5}{16}$ reduced to a decimal is .3125, since

$$
\begin{array}{l}
\frac{5}{16})\; 5.0000 \;(.3125 \\
\quad\; 48 \\
\hline
\quad\;\; 20 \\
\quad\;\; 16 \\
\hline
\quad\;\;\; 40 \\
\quad\;\;\; 32 \\
\hline
\quad\;\;\;\; 80 \\
\quad\;\;\;\; 80 \\
\hline
\end{array}
$$

Point off four decimal places, since four ciphers were annexed to the dividend.

Then, $7\frac{5}{16} = 7.3125$, and $13\frac{1}{4} = 13.25$, since $\frac{1}{4} = \frac{1}{4}$) 1.00 = .25

$$\begin{array}{r} 13.25 \\ -\ \ 7.3125 \\ \hline 5.9375 \end{array} \qquad \begin{array}{r} 5.9375 \\ \times\ \ 1.093 \\ \hline 178125 \\ 534375 \\ 593750 \\ \hline 6.4896875 \end{array} \text{ Ans.}$$

(14) $\dfrac{1.25 \times 20 \times 3}{\dfrac{87 + (11 \times 8)}{459 + 32}} = ?$ In this problem, $1.25 \times 20 \times 3$ constitutes the numerator of the complex fraction.

$$\begin{array}{r} 1.25 \\ \times\ \ 20 \\ \hline 25.00 \\ \times\ 3 \\ \hline 75 \end{array}$$

Multiplying the factors of the numerator together, we find their product to be 75.

The fraction $\dfrac{87 + (11 \times 8)}{459 + 32}$ constitutes the denominator of the complex fraction. The value of the numerator of this fraction equals $87 + 88 = 175$.

The numerator is combined as though it were written $87 + (11 \times 8)$, and its result is

$$\begin{array}{r} 11 \\ \times\ 8 \\ \hline 88 \\ +\ 87 \\ \hline 175 \end{array}$$

The value of the denominator of this fraction is equal to $459 + 32 = 491$. The problem then becomes

$$\frac{75}{\frac{175}{491}} = \frac{75}{1} \div \frac{175}{491} = \frac{75}{1} \times \frac{491}{175} = \frac{\overset{3}{\cancel{75}} \times 491}{\underset{7}{\cancel{175}}} = \frac{1,473}{7} = 210\frac{3}{7}. \text{ Ans.}$$

(15) $\$2,632 \div 60 = \$43.866+$ or $\$43.87$, nearly. Ans.

(16) $62.5 \times 2.4 = 150$ pounds. Ans.

(17) $.49175 \div .03617 = 13.5955$ cu. in. Ans.

(18) $2,000 \div 7.48 = 267.4$ cu. ft., nearly. Ans.

(19) $267.3 \times 92.5 = 24,725.25$ pounds.

$$\frac{24,725.25}{2000} = 12.36 \text{ tons. Ans.}$$

(20) $267.3 \times .18 = 48.11$ cu. ft., amount of shrinkage.

$267.3 - 48.11 = 219.19$ cu. ft. Ans.

ARITHMETIC.

(PART 4.)

(1) Here 50 is the base, 2 is the percentage, and it is required to find the rate. Applying rule, Art. **193,**

$$\text{rate} = \text{percentage} \div \text{base};$$
$$\text{rate} = 2 \div 50 = .04 \text{ or } 4\%. \quad \text{Ans.}$$

(2) In this problem, \$5,500 is the amount, since it equals what he paid for the farm + what he gained; 15% is the rate, and the cost (to be found) is the base. Applying rule, Art. **197,**

$$\text{base} = \text{amount} \div (1 + \text{rate}); \text{ hence,}$$
$$\text{base} = \$5{,}500 \div (1 + .15) = \$4{,}782.61. \quad \text{Ans.}$$

```
1.15 ) 5.500.0000 ( 4782.61
       460
       ---
        900
        805
        ---
         950
         920
         ---
          300
          230
          ---
           700
           690
           ---
            100
            115
            ---
```

The example can also be solved as follows: $100\% =$ cost; if he gained 15%, then $100 + 15 = 115\% = \$5,500$, the selling price.

If $115\% = \$5,500$, $1\% = \frac{1}{115}$ of $\$5,500 = \$47.8261$, and 100%, or the cost, $= 100 \times \$47.8261 = \$4,782.61$. Ans.

(3)
$$\begin{array}{lr}
24\ \% \text{ of } \$950 = .24 \times 950 = & \$228 \\
12\tfrac{1}{2}\% \text{ of } \$950 = .125 \times 950 = & 118.75 \\
17\ \% \text{ of } \$950 = .17 \times 950 = & 161.50 \\
\hline
53\tfrac{1}{2}\% \text{ of } \$950 \qquad\qquad = & \$508.25
\end{array}$$

The total amount of his yearly expenses, then, is \$508.25; hence, his savings are $\$950 - \$508.25 = \$441.75$. Ans.

Or, as above, $24\% + 12\frac{1}{2}\% + 17\% = 53\frac{1}{2}\%$, the total percentage of expenditures; hence, $100\% - 53\frac{1}{2}\% = 46\frac{1}{2}\% =$ per cent. saved. And $\$950 \times .465 = \$441.75$, his yearly savings. Ans.

(4) The percentage is 961.38 and the rate is $37\frac{1}{2}$. By Art. **192**,

$$\begin{aligned}\text{Base} &= \text{percentage} \div \text{rate} \\ &= 961.38 \div .375 = 2,563.68, \text{ the number. Ans.}\end{aligned}$$

(5) Here \$4,810 is the difference and 35% the rate. By Art. **198**,

$$\begin{aligned}\text{Base} &= \text{difference} \div (1 - \text{rate}) \\ &= \$4,810 \div (1 - .35) = \$4,810 \div .65 = \$7,400. \text{ Ans.}\end{aligned}$$

$$\begin{array}{l}
.65\,)\,4810.00\,(\,7400 \\
\quad\ \ 455 \\
\quad\ \ \overline{\ \ 260} \\
\quad\ \ \ \ 260 \\
\quad\ \ \ \ \overline{\ \ \ 00}
\end{array}
\qquad\qquad
\begin{array}{r}
1.00 \\
.35 \\
\hline
.65
\end{array}$$

Solution can also be effected as follows: $100\% =$ the sum diminished by 35%; then $(1 - .35) = .65$, which is \$4,810.

If 65% = \$4,810, 1% = $\frac{1}{65}$ of 4,810 = \$74; and 100% = 100 × \$74 = \$7,400. Ans.

(6) 16.5 miles = $12\frac{1}{2}$% of the entire length of the road. We wish to find the *entire* length.

16.5 miles is the percentage, $12\frac{1}{2}$% is the rate, and the entire length will be the base. By Art. **192**,

$$\text{Base} = \text{percentage} \div \text{rate} = 16.5 \div .12\tfrac{1}{2}.$$

```
.125 ) 16.500 ( 132 miles.   Ans.
       125
       ---
        400
        375
        ---
         250
         250
         ---
```

(7) 28 rd. 4 yd. 2 ft. 10 in. to inches.

```
×     5½
    ----
     154
+      4
    ----
     158 yards
×      3
    ----
     474
+      2
    ----
     476 feet
×     12
    ----
    5712
+     10
    ----
    5722 inches.  Ans.
```

Since there are $5\frac{1}{2}$ yards in one rod, in 28 rods there are $28 \times 5\frac{1}{2}$ or 154 yards; 154 yards plus 4 yards = 158 yards. There are 3 feet in one yard; therefore, in 158 yards there are 3×158 or 474 feet; 474 feet + 2 feet = 476 feet. There are 12 inches in one foot, and in 476 feet there are 12×476, or 5,712 inches; 5,712 inches + 10 inches = 5,722 in.

(8)

```
  12 ) 5722 inches.
   3 ) 476 + 10 inches.
  5½ ) 158 + 2 feet.
        28 + 4 yards.
```

Ans. = 28 rd. 4 yd. 2 ft. 10 in.

EXPLANATION.—There are 12 inches in 1 foot; hence, in 5,722 inches there are as many feet as 12 is contained times in 5,722 inches, or 476 ft. and 10 inches remaining. Write these 10 inches as a remainder. There are 3 feet in 1 yard; hence, in 476 feet there are as many yards as 3 is contained times in 476 feet, or 158 yards and 2 feet remaining. There are $5\frac{1}{2}$ yards in 1 rod; hence, in 158 yards there are 28 rods and 4 yards remaining. Then, in 5,722 inches there are 28 rd. 4 yd. 2 ft. 10 in.

(9) Since there are 24 gr. in 1 pwt., in 13,750 gr. there are as many pennyweights as 24 is contained times in 13,750, or 572 pwt. and 22 gr. remaining. Since there are 20 pwt. in 1 oz., in 572 pwt. there are as many ounces as 20 is contained times in 572, or 28 oz. and 12 pwt. remaining.

Since there are 12 oz. in 1 lb. (Troy), in 28 oz. there are as many pounds as 12 is contained times in 28, or 2 lb. and 4 oz. remaining. We now have the pounds and ounces required by the problem; therefore, in 13,750 gr. there are 2 lb. 4 oz. 12 pwt. 22 gr.

$$\begin{array}{r} 24\,)\,\underline{13750} \text{ gr.} \\ 20\,)\,\underline{572} \text{ pwt.} + 22 \text{ gr.} \\ 12\,)\,\underline{28} \text{ oz.} + 12 \text{ pwt.} \\ 2 \text{ lb.} + 4 \text{ oz.} \end{array}$$

Ans. = 2 lb. 4 oz. 12 pwt. 22 gr.

(10) We write the compound numbers so that the units of the same denomination shall stand in the same column. Beginning to add with the lowest denomination, we find that the sum of the gills is $1 + 2 + 3 = 6$. Since there are 4 gi. in 1 pint, in 6 gi. there are as many pints as 4 is contained times in 6, or 1 pt. and 2 gi. We place 2 gi. under the gills column and reserve the 1 pt. for the pints column; the sum of the pints is 1 (reserved) $+ 5 + 1 + 1 = 8$. Since there are 2 pt.

gal.	qt.	pt.	gi.
3	3	1	3
6	0	1	2
4	0	0	1
	8	5	0
16 gal.	3 qt.	0 pt.	2 gi.

in 1 quart, in 8 pt. there are as many quarts as 2 is contained times in 8, or 4 qt. and 0 pt. We place the cipher under the column of pints and reserve the 4 for the quarts column. The sum of the quarts is 4 (reserved) + 8 + 3 = 15. Since there are 4 qt. in 1 gallon, in 15 qt. there are as many gallons as 4 is contained times in 15, or 3 gal. and 3 qt. remaining. We now place the 3 under the quarts column and reserve the 3 gal. for the gallons column. The sum of the gallons column is 3 (reserved) + 4 + 6 + 3 = 16 gal. Since we cannot reduce 16 gal. to any higher denominations, we have 16 gal. 3 qt. 0 pt. and 2 gi. for the answer.

(11) Since "seconds" is the lowest denomination in this problem, we find their sum first, which is 11 + 29 + 25 + 30 + 12, or 107 seconds. Since there are 60 seconds in 1 minute, in 107″ there are as many minutes as 60 is contained times in 107, or 1 minute and 47 seconds remaining. We place the 47 under the seconds column and reserve the 1 for the minutes column. The sum of the minutes is 1 (reserved) + 17 + 26 + 19 + 16, or 79. Since there are 60 minutes in 1 degree, in 79 minutes there are as many degrees as 60 is contained times in 79, or 1 degree and 19 minutes remaining. We place the 19 under the minutes column and reserve the 1 degree for the degrees column. The sum of the degrees is 1 (reserved) + 10 + 20 + 13 + 11, or 55 degrees. Since we cannot reduce 55 degrees to any higher denominations, we have 55° 19′ 47″ for the answer.

deg.	min.	sec.
11	16	12
13	19	30
20	0	25
0	26	29
10	17	11
55°	19′	47″

(12) Since "inches" is the lowest denomination in this problem, we find their sum first, which is 11 + 8 + 6, or 25 inches. Since there are 12 inches in 1 foot, in 25 inches there are as many feet as 12 is contained times in 25, or 2 feet and 1 inch remaining. Place the 1 inch under the inches column, and reserve the 2 feet to add to the column

of feet. The sum of the feet is 2 feet (reserved) + 2 + 1 = 5 feet. Since there are 3 feet in 1 yard, in 5 feet there are as many yards as 3 is contained times in 5 feet, or 1 yard and 2 feet remaining. Place the 2 feet under the column of feet, and reserve the 1 yard to add to the column of yards. The sum of the yards is 1 yard (reserved) + 4 + 5 = 10 yards. Since there are $5\frac{1}{2}$ yards in 1 rod, in 10 yards there are as many rods as $5\frac{1}{2}$ is contained times in 10, or 1 rod and $4\frac{1}{2}$ yards remaining. Place the $4\frac{1}{2}$ yards under the column of yards, and reserve the 1 rod for the column of rods. The sum of the rods is 1 (reserved) + 304 + 215 + 130 = 650 rods. Place 650 rods under the column of rods. Therefore, the sum is 650 rd. $4\frac{1}{2}$ yd. 2 ft. 1 in. Or, since $\frac{1}{2}$ yard = 1 ft. 6 in., and since there are 320 rods in 1 mile, the sum may be expressed as 2 mi. 10 rd. 5 yd. 0 ft. 7 in. Ans.

	mi.	rd.	yd.	ft.	in.	
		130	5	1	6	
		215	0	2	8	
		304	4	0	11	
		650	$4\frac{1}{2}$	2	1	
or,	2	10	5	0	7	Ans.

(13) Since "square links" is the lowest denomination in this problem, we find their sum first, which is 21 + 23 + 18 + 16 + 23 + 21, or 122 square links. Place 122 square links under the column of square links. The sum of the square rods is 2 + 3 + 2 + 2 + 2 + 3, or 14 square rods. Place 14 square rods under the column of square rods. The sum of the square chains is 323 square chains. Since there are 10 square chains in 1 acre, in 323 square chains there are as many acres as 10 is contained times in 323 square chains, or 32 acres and 3 square chains remaining. Place 3 square chains under the column of square chains, and reserve the 32 acres to add to the

A.	sq. ch.	sq. rd.	sq. li.
21	67	3	21
28	78	2	23
47	6	2	18
56	59	2	16
25	38	3	23
46	75	2	21
255	3	14	122

column of acres. The sum of the acres is 32 acres (reserved) + 46 + 25 + 56 + 47 + 28 + 21, or 255 acres. Place 255 acres under the column of acres. Therefore, the sum is 255 A. 3 sq. ch. 14 sq. rd. 122 sq. li. Ans.

(14) Before we can subtract 300 ft. from 20 rd. 2 yd. 2 ft. and 9 in., we must reduce the 300 ft. to higher denominations.

Since there are 3 feet in 1 yard, in 300 feet there are as many yards as 3 is contained times in 300, or 100 yards. There are $5\frac{1}{2}$ yards in 1 rod; hence, in 100 yards there are as many rods as $5\frac{1}{2}$ or $\frac{11}{2}$ is contained times in $100 = 18\frac{2}{11}$ rods.

$$100 \div \frac{11}{2} = 100 \times \frac{2}{11} = \frac{100 \times 2}{11} = \frac{200}{11}$$

$$\begin{array}{r} 11\,)\,200\,(\,18\tfrac{2}{11}\text{ rd.} \\ \underline{11}\phantom{0\,(\,18\tfrac{2}{11}\text{ rd.}} \\ 90\phantom{\,(\,18\tfrac{2}{11}\text{ rd.}} \\ \underline{88}\phantom{\,(\,18\tfrac{2}{11}\text{ rd.}} \\ 2\phantom{\,(\,18\tfrac{2}{11}\text{ rd.}} \end{array}$$

Since there are $5\frac{1}{2}$ or $\frac{11}{2}$ yards in 1 rod, in $\frac{2}{11}$ rods there are $\frac{\cancel{2}}{\cancel{11}} \times \frac{\cancel{11}}{\cancel{2}}$, or 1 yard, so we find that 300 feet equals 18 rods and 1 yard. The problem now is as follows: From 20 rd. 2 yd. 2 ft. and 9 in. take 18 rd. and 1 yd.

We place the smaller number under the larger one, so that units of the same denomination fall in the same column. Beginning with the lowest denomination, we see that 0 inches from 9 inches leaves 9 inches. Going to the next higher denomination, we see that 0 feet from 2 feet leaves 2 feet. Subtracting 1 yard from 2 yards, we have 1 yard remaining, and 18 rods from 20 rods leaves 2 rods. Therefore, the difference is 2 rd. 1 yd. 2 ft. 9 in. Ans.

rd.	yd.	ft.	in.
20	2	2	9
18	1	0	0
2	1	2	9

(15) If a note given Aug. 5, 1890, were paid June 3, 1892, in order to find the length of time it was due, subtract the earlier date from the later date.

Beginning with the lowest denomination, we find that 5 cannot be subtracted from 3, so we take a unit from the next higher denomination, which is months. The 1 month which we take equals 30 days. Adding the 30 days to the 3 days, we have 33 days. 5 days from 33 days leaves 28 days. Since we took 1 month from the months column, only 4 months remain. 7 months cannot be taken from 4 months, so we take 1 year from the years column, which equals 12 months. 12 months + 4 months = 16 months. 7 months from 16 months = 9 months. Since we took 1 year from the years column, we have 1892 − 1, or 1891 remaining. 1890 from 1891 leaves 1 year. Hence, the note ran 1 year 9 months and 28 days. Ans.

yr.	mo.	da.
1892	5	3
1890	7	5
1	9	28

(16) In multiplication of denominate numbers, we place the multiplier under the lowest denomination of the multiplicand, as

$$\begin{array}{rr} 17 \text{ ft.} & 3 \text{ in.} \\ & 51 \\ \hline 879 \text{ ft.} & 9 \text{ in.} \end{array}$$

and begin at the right to multiply. $51 \times 3 = 153$ in. As there are 12 inches in 1 foot, in 153 in. there are as many feet as 12 is contained times in 153, or 12 feet and 9 inches remaining. Place the 9 inches under the inches, and reserve the 12 feet. 51×17 ft. = 867 ft. 867 ft. + 12 ft. (reserved) = 879 ft.

879 feet can be reduced to higher denominations by dividing by 3 feet to find the number of yards, and by $5\frac{1}{2}$ yards to find the number of rods.

$$\begin{array}{rl} 3 \,) & \underline{8\,7\,9} \text{ ft. } 9 \text{ in.} \\ 5.5 \,) & \underline{2\,9\,3} \text{ yd.} \\ & \quad 5\,3 \text{ rd. } 1\tfrac{1}{2} \text{ yd.} \end{array}$$

Then, answer = 53 rd. $1\frac{1}{2}$ yd. 0 ft. 9 in.; or 53 rd. 1 yd. 2 ft. 3 in.

(**17**) (3 lb. 10 oz. 13 pwt. 12 gr.) × 1.5 = ?

```
        3 lb. 10 oz. 13 pwt. 12 gr.
  × 12
    ——
    36   oz.
  + 10
    ——
    46   oz.
  ×  20
    ———
    920  pwt.
  +  13
    ———
    933  pwt.
  ×  24
  —————
  22392  gr.
  +  12
  —————
  22404  gr.
```

22,404 gr. × 1.5 = 33,606 gr.

```
24)33606 gr.
  20)1400 pwt. + 6 gr.
     12)70 oz. + 0 pwt.
         5 lb. + 10 oz.
```

Since there are 24 gr. in 1 pwt., in 33,606 gr. there are as many pwt. as 24 is contained times in 33,606, or 1,400 pwt. and 6 gr. remaining. This gives us the number of grains in the answer. We now reduce 1,400 pwt. to higher denominations. Since there are 20 pwt. in 1 oz., in 1,400 pwt. there are as many ounces as 20 is contained times in 1,400, or 70 oz. and 0 pwt. remaining; therefore, there are 0 pwt. in the answer. We reduce 70 oz. to higher denominations. Since there are 12 oz. in 1 lb., in 70 oz. there are as many pounds as 12 is contained times in 70, or 5 lb. and 10 oz. remaining. We cannot reduce 5 lb. to any higher denominations. Therefore, our answer is 5 lb. 10 oz. 6 gr.

Another but more complicated way of working this problem is as follows:

To get rid of the decimal in the pounds, reduce .5 of a pound to ounces. Since 1 lb. = 12 oz., .5 of a pound equals .5 lb. × 12 = 6 oz. 6 oz. + 15 oz. = 21 oz. We

now have 4 lb. 21 oz. 19.5 pwt. and 18 gr., but we still have a decimal in the column of pwt., so we reduce .5 pwt. to grains to get rid of it. Since 1 pwt. = 24 gr., .5 pwt. = .5 pwt. × 24 = 12 gr. 12 gr. + 18 gr. = 30 gr. We now have 4 lb. 21 oz. 19 pwt. and 30 gr. Since there are 24 gr. in 1 pwt., in 30 gr. there is 1 pwt. and 6 gr. remaining. Place 6 gr. under the column of grains and add 1 pwt. to the pwt. column. Adding 1 pwt., we have 19 + 1 = 20 pwt. Since there are 20 pwt. in 1 oz., we have 1 oz. and 0 pwt. remaining. Write the 0 pwt. under the pwt. column, and reserve the 1 oz. to add to the oz. column. 21 oz. + 1 oz. = 22 oz. Since there are 12 oz. in 1 lb., in 22 oz. there is 1 lb. and 10 oz. remaining. Write the 10 oz. under the ounce column, and reserve the 1 lb. to add to the lb. column. 4 lb. + 1 lb. (reserved) = 5 lb. Hence, the answer equals 5 lb. 10 oz. 6 gr.

	lb.	oz.	pwt.	gr.	
	3	10	13	12	
				1.5	
	4.5	15	19.5	18	
or,	4	21	19	30	
or,	5	10	0	6	Ans.

(18) (7 T. 15 cwt. 10.5 lb.) × 1.7 = ? When the multiplier is a decimal, instead of multiplying the denominate numbers as in the case when the multiplier is a whole number, it is much easier to reduce the denominate numbers to the lowest denomination given; then, multiply that result by the decimal, and lastly reduce the product to higher denominations. Although the correct answer can be obtained by working examples involving decimals in the manner as in the last example, it is much more complicated than this method.

$$\begin{array}{rl}
 & 7\text{ T. } 15\text{ cwt. } 10.5\text{ lb.} \\
\times & \underline{20} \\
 & 140 \text{ cwt.} \\
+ & \underline{15} \\
 & 155 \text{ cwt.} \\
\times & \underline{100} \\
 & 15500 \text{ lb.} \\
+ & \underline{10.5} \\
 & 15510.5 \text{ lb.}
\end{array}$$

15,510.5 lb. × 1.7 = 26,367.85 lb.

There are 100 lb. in 1 cwt., and in 26,367.85 lb. there are as many cwt. as 100 is contained times in 26,367.85, which equals 263 cwt. and 67.85 lb. remaining. Since we have the number of pounds for our answer, we reduce 263 cwt. to higher denominations.

$$\begin{array}{r|l} 100 & 2\,6\,3\,6\,7.8\,5 \text{ lb.} \\ 20 & 2\,6\,3 \text{ cwt.} + 67.85 \text{ lb.} \\ & 1\,3 \text{ T.} + 3 \text{ cwt.} \end{array}$$

There are 20 cwt. in 1 ton, and in 263 cwt. there are as many tons as 20 is contained times in 263, or 13 tons and 3 cwt. remaining. Since we cannot reduce 13 tons any higher, our answer is 13 T. 3 cwt. 67.85 lb. Or, since .85 lb. = .85 lb. $\times$ 16 = 13.6 oz., the answer may be written 13 T. 3 cwt. 67 lb. 13.6 oz.

(19)

$$\begin{array}{r|llll} 12 & 282 \text{ bu.} & 3 \text{ pk.} & 1 \text{ qt.} & 1 \text{ pt.} \\ \hline & 23 \text{ bu.} & 2 \text{ pk.} & 2 \text{ qt.} & \frac{1}{4} \text{ pt. Ans.} \end{array}$$

12 is contained in 282 bu. 23 times and 6 bu. remaining. We write 23 bu. under the 282 bu. in the dividend, and reduce the remaining 6 bu. to pecks = 24 pk. + the 3 pk. in the dividend = 27 pk. 12 is contained in 27 pk. 2 times and 3 pk. remaining. We write 2 pk. under the 3 pk. in the dividend, and reduce the remaining 3 pk. to quarts. 3 pk. = 24 qt.; 24 qt. + the 1 qt. in the dividend = 25 qt. 12 is contained in 25 qt. 2 times and 1 qt. remaining. We write 2 qt. under the 1 qt. in the dividend, and reduce 1 qt. to pints = 2 pt. + the 1 pt. in the dividend = 3 pt. $3 \div 12 = \frac{3}{12}$ or $\frac{1}{4}$ pt.

(20) We must first reduce 16 square miles to acres.

In 1 sq. mi. there are 640 A., and in 16 sq. mi. there are 16 $\times$ 640 A. = 10,240 A.

$$\begin{array}{r|l} 62 & 1\,0\,2\,4\,0 \text{ A.} \\ \hline & 1\,6\,5 \text{ A. } 25 \text{ sq.rd. } 24 \text{ sq.yd. } 3 \text{ sq.ft. } 80+ \text{ sq.in. Ans.} \end{array}$$

62 is contained in 10,240 A. 165 times and 10 A. remaining. We write 165 A. under the 10,240 A. in the dividend and reduce 10 A. to sq. rd. In 1 A. there are 160 sq. rd. and in 10 A. there are 10 $\times$ 160 = 1,600 sq. rd. 62 is contained in 1,600 sq. rd. 25 times and 50 sq. rd. remaining.

We write 25 sq. rd. in the quotient and reduce 50 sq. rd. to sq. yd. In 1 sq. rd. there are $30\frac{1}{4}$ sq. yd., and in 50 sq. rd. there are 50 times $30\frac{1}{4}$ sq. yd. $= 1,512\frac{1}{2}$ sq. yd. 62 is contained in $1,512\frac{1}{2}$ sq. yd. 24 times and $24\frac{1}{2}$ sq. yd. remaining. In 1 sq. yd. there are 9 sq. ft., and in $24\frac{1}{2}$ sq. yd. there are $24\frac{1}{2} \times 9 = 220\frac{1}{2}$ sq. ft. 62 is contained in $220\frac{1}{2}$ sq. ft. 3 times and $34\frac{1}{2}$ sq. ft. remaining. We write 3 sq. ft. in the quotient and reduce $34\frac{1}{2}$ sq. ft. to sq. in. In 1 sq. ft. there are 144 sq. in., and in $34\frac{1}{2}$ sq. ft. there are $34\frac{1}{2} \times 144 = 4,968$ sq. in. 62 is contained in 4,968 sq. in. 80 times and 8 sq. in. remaining.

We write 80 sq. in. in the quotient.

It should be borne in mind that it is only for the purpose of illustrating the method that this problem is carried out to square inches. It is not customary to reduce any lower than square rods in calculating the area of a farm.

ARITHMETIC.

(PART 5.)

(1) To square a number, we must multiply the number by itself once, that is, use the number twice as a factor. Thus, the second power of 108 is $108 \times 108 = 11,664$. Ans.

$$\begin{array}{r} 108 \\ 108 \\ \hline 864 \\ 1080 \\ \hline 11664 \end{array}$$

(2) $9^5 = 9 \times 9 \times 9 \times 9 \times 9 = 59,049$. Ans.

$$\begin{array}{r} 9 \\ 9 \\ \hline 81 \\ 9 \\ \hline 729 \\ 9 \\ \hline 6561 \\ 9 \\ \hline 59049 \end{array}$$

(3) (*a*) $.0133^3 = .0133 \times .0133 \times .0133 = .000002352637$. Ans.

Since there are four decimal places in the multiplicand and 4 in the multiplier, we must point off $4 + 4 = 8$ decimal places in the product; but as there are only 5 figures in the product, we prefix three ciphers to form the eight necessary decimal places in the first product.

$$\begin{array}{r} .0133 \\ .0133 \\ \hline 399 \\ 399 \\ 133 \\ \hline .00017689 \\ .0133 \\ \hline 53067 \\ 53067 \\ 17689 \\ \hline .000002352637 \end{array}$$

Since there are 8 decimal places in the multiplicand and 4 in the multiplier, we must point off $8 + 4 = 12$ decimal places in the product; but as there are only 7 figures in the product, we prefix 5 ciphers to make the 12 necessary decimal places in the final product.

(**4**) Evolution is the reverse of involution. In involution we find the *power* of a number by multiplying the number by itself one or more times, while in evolution we find the *number* or *root* which was multiplied by itself one or more times to make the power.

(**5**) $\sqrt{90} = ?$ The root is evidently 9 plus an interminable decimal. Trying 9 for one factor, the other is $90 \div 9 = 10$, and the first approximation is $\frac{9 + 10}{2} = 9.5$. $90 \div 9.5 = 9.473+$, and the second approximation is $\frac{9.5 + 9.473}{2} = 9.486+$, or 9.49 to three figures. Using 9.49 for one factor, the other is $90 \div 9.49 = 9.48366+$, and the third approximation is $\frac{9.49 + 9.48366}{2} = 9.48683$ or $9.4868+$ to five figures. Ans.

This solution may be shortened by using the table and applying the method described in Art. **38** to find the first three significant figures of the root. Referring to the table, the first two significant figures of the root are 9.4; the first difference is $90.25 - 88.36 = 1.89$; the second difference is

$90 - 88.36 = 1.64$; $1.64 \div 1.89 = .86+$. Hence, the first three figures are 9.48.

(6) To find any power of a mixed number, first reduce it to an improper fraction, and then multiply the numerators together for the numerator of the answer, and multiply the denominators together for the denominator of the answer.

$$\left(3\frac{3}{4}\right)^3 = \frac{15}{4} \times \frac{15}{4} \times \frac{15}{4} = \frac{15 \times 15 \times 15}{4 \times 4 \times 4} = \frac{3.375}{64} = 52\frac{47}{64}$$

$= 52.734375$. Ans.

$$3\frac{3}{4} = \frac{3 \times 4 + 3}{4} = \frac{12 + 3}{4} = \frac{15}{4}.$$

```
   15                64 ) 3375 ( 52 47/64
   15                     320
   --                     ---
   75                      175
  15                       128
  ---                      ---
  225                       47
   15
 ----
 1125
 225
 ----
 3375
```

```
64 ) 47.000000 ( .734375
     448
     ---
      220
      192
      ---
       280
       256
       ---
        240
        192
        ---
         480
         448
         ---
          320
          320
          ---
```

Since six ciphers were annexed to the dividend, six decimal places must be pointed off in the quotient.

(7) $\sqrt[3]{92{,}416} = ?$ Pointing off into periods, the result is 92'416. Since 92 lies between $4^3 = 64$ and $5^3 = 125$, the root is 4+. Trying 4 for one of the two equal factors, the third factor is $92 \div 4^2 = 92 \div 16 = 5.75$. Trying 5 for one of the two equal factors. the third factor is $92 \div 5^2 = 92 \div 25 = 3.68$. Difference between 4 and 5.75 is 1.75, and between 5 and 3.68 is 1.32; hence, use 5, the first approximation being $\frac{2 \times 5 + 3.68}{3} = 4.56$, or 4.6 to two figures.

Using 46 for one of the two equal factors, the third factor is $92416 \div 46^2 = 92416 \div 2116 = 43.67+$, and the second approximation is $\frac{2 \times 46 + 43.67}{3} = 45.22+$, or 45.2 to three figures.

Using 45.2 for one of the two equal factors, the third factor is $92416 \div 45.2^2 = 92416 \div 2043.04 = 45.2345+$, and the third approximation is $\frac{2 \times 45.2 + 45.2345+}{3} = 45.2115+$, or 45.212− to five figures. Ans.

The first three significant figures may also be found by the aid of the table and the method described in Art. **39.**

In order to obtain two figures of the root from the table, we place a decimal point between the first and second significant periods; the result is 92.416. Referring to the table, the first two figures of the root are 4.5; the first difference is $97.336 - 91.125 = 6.211$; the second difference is $92.416 - 91.125 = 1.291$; $1.291 \div 6.211 = .20+$. Therefore, $\sqrt[3]{92.416} = 4.52$ and $\sqrt[3]{92{,}416} = 45.2$ to three significant figures.

(8) $\sqrt{502{,}681} = ?$ Pointing off into periods, we have 50'26'81. The first figure of the root is evidently 7, since $7^2 = 49$ and $8^2 = 64$. The two factors then are 7, and $50 \div 7 = 7.14+$. The first approximation is $\frac{7 + 7.14+}{2} = 7.07+$, or 7.1 to two figures. To find the second approximation, we use the first two periods and drop the decimal point in the first

approximation. One factor is then 71 and the other 5026 $\div$ 71 = 70.78+. The second approximation is therefore $\frac{71 + 70.78}{2} = 70.89$, or 70.9 to three figures. Using 709 for one factor, the other is 502681 $\div$ 709 = 709. Hence, the number is a perfect power and the root is 709. Ans.

Using the table and placing a decimal point between the first and second periods so that we may obtain two figures of the root from the table, the number becomes $\sqrt{50.2681}$ or to four figures, $\sqrt{50.27}$. Referring to the table, the first two figures of the root are 7.0; the first difference is 50.41 − 49.00 = 1.41; the second difference is 50.27 − 49.00 = 1.27; 1.27 $\div$ 1.41 = .9+. Therefore, $\sqrt{502,681} = 709$ to three figures.

(9) $$\sqrt[3]{\frac{27}{64}} = \frac{\sqrt[3]{27}}{\sqrt[3]{64}} = \frac{3}{4}. \quad \text{Ans.}$$

(10) $4^3 = 4 \times 4 \times 4 = 64.$

$$\sqrt[3]{8} = 2.$$

$$4^3 - \sqrt[3]{8} = 64 - 2 = 62. \quad \text{Ans.}$$

(11) Since $\frac{3}{8} = .375$, $\sqrt[3]{\frac{3}{8}} = \sqrt[3]{.375}$. Moving the decimal point three places to the right, the number becomes 375. Since 375 lies between $7^3 = 343$ and $8^3 = 512$, the root is 7+. Trying 7 for one of the two equal factors, the third factor is $375 \div 7^2 = 375 \div 49 = 7.65+$, and the first approximation is $\frac{2 \times 7 + 7.65}{3} = 7.21+$, or 7.2 to two figures. As the difference between the equal and unequal factors is very slight, it is not necessary to try 8.

Using 7.2 for one of the two equal factors, the third factor is $375 \div 7.2^2 = 375 \div 51.84 = 7.233+$, and the second approximation is $\frac{2 \times 7.2 + 7.233}{3} = 7.211+$, or 7.21 to three figures.

Using 7.21 for one of the two equal factors, the third factor is $375 \div 7.21^2 = 375 \div 51.9841 = 7.21374+$, and the third approximation is $\frac{2 \times 7.21 + 7.21374}{3} = 7.21124+$, or

7.2112+ to five figures. Since the number is entirely decimal, the root is wholly decimal; hence, locating the decimal point, the $\sqrt[3]{.375} = .72112+$. Ans.

By aid of the table, the first three figures are determined as follows: Move the decimal point three places to the right so it will fall between the first period 375 and the cipher period that follows.

Referring to the table, the first two figures of the root are 7.2; the first difference is $389.017 - 373.248 = 15.769$; the second difference is $375.000 - 373.248 = 1.752$; $1.752 \div 15.769 = .11+$, or .1 to one figure. Hence, the first three figures are 7.21, or the $\sqrt[3]{.375} = .721$. The fourth and fifth figures are then determined as previously indicated.

(12) $\sqrt{.3364} = ?$

Moving the decimal point two places to the right, so that the first period may be integral, the result is 33.64. The first two factors are evidently 5 and $33 \div 5 = 6.6$, and the first approximation is $\frac{5 + 6.6}{2} = 5.8$. $33.64 \div 5.8 = 5.8$. Hence, the given number is a perfect power, and as it is wholly decimal, $\sqrt{.3364} = .58$. Ans.

(13) $\sqrt{3.1416} = ?$

Pointing off, we obtain 3.14'16. The first two significant figures are 3.1. It is evident that the first figure of the root is 1, since $1^2 = 1$ and $2^2 = 4$. Using 1 as one factor, the other is $3.1 \div 1 = 3.1$, and the first approximation is $\frac{1 + 3.1}{2} = 2.05$. Had 2 been used as one factor, the other would have been $3.1 \div 2 = 1.55$, and the first approximation would have been $\frac{2 + 1.55}{2} = 1.77+$, or 1.8 to two figures. In the first case, the difference between the two factors is $3.1 - 1 = 2.1$; in the second case, the difference is $2 - 1.55 = .45$. As the factors are more nearly equal in the second case than in the first, it is evident that 1.8 is more nearly equal to the correct value of the root than 2.05 is; hence, 1.8 will be used for the first approximation.

For the second approximation, use the first two periods and 1.8 for one factor, the other factor is $3.14 \div 1.8 = 1.74+$; hence, the second approximation $= \frac{1.8 + 1.74}{2} = 1.77$. Using 1.77 for one factor, the other is $3.1416 \div 1.77 = 1.77491+$. The third approximation is $\frac{1.77 + 1.77491}{2} = 1.772455$, or $1.7725-$ to five figures. Ans.

Using the table to find the first three figures, the first two figures of the root are 1.7; the first difference is $3.24 - 2.89 = .35$; the second difference is $3.14 - 2.89 = .25$; $.25 \div .35 = .71+$. Therefore, $\sqrt{3.1416} = 1.77$ to three significant figures.

(**14**) Since some number multiplied by itself equals 114.9184, then the number is $\sqrt{114.9184}$. Pointing off into periods and placing the decimal point between the first and second periods, we have 1.14'91'84. Considering the first two figures, it is evident that the first figure of the root is 1. Using 1 as one factor, the other is $1.1 \div 1 = 1.1$, and the first approximation is $\frac{1 + 1.1}{2} = 1.05$, or 1.1 to two figures. Using the first three figures and 1.1 for one factor, the other factor is $1.15 \div 1.1 = 1.045+$, and the second approximation is $\frac{1.1 + 1.045}{2} = 1.072+$, or 1.07 to three figures. Using 1.07 for one factor, the other is $1.149184 \div 1.07 = 1.074003+$, and the third approximation is $\frac{1.07 + 1.074003}{2} = 1.072001+$, or 1.0720 to five figures. Noticing that the square of the last significant figure is $2^2 = 4$, which corresponds to the last figure of the given number, and that the fifth and sixth figures of the third approximation are ciphers, we suspect that the given number is a perfect power. We find such to be the case on squaring 1.072. Since there are two periods in the integral part of the number, there are two figures in the integral part of the root, and $\sqrt{114.9184} = 10.72$. Ans.

Using the table to find the first three figures of the root, the first two figures of the root are 1.0; the first difference

is $1.21 - 1.00 = .21$; the second difference is $1.15 - 1.00 = .15$; $.15 \div .21 = .7$. Therefore, the first three figures are 10.7.

(15) $\sqrt{3{,}486{,}784} = ?$

Pointing off into periods, we have 3'48'67'84. Placing a decimal point between the first and second periods and using the first two figures, we obtain 3.5. Considering 2 as one factor, the other is $3.5 \div 2 = 1.75$, and the first approximation is $\frac{2+1.75}{2} = 1.87+$, or 1.9 to two figures. For the second approximation, we use the first two periods and 19 for one factor, the other factor being $349 \div 19 = 18.37+$. We used 349 instead of 348 because the fourth figure was 6, and the number correct to three figures is 349. The second approximation is $\frac{19+18.37}{2} = 18.68+$, or 18.7 to three figures. $34868 \div 187 = 186.459+$; $\frac{187 + 186.459}{2} = 186.729+$, or 1867.3− to five figures. Ans.

In order to obtain three figures from the table, we place the decimal point between the first and second periods, and use the first two periods only; that is, we find the value of $\sqrt{3.49}$. Referring to the table, the first two figures of the root are 1.8; the first difference is $3.61 - 3.24 = .37$; the second difference is $3.49 - 3.24 = .25$; $.25 \div .37 = .67+$, or .7 to one figure. Therefore, $\sqrt{3{,}486{,}784} = 187$ to three significant figures.

The fourth and fifth figures may be found as previously indicated.

(16) $\sqrt{.00041209} = ?$

Pointing off into periods, the result is .00'04'12'09. Placing the decimal point between the first and second periods of the significant part of the number, we obtain 4.1 for the first two figures. The first factor is evidently 2 and the second factor $4.1 \div 2 = 2.05$. The first approximation is $\frac{2+2.05}{2} = 2.02+$, or 2.0 to two figures. Using 20 for one factor, the other is

$412 \div 20 = 20.6$, and the second approximation is $\frac{20 + 20.6}{2}$ $= 20.3$.

Using 203 for one factor, the other is $41209 \div 203 = 203$. Hence the number is a perfect power and the significant figures of the root are 203. There being one full cipher period following the decimal point, the root is .0203. Ans.

Using the table, we find that the first two figures of the root are 2.0; the first difference is $4.41 - 4.00 = .41$; the second difference is $4.12 - 4.00 = .12$; $.12 \div .41 = .3$, nearly. Therefore, $\sqrt{.00041209} = .0203$ to three significant figures.

(17) $\sqrt[5]{4,558.4} = ?$

The first period 4558 lies between $5^5 = 3125$ and $6^5 = 7776$; hence, the root is 5 plus an interminable decimal. Trying 5 as one of the four equal factors and dividing the first period by their product to find the fifth factor, we have $\frac{4558}{5 \times 5 \times 5 \times 5}$ $= 4558 \div 625 = 7.29+$. Trying 6 as one of the four equal factors, the fifth factor is $4558 \div 6^4 = 4558 \div 1296 = 3.51$. Since the difference between 5 and 7.29 is less than the difference between 6 and 3.51, we use 5 as one of the equal factors. Then, $4558 = 5 \times 5 \times 5 \times 5 \times 7.29$. The first approximation is $\frac{5 + 5 + 5 + 5 + 7.29}{5} = \frac{4 \times 5 + 7.29}{5} = \frac{27.29}{5}$ $= 5.45+$, or 5.5 to two figures.

Using 5.5 for one of the four equal factors, the fifth factor is $4558 \div 5.5^4 = 4558 \div 915.1 = 4.981-$, and the second approximation is $\frac{4 \times 5.5 + 4.981}{5} = 5.396+$, or 5.39 to three figures.

Using 5.39 for one of the four equal factors, the fifth factor is $4558.4 \div 5.39^4 = 4558.4 \div 844.019 = 5.40082+$, and the third approximation is $\frac{4 \times 5.39 + 5.40082}{5} = 5.39216+$, or $5.3922-$ to five figures. Ans.

Using the table to determine the first three figures, the first two figures are 5.3; the first difference is $4591.7 - 4182.0 = 409.7$, the second difference is $4558.4 - 4182.0$

$= 376.4$; $376.4 \div 409.7 = .91+$, or $.9$ to one figure. Therefore, $\sqrt[5]{4558.4} = 5.39$ to three figures.

The fourth and fifth figures may be found as previously indicated.

(18) $\sqrt[5]{.127} = ?$

Completing the period and neglecting the decimal point, we have 12700. Since 12700 lies between $6^5 = 7776$ and $7^5 = 16807$, the root is evidently $6+$. Trying 6 as one of the four equal factors and dividing the first period by their product to find the fifth factor, we have $\frac{12700}{6 \times 6 \times 6 \times 6} = 12700 \div 1296 = 9.79+$. Trying 7 as one of the four equal factors, the fifth factor is $12700 \div 7^4 = 12700 \div 2401 = 5.28+$. Since the difference between 7 and 5.28 is less than the difference between 6 and 9.79, use 7 as one of the equal factors. The first approximation is $\frac{7 + 7 + 7 + 7 + 5.28}{5} = 6.65+$. Using 6.7 as one of the four equal factors, the fifth factor is $12700 \div 6.7^4 = 12700 \div 2015.1 = 6.302$, and the second approximation is $\frac{4 \times 6.7 + 6.302}{5} = 6.6204+$, or 6.62 to three figures.

Using 6.62 as one of the equal factors, the fifth factor is $12700 \div 6.62^4 = 12700 \div 1920.54 = 6.61272+$, and the third approximation is $\frac{4 \times 6.62 + 6.61272}{5} = 6.61854+$, or $6.6185+$ to five figures.

Since the number is entirely decimal, the root is wholly decimal; hence, locating the decimal point, $\sqrt[5]{.127} = .66185+$. Ans.

Using the table to find the first three figures, the first two figures of the root are 6.6; the first difference is $13501 - 12523 = 978$; the second difference is $12700 - 12523 = 177$; $177 \div 978 = .18+$. Hence, the first three figures of the root are .662.

(19) $\sqrt[5]{72.415} = ?$

The first period is 72. Considering 2 as one of the four equal factors, the fifth factor is $72 \div 2^4 = 72 \div 16 = 4.5$. Considering 3 as one of the four equal factors, the fifth factor is

$72 \div 3^4 = .88+$. Since the difference between 2 and 4.5 is less than the difference between 3 and .88, use 2 for one of the equal factors. The first approximation is $\frac{2 \times 2 \times 2 \times 2 + 4.5}{5} = 2.5$.

Using 2.5 as one of the equal factors, the fifth factor is $72.415 \div 2.5^4 = 72 \div 39.06 = 1.854$ and the second approximation is $\frac{4 \times 2.5 + 1.854}{5} = 2.3708$, or 2.37 to three figures.

Since the difference between 2.5, the first approximation, and 2.37, the second approximation, is greater than one unit in the second figure, try 2.4 for one of the equal factors and recalculate the second approximation. $72.415 \div 2.4^4 = 72.415 \div 33.1776 = 2.183-$; $\frac{4 \times 2.4 + 2.183}{5} = 2.356+$, or 2.36 to three figures. Using 2.36 as one of the equal factors, the fifth factor is $72.415 \div 2.36^4 = 72.415 \div 31.0204 = 2.33443+$ and the third approximation is $\frac{4 \times 2.36 + 2.33443}{5} = 2.35488+$, or $2.3549-$ to five figures. Ans.

Using the table to find the first three figures, the first two figures of the root are 2.3; the first difference is $79.626 - 64.363 = 15.263$; the second difference is $72.415 - 64.363 = 8.052$; $8.052 \div 15.263 = .52+$, or .5 to one figure; hence, $\sqrt[5]{72\ 415} = 2.35$, to three figures. The fourth and fifth figures may be found as previously indicated.

ARITHMETIC.

(PART 6.)

---

(1) $11.7 : 13 :: 20 : x.$ The product of the means equals the product of the extremes.

$$11.7\,x = 13 \times 20$$
$$11.7\,x = 260$$

$$x = \frac{260}{11.7}$$

```
11.7) 260.000 ( 22.22+   Ans.
      234
      ---
       260
       234
       ---
        260
        234
        ---
         260
         234
         ---
          26
```

(2) (*a*) $20 + 7 : 10 + 8 :: 3 : x.$

$$27 : 18 :: 3 : x$$
$$27\,x = 18 \times 3$$
$$27\,x = 54$$
$$x = \tfrac{54}{27} = 2. \quad \text{Ans.}$$

(*b*) $12^2 : 100^2 :: 4 : x.$

$$144 : 10{,}000 :: 4 : x$$
$$144\,x = 10{,}000 \times 4$$
$$144\,x = 40{,}000$$

$$x = \frac{40{,}000}{144}$$

```
144 ) 40000.0 ( 277.7+   Ans.
      288
      ----
      1120
      1008
      -----
       1120
       1008
       -----
        1120
        1008
        -----
         112
```

(3) (*a*) $\frac{4}{x} = \frac{7}{21}$ is equivalent to $4 : x :: 7 : 21$. The product of the means equals the product of the extremes. Hence,

$$7\,x = 4 \times 21$$
$$7\,x = 84$$
$$x = \tfrac{84}{7} \text{ or } 12. \quad \text{Ans.}$$

(*b*) In like manner,

$\frac{x}{24} = \frac{8}{16}$ is equivalent to $x : 24 :: 8 : 16$.

$$16\,x = 24 \times 8$$
$$16\,x = 192$$
$$x = \tfrac{192}{16} = 12. \quad \text{Ans.}$$

(*c*) $\frac{2}{10} = \frac{x}{100}$ is equivalent to $2 : 10 :: x : 100$.

$$10\,x = 2 \times 100$$
$$10\,x = 200$$
$$x = \tfrac{200}{10} = 20. \quad \text{Ans.}$$

(*d*) $\frac{15}{45} = \frac{60}{x}$ is equivalent to $15 : 45 :: 60 : x$.

$$15\,x = 45 \times 60$$
$$15\,x = 2{,}700$$
$$x = \frac{2{,}700}{15} = 180. \quad \text{Ans.}$$

(*e*) $\frac{10}{150} = \frac{x}{600}$ is equivalent to $10 : 150 :: x : 600$.

$$150\,x = 10 \times 600$$
$$150\,x = 6{,}000$$
$$x = \frac{6{,}000}{150} = 40. \quad \text{Ans.}$$

(4) $45 : 60 :: x : 24.$

$$60\,x = 45 \times 24$$
$$60\,x = 1{,}080$$
$$x = \frac{1{,}080}{60} = 18. \quad \text{Ans.}$$

(5) $x : 35 :: 4 : 7.$

$$7\,x = 35 \times 4$$
$$7\,x = 140$$
$$x = \frac{140}{7} = 20. \quad \text{Ans.}$$

(6) $\sqrt[3]{1{,}000} : \sqrt[3]{1{,}331} :: 27 : x.$

Referring to the table, we find that the $\sqrt[3]{1{,}000} = 10$ and $\sqrt[3]{1{,}331} = 11$.

$$10 : 11 :: 27 : x.$$
$$10\,x = 297$$
$$x = \tfrac{297}{10} = 29.7. \quad \text{Ans.}$$

(7) $64 : 81 = 21^2 : x^2.$

Extracting the square root of each term of any proportion does not change its value, so we find that $\sqrt{64} : \sqrt{81} = \sqrt{21^2} : \sqrt{x^2}$ is the same as

$$8 : 9 = 21 : x$$
$$8\,x = 189$$
$$x = 23.625. \quad \text{Ans.}$$

(8) $7 + 8 : 7 = 30 : x$ is equivalent to

$$15 : 7 = 30 : x$$
$$15\,x = 7 \times 30$$
$$15\,x = 210$$
$$x = \tfrac{210}{15} = 14. \quad \text{Ans.}$$

(9) This is also a direct proportion; hence,

$$27.63 : 29.4 = .76 : x,$$

or $27.63\,x = 29.4 \times .76 = 22.344;$

whence, $x = 22.344 \div 27.63 = .808+$ lb. Ans.

(10) First cause, 5 men and 8 hours; second cause, x men, 10 hours. The effect is the amount of work, which is the same in each case.

$$\begin{array}{c|c|c}
\cancel{5} & x & \\
\cancel{8} & \cancel{10} = \text{work} & \text{work} \\
4 & \cancel{2} &
\end{array}$$

$x = 4$ men. Ans.

(11) Taking the times as the causes,

$$\begin{array}{c|c c|l}
\cancel{20} & \cancel{25} & & 14 \\
 & \cancel{5} & & \cancel{70} \\
 & & = \cancel{540} & \cancel{630} \\
\cancel{40} & x & \cancel{27} & \\
2 & & 3 &
\end{array}$$

; hence, $3x = 2 \times 14 = 28$, or $x = 9\frac{1}{3}$ hr.

(12) The proportion of lead in the ore is the ratio of $2.5 : 5$, or $2.5 \div 5 = .5$. Ans.

(13) The proportion of zinc is the ratio of $.6492 : 3.042$, or $.6492 \div 3.042 = .2134$ part zinc. Ans.

(14) The atomic weight of both is $56 + 16 = 72$. Then the proportion of oxygen is the ratio of 16 to 72, or $16 \div 72 = 22.22$ parts oxygen. Ans.

(15) The proportion of carbon is the ratio of 12 to 28, or $12 \div 28 = 42.86$ parts of carbon. Number of parts of oxygen in the gas is the ratio of $16 : 28$, or $16 \div 28 = 57.14$. Ans.

MENSURATION AND USE OF LETTERS IN FORMULAS.

(1) Substituting for D, x, B, and i their values,

$$C = \frac{D - x}{B + i} = \frac{120 - 12}{10 + 3.5} = \frac{108}{13.5} = 8. \quad \text{Ans.}$$

A line between two numbers signifies that the one above the line is to be divided by the one below the line.

(2) Substituting for A, h, D, and x their values,

$$\frac{A\,h + D}{2\,x + 6} = \frac{(5 \times 200) + 120}{(2 \times 12) + 6} = \frac{1{,}000 + 120}{24 + 6} = \frac{1{,}120}{30} = 37\tfrac{1}{3}$$

$$37\tfrac{1}{3} + D = 37\tfrac{1}{3} + 120 = 157\tfrac{1}{3}. \quad \text{Ans.}$$

When there is no sign between the letters, multiplication is understood.

(3) Substituting for A, D, i, and B their values,

$$v = \sqrt{\frac{A\,D}{i\,B + 1.5}} = \sqrt{\frac{5 \times 120}{(3.5 \times 10) + 1.5}} = \sqrt{\frac{600}{36.5}}$$

$$= \sqrt{16.4383} = 4.05+. \quad \text{Ans.}$$

The square root sign extends over both numerator and denominator, thus indicating that the square root of the entire fraction is to be extracted.

(4) Substituting for A, B, D, and h their values,

$$g = \frac{(B - A)^2 - \sqrt{h + 2\,B + A}}{A^3 - (1 + D)} = \frac{(10 - 5)^2 - \sqrt{200 + 2 \times 10 + 5}}{5^3 - (1 + 120)}$$

$$= \frac{5^2 - \sqrt{225}}{125 - 121} = \frac{25 - 15}{4} = \frac{10}{4} = 2\tfrac{1}{2}. \quad \text{Ans.}$$

(5) When one straight line meets another straight line, two angles are formed which together equal 180°. Hence, if one of the angles = 152° 3′, the other angle = 180° − 152° 3′, or

$$\begin{array}{rr} 180^\circ = & 179^\circ\ 60' \\ \text{subtracting,} & 152^\circ\ \ 3' \\ \hline & 27^\circ\ 57' \end{array} \quad \text{Ans.}$$

(6) See Arts. **26-28.**

(7) See Art. **41.** A rectangle with the same area would have the same base and altitude.

(8) Since the area is to be found in square inches, the $2\frac{1}{2}$ feet must be reduced to inches. $2\frac{1}{2}$ ft. = 30 in. Area $= 30 \times 11\frac{1}{2} = 345$ sq. in. Ans.

(9) It will take $1\frac{1}{2}$ boards to reach lengthways of the room. Since the room is 15 feet wide and each board is 5 *inches* wide, it will take $15 \div \frac{5}{12} = 36$ boards, laid side by side, to extend across the width of the room. Hence, number of boards required $= 36 \times 1\frac{1}{2} = 54$. Ans.

(10) The total area of the floor of the station $= 55 \times 58$ ft. = 3,190 sq. ft. − 25×26 ft. = 650 sq. ft., the area represented by the lower right-hand corner of the figure. Hence, total area of floor = 3,190 − 650 = 2,540 sq. ft.

From this we have to deduct the following areas:

$$\begin{array}{lcrl} \text{2 boilers} & = 2 \times 8 \times 19 = & 304 & \text{sq. ft.} \\ \text{Feed-pump} & = \quad 2\frac{1}{2} \times 5 \quad = & 12.5 & \text{sq. ft.} \\ \text{2 engines} & = 2 \times 4\frac{1}{2} \times 10 = & 90 & \text{sq. ft.} \\ \text{2 dynamos} & = 2 \times 5\frac{1}{2} \times 6\frac{1}{2} = & 71.5 & \text{sq. ft.} \\ \text{Switchboard} & = \dfrac{10 \times 3.5}{12} \quad = & 2.92 & \text{sq. ft.} \\ \hline & & 480.92 & \text{sq. ft.} \end{array}$$

The unoccupied floor space, therefore, equals

$$2{,}540 - 480.92 = 2{,}059.08 \text{ sq. ft.} \quad \text{Ans.}$$

(11) A triangle with three equal angles has three equal sides, and is therefore an equilateral triangle.

(12) A triangle with two equal angles has two equal sides, and is therefore an isosceles triangle.

(13) The sum of the three angles in any triangle = 2 right angles, or 180°. In the given triangle, the sum of two angles $= 23° + 32° \ 32' = 55° \ 32'$, and the third angle $= 180° - 55° \ 32'$, or

$$\begin{array}{rr} 180° = & 179° \ 60' \\ \text{subtracting,} & 55° \ 32' \\ \hline & 124° \ 28' \quad \text{Ans.} \end{array}$$

(14) In Fig. I we have the proportion $AD : DE :: AB : BC$, in which $AD = 10$ in., $AB = 24$ in., and $BC = 13\frac{1}{2}$ in., to find DE.

Substituting the given values,

$$10 : DE :: 24 : 13\tfrac{1}{2}, \text{ or}$$

$$DE = \frac{10 \times 13.5}{24} = 5.625 \text{ in. Ans.}$$

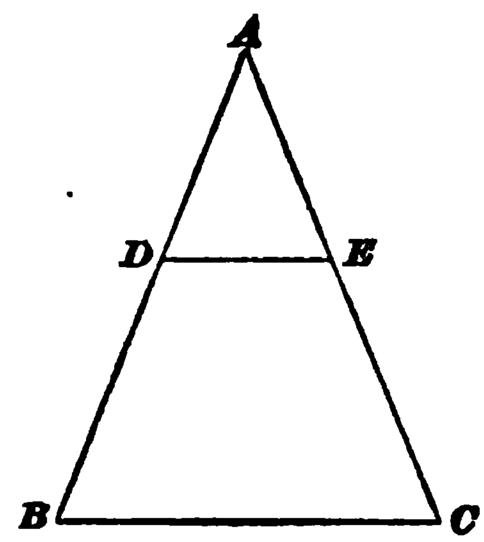


FIG. I.

(15) A line drawn diagonally from one corner to the opposite one would form the hypotenuse of a right triangle, whose two sides are 39 and 52 feet. By rule **6**, Art. **58**, the length of the diagonal $= \sqrt{52^2 + 39^2} = 65$ ft. Ans.

(16) See example, Art. **64.** The process is simply to find one of the angles of the polygon, and then to divide it by 2. By rule **10**, Art. **64**, one of the interior angles $= \frac{180 \times (8 - 2)}{8} = 135°$. This divided by $2 = 67\frac{1}{2}°$. Ans.

(17) Since this is a regular hexagon, it may be inscribed in a circle (Fig. II), and the radius of the inscribing circle will be equal to one side of the hexagon. Since the diameter $EF = 2$ inches, the radii AB and AC, and the side BC each = 1 inch, and the triangle ABC is equilateral. Draw the line AD perpendicular to the side BC; it will bisect BC. Then, in the right-angled triangle ADB, $AB = 1''$ and $BD = \frac{1}{2}''$,

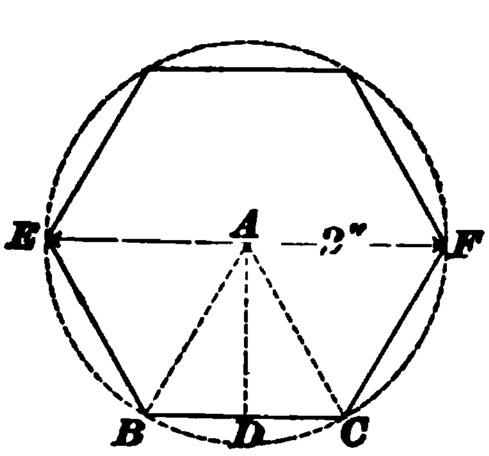


FIG. II.

to find $A D$. According to rule **7**, Art. **59**, $A D = \sqrt{1^2 - .5^2} = \sqrt{.75} = .866''$. Hence, the distance between two opposite sides of the hexagon $= A D \times 2 = .866 \times 2 = 1.732''$. Ans.

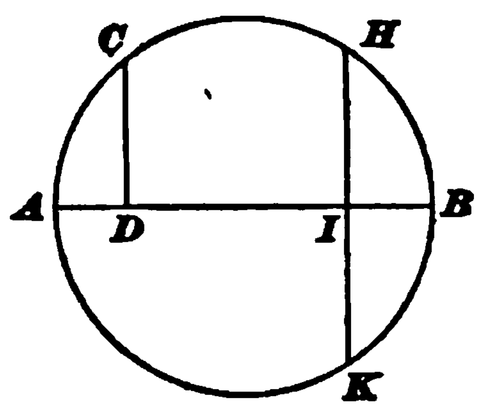


FIG. III.

(18) In Fig. III, we have the proportion $B I : H I :: H I : I A$, in which $B I = 6$ and $H I = \frac{1}{2}$ of $H K = \frac{18}{2} = 9$. Substituting, $6 : 9 :: 9 : I A$, or $I A = \frac{81}{6} = 13.5$ in. Hence, the diameter $A B = I A + B I = 13.5 + 6 = 19.5$ in. Ans.

(19) One mile = 5,280 feet. The circumference of the wheel in feet $= \frac{72 \times 3.1416}{12} = 18.8496$. (See rule **12**, Art. **77**.) Number of revolutions in going 1 mile $= 5{,}280 \div 18.8496 = 280.112$. Ans.

(20) Using rule **15**, Art. **80**, area = diameter squared $\times .7854$. $6.06^2 = 36.7236$; $36.7236 \times .7854 = 28.8427$ sq. in. Ans.

(21) Since the radius of the circle = 6 in., its diameter = 12 in., and its circumference $= 12 \times 3.1416 = 37.6992$ in. There are 360° in the circumference, and the length of an arc of $12° = 37.6992 \times \frac{12}{360} = 1.25664$ in. Ans.

(22) The area of a circle 15 in. in diameter $= 15^2 \times .7854 = 176.715$ sq. in. Hence, the area of a sector of this circle whose angle is $12\frac{1}{2}° = 176.715 \times \frac{12\frac{1}{2}}{360} = \frac{2{,}208.937}{360} = 6.1359$ sq. in. Ans. (See rule **17**, Art. **82**.)

(23) (*a*) The side of a square whose area = 103.8691 sq. in. $= \sqrt{103.8691} = 10.1916$ in. Ans.

(*b*) By rule **16**, Art. **81**, the diameter of a circle having the same area $= \sqrt{\frac{103.8691}{.7854}} = 11\frac{1}{2}$ in. Ans.

(*c*) Perimeter of the square $= 10.1916 \times 4 = 40.7664$ in.; circumference of the circle $= 11.5 \times 3.1416 = 36.1284$ in.; difference $= 40.7664 - 36.1284 = 4.638$ in. Ans.

(24) The perimeter of the base $= 4 \times 6 = 24$ in. $= 2$ ft. Convex area $= 2 \times 12 = 24$ sq. ft. The area of the bases is found as follows: In Fig. IV, $AB = 4$ in. and $AC = 2$ in.; since this is a regular hexagon, $AO = AB = 4$ in. By rule **7**, Art. **59**, $OC = \sqrt{4^2 - 2^2} = \sqrt{12} = 3.4641$ in.; area of triangle $AOB = \frac{4 \times 3.4641}{2} = 6.9282$ sq. in.; area of base $= 6.9282 \times 6 = 41.5692$; and the area of both bases $= 41.5692 \times 2 = 83.1384$ sq. in. This reduced to square feet $= \frac{83.1384}{144} = .5774$. Hence, the area of the entire surface of the column is $24 + .5774 = 24.5774$ sq. ft. Ans.

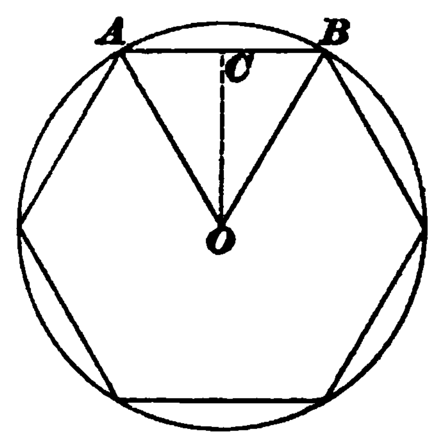


FIG. IV.

(25) The cubical contents in cubic inches = area of base in square inches $\times$ altitude in inches. The area of the base in the last example was found to be 41.5692 sq. in.; altitude $= 12 \times 12 = 144$ in. Hence, the cubical contents $= 41.5692 \times 144 = 5{,}985.9648$ cu. in. Ans.

(26) This example is solved by combining the rules for the circular ring (see example, Art. **81**) and for the cylinder. To obtain the area of one end of the tube, we have $4^2 \times .7854 = 12.5664$ = area of a circle 4 inches in diameter; $3.73^2 \times .7854 = 10.9272$ = area of a circle 3.73 inches in diameter; difference $= 12.5664 - 10.9272 = 1.6392$ = area of one end of the tube. The cubical contents $= 1.6392 \times 12 = 19.6704$ cu. in.; the weight $= 19.6704 \times .28 = 5.5$, or $5\frac{1}{2}$ lb. Ans.

(27) This example is done exactly like the one in Art. **92**, and the solution is given here without explanation.

(*a*) In the formula of rule **18**, Art. **83**, $\frac{4h^2}{3}\sqrt{\frac{D}{h} - .608}$, h in this case $= 18$, and $D = 60$.

Substituting, area =

$$\frac{4 \times 18^2}{3} \sqrt{\frac{60}{18} - .608} = \frac{4 \times 324}{3} \sqrt{3.333 - .608} = 432 \times \sqrt{2.725}$$

$= 432 \times 1.65 = 712.8$ sq. in. This reduced to square feet $= 712.8 \div 144 = 4.95$. Hence, the steam space $= 4.95 \times 16 = 79.2$ cu. ft. Ans.

(*b*) Total area of one end of boiler in square inches $= 60^2 \times .7854 = 2{,}827.44$. From this is to be subtracted the area of the tube ends and of the segment found above.

Area of ends of tubes $= 3.5^2 \times .7854 \times 64 =$	615.75 sq. in.
Area of segment	712.8 sq. in.
	1,328.55 sq. in.

Area of water space $= 2{,}827.44 - 1{,}328.55 = 1{,}498.89$ sq. in.

Contents of water space $= 1{,}498.89 \times 16 \times 12 = 287{,}786.88$ cu. in., and $287{,}786.88 \div 231 = 1{,}245.83$, number of gallons, or say 1,246 gal. Ans.

(28) The area of the convex surface = circumference of base $\times \frac{1}{2}$ slant height $= 18.8496 \times \frac{10}{2} = 94.248$ sq. in. (See rule **21**, Art. **97**.) The area of the entire surface = 94.248 sq. in. + the area of the base. The diameter of the base $= \frac{18.8496}{3.1416} = 6$ in.; hence, the area of the base $= 6^2 \times .7854 = 28.2744$ (rules **13** and **15**, Arts. **78** and **80**); therefore, the area of the entire surface $= 94.248 + 28.2744 = 122.5224$ sq. in. Ans.

(29) Using rule **22**, Art. **98**, volume = area of base $\times \frac{1}{3}$ altitude $= 28.2744 \times \frac{9}{3} = 84.8232$ cu. in. Ans.

(30) The vat has the form of an inverted frustum of a pyramid. Area of larger base $= 15^2 = 225$ sq. ft.; area of smaller base $= 12^2 = 144$ sq. ft. Hence, by rule **24**, Art. **102**, the contents of the vat in cubic feet =

$$(225 + 144 + \sqrt{225 \times 144})\frac{11}{3} = (369 + 180) \times \frac{11}{3} = 549 \times \frac{11}{3}$$

$= 2{,}013$ cu. ft. This should be reduced to cubic inches by

multiplying by 1,728, the number of cubic inches in a cubic foot. $2,013 \times 1,728 = 3,478,464$ cu. in. Since there are 231 cubic inches in a gallon, the number of gallons that the vat will hold $= \frac{3,478,464}{231} = 15,058.29$. Ans.

(31) The pail is in the form of a frustum of a cone. Area of larger base $= 12^2 \times .7854 = 113.0976$ cu. in. Area of smaller base $= 63.6174$ cu. in. Hence, the contents in cubic inches =

$$(113.0976 + 63.6174 + \sqrt{113.0976 \times 63.6174}) \times \frac{11}{3}$$

$$= (176.715 + \sqrt{7,194.9753})\frac{11}{3} = (176.715 + 84.8232) \times \frac{11}{3}$$

$$= 261.5382 \times \frac{11}{3} = 958.9734.$$

The contents of the vat in cubic inches were found in the last example to be 3,478,464. Hence, the number of pails of water required to fill the vat $= 3,478,464 \div 958.9734 = 3,627.28$. Ans.

(32) (*a*) By rule **25**, Art. **104**, area of the surface $= 22.5^2 \times 3.1416 = 506.25 \times 3.1416 = 1,590.435$ sq. in. Ans.

(*b*) Using rule **26**, Art. **105**, the cubical contents = the cube of the diameter $\times .5236 = 11,390.625 \times .5236 = 5,964.1313$ cu. in. Ans.

(33) (*a*) Given $OB = \frac{16}{2}$, or 8 inches, and $OA = \frac{13}{2}$, or $6\frac{1}{2}$ inches, to find the volume, area, and weight (see Fig. V):

Radius of center circle equals $\frac{8 + 6.5}{2}$, or $7\frac{1}{4}$ inches.

Length of center line $= 2 \times 3.1416 \times 7\frac{1}{4} = 45.5532$ inches.

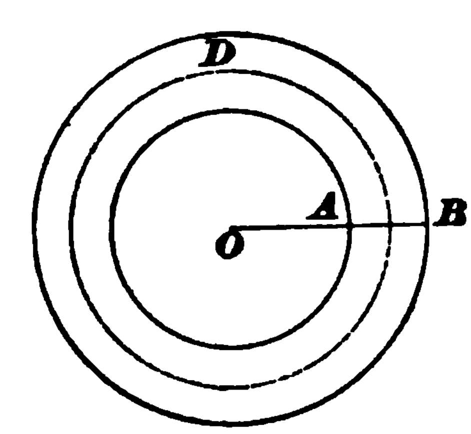


FIG. V.

The radius of the inner circle is $6\frac{1}{2}$ inches and of the outer circle 8 inches; therefore, the diameter of the cross-section on the line AB is $1\frac{1}{2}$ inches. Then, according to rule **27**, Art. **106**,

the area of the ring is $1\frac{1}{2} \times 3.1416 \times 45.553 = 214.665$ square inches. Ans.

Diameter of cross-section of ring $= 1\frac{1}{2}$ inches.

Area of cross-section of ring $= (1\frac{1}{2})^2 \times .7854 = 1.76715$ sq. in. Ans.

By rule **28**, Art. **107**, volume of ring $= 1.76715 \times 45.553$ $= 80.499$ cu. in. Ans.

(*b*) Weight of ring $= 80.499 \times .261 = 21$ lb. Ans.

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---

(1) See Art. **531**.

(2)

$$
\begin{array}{r}
a^3 - a^2 - 2a - 1)\, 2a^6 - 4a^5 - 5a^4 + 3a^3 + 10a^2 + 7a + 2\,(2a^3 - 2a^2 - 3a - 2. \\
\underline{2a^6 - 2a^5 - 4a^4 - 2a^3} \qquad\qquad\qquad\qquad\qquad\qquad \text{Ans.} \\
-2a^5 - a^4 + 5a^3 + 10a^2 \qquad\qquad\qquad\qquad\qquad\qquad\; \\
\underline{-2a^5 + 2a^4 + 4a^3 + 2a^2} \qquad\qquad\qquad\qquad\qquad\qquad\; \\
-3a^4 + a^3 + 8a^2 + 7a \qquad\qquad\qquad\qquad\qquad \\
\underline{-3a^4 + 3a^3 + 6a^2 + 3a} \qquad\qquad\qquad\qquad\qquad \\
-2a^3 + 2a^2 + 4a + 2 \qquad\qquad\qquad\qquad \\
\underline{-2a^3 + 2a^2 + 4a + 2} \qquad\qquad\qquad\qquad
\end{array}
$$

(3) (*a*)
$$
\begin{array}{l}
2 + 4a - 5a^2 - 6a^3 \\
\underline{7a^3 \qquad\qquad\qquad\qquad} \\
14a^3 + 28a^4 - 35a^5 - 42a^6.
\end{array}
$$
Ans. (Art. **493**.)

(*b*)
$$
\begin{array}{l}
4x^2 - 4y^2 + 6z^2 \\
\underline{3x^2y \qquad\qquad\qquad\quad} \\
12x^4y - 12x^2y^3 + 18x^2yz^2.
\end{array}
$$
Ans.

(*c*)
$$
\begin{array}{l}
3b + 5c - 2d \\
\underline{6a \qquad\qquad\quad} \\
18ab + 30ac - 12ad.
\end{array}
$$
Ans.

(4) Let x = number of miles he traveled per hour.

Then, $\frac{48}{x}$ = time it took him.

$\frac{48}{x+4}$ = time it would take him if he traveled 4 miles more per hour.

In the latter case the time would have been 6 hours; hence the equation

$$\frac{48}{x+4} = 6.$$

Clearing of fractions, $48 = 6\,(x + 4)$.
Dividing by 6, $8 = x + 4$.
Transposing, $8 - 4 = x$.
$x = 4$ miles. Ans.

(**5**) The square root of the fraction a plus b plus c over n, plus the square root of a, plus the fraction b plus c over n, plus the square root of the quantity a plus b, plus the fraction c over n, plus the parenthesis a plus b, times c, plus a plus bc.

(**6**) (*a*) Writing the work as follows, and canceling common factors in both numerator and denominator (Arts. **545** and **546**), we have

$$\frac{9m^2n^2}{8p^3q^2} \times \frac{5p^2q}{2xy} \times \frac{24x^2y^2}{90mn}$$
$$= \frac{9 \times 5 \times 24 \times m^2 \times n^2 \times p^2 \times q \times x^2 \times y^2}{8 \times 2 \times 90 \times m \times n \times p^3 \times q^2 \times x \times y} = \frac{3mnxy}{4pq^2}. \quad \text{Ans.}$$

(*b*) This problem may be written as follows, according to Art. **529**,

$$\frac{3ax + 4}{1} \times \frac{a^2}{a(3ax + 4)\;(3ax + 4)}.$$

Canceling a and $(3ax + 4)$, we have $\frac{a}{3ax + 4}$. Ans.

(**7**) Let x = the capacity.
Then, $x - 42$ = amount held at first.

$$7(x - 42) = x.$$
$$7x - 294 = x.$$
$$6x = 294.$$
$x = 49$ gallons. Ans.

(8) $\frac{c(a+b)+cd}{(a+b)c} = \frac{ac+bc+cd}{ac+bc}$. Canceling c, which is common to each term, we have $\frac{a+b+d}{a+b} = 1 + \frac{d}{a+b}$.
Ans.

(9) $\frac{3+2x}{2-x} - \frac{2-3x}{2+x} + \frac{16x-x^2}{x^2-4}$. If the denominator of the third fraction were written $4-x^2$, instead of x^2-4, the common denominator would then be $4-x^2$.

By Art. **531**, $\frac{16x-x^2}{x^2-4}$ becomes $-\frac{16x-x^2}{-x^2+4} = -\frac{16x-x^2}{4-x^2}$.

Hence, $\frac{3+2x}{2-x} - \frac{2-3x}{2+x} - \frac{16x-x^2}{4-x^2}$, when reduced to a common denominator, becomes

$$\frac{(3+2x)(2+x)-(2-3x)(2-x)-(16x-x^2)}{4-x^2}$$
$$= \frac{(6+7x+2x^2)-(4-8x+3x^2)-(16x-x^2)}{4-x^2}.$$

Removing the parentheses (Art. **482**), we have

$$\frac{6+7x+2x^2-4+8x-3x^2-16x+x^2}{4-x^2}.$$

Combining like terms in the numerator, we have

$$\frac{2-x}{4-x^2}.$$

Factoring the denominator (Art. **523**), we have

$$\frac{2-x}{(2+x)(2-x)}.$$

Canceling the common factor $(2-x)$, the result is equal to

$$\frac{1}{2+x}, \text{ or } \frac{1}{x+2}. \quad \text{Ans.} \quad (\text{Art. } \mathbf{456}.)$$

(10) If none of the terms is similar, the subtraction of one expression from another may be represented only, by connecting the subtrahend with the minuend by means of the sign $-$. Thus, if it is required to subtract $5a^2b - 7a^2b^2 + 5ab^3$ from $a^4 - b^4$, the result will be represented by $a^4 - b^4$

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$-(5a^3b - 7a^2b^2 + 5ab^3)$, which, on removing the parenthesis (Art. **482**), becomes $a^4 - b^4 - 5a^3b + 7a^2b^2 - 5ab^3$. From this result subtract $3a^4 - 4a^3b + 6a^2b^2 + 5ab^3 - 3b^4$.

$$\begin{array}{rl} a^4 - b^4 - 5a^3b + 7a^2b^2 - 5ab^3 & \textit{minuend.} \\ -3a^4 + 3b^4 + 4a^3b - 6a^2b^2 - 5ab^3 & \textit{subtrahend}\text{, with signs changed.} \\ \hline -2a^4 + 2b^4 - a^3b + a^2b^2 - 10ab^3 & \textit{remainder.} \qquad \text{(Art. 479.)} \end{array}$$

Or, $-2a^4 - a^3b + a^2b^2 - 10ab^3 + 2b^4$. Ans., arranged according to the decreasing powers of a.

(11) (*a*) If the work done *on* a piston by the confined gases be considered positive, then the work done *by* the piston in pushing the gas out of the exhaust port may be considered negative.

(*b*) While in arithmetic we can add and subtract only positive quantities, in algebra we can perform these operations on both positive and negative quantities.

(12) (*a*) The value of a^0 is 1. (Art. **503.**)

(*b*) $\frac{a^0}{a^{-1}} = a$. Ans. (Art. **565.**)

(13) $-\frac{c-(a-b)}{c+(a+b)} = \frac{(a-b)-c}{c+(a+b)}$. Ans. (Art. **531.**)

(14) (*a*) By Art. **530,** the reciprocal of $\frac{24}{49} = 1 \div \frac{24}{49} = 1 \times \frac{49}{24} = \frac{49}{24}$. Ans.

(*b*) Since, by Art. **530,** a number may be found from its reciprocal by dividing 1 by the reciprocal, the number $= 1 \div 700 = .0014\frac{2}{7}$. Ans.

(15) (*a*) See Art. **445.**

(*b*) In multiplication, coefficients are multiplied, and exponents are added. In division, the coefficients of the dividend are divided by those of the divisor, and the exponents of the divisor are subtracted from those of the dividend. See rules of multiplication and division.

(*c*) See Art. **487.**

(16) (*a*) $$\frac{9x+20}{36} = \frac{4(x-3)}{5x-4} + \frac{x}{4}.$$

When the denominators contain both simple and compound expressions, it is best to remove the simple expressions first, and then remove each compound expression in order. Then, after each multiplication, the result should be reduced to the simplest form.

Multiplying both sides by 36,

$$9x + 20 = \frac{144\,(x - 3)}{5x - 4} + 9x,$$

or

$$\frac{144x - 432}{5x - 4} = 20.$$

Clearing of fractions,

$$144x - 432 = 100x - 80.$$

Transposing and combining,

$$44x = 352;$$

whence

$$x = 8. \quad \text{Ans.}$$

(*b*) $ax - \dfrac{3a - bx}{2} = \dfrac{1}{2}$ becomes, when cleared of fractions,

$$2ax - 3a + bx = 1.$$

Transposing and uniting terms,

$$2ax + bx = 3a + 1.$$

Factoring,

$$(2a + b)x = 3a + 1;$$

whence

$$x = \frac{3a + 1}{2a + b}. \quad \text{Ans.}$$

(*c*) $am - b - \dfrac{ax}{b} + \dfrac{x}{m} = 0$ becomes, when cleared of fractions,

$$abm^2 - b^2m - amx + bx = 0.$$

Transposing,

$$bx - amx = b^2m - abm^2.$$

Factoring,

$$(b - am)x = bm(b - am);$$

whence

$$x = \frac{bm(b - am)}{(b - am)} = bm. \quad \text{Ans.}$$

(17) (*a*) *a* square *x* square, plus two *a* cube *b* fifth, minus the parenthesis *a* plus *b*.

(*b*) The cube root of *x*, plus *y* times the two-thirds power of the parenthesis *a* minus *n* square.

(*c*) The parenthesis *m* plus *n*, times the square of the parenthesis *m* minus *n*, times the parenthesis *m* minus the fraction *n* over two.

(18) (*a*) $16a^2b^2$; a^4+4ab; $4a^2-16a^2b+5a^6+7ax$.

(*b*) Since the terms are not alike, we can only indicate the sum, connecting the terms by their proper signs. (Art. **476.**)

(*c*) Multiplication: $4ac^2d$ means $4\times a\times c^2\times d$. (Art. **441.**)

(19) (*a*) $45x^7y^{10}-90x^5y^7-360x^4y^8=$
$45x^4y^7(x^3y^3-2x-8y)$. Ans. (Art. **516.**)

(*b*) $a^2b^2+2abcd+c^2d^2=(ab+cd)^2$. Ans. (Art. **519.**)

(*c*) $(a+b)^2-(c-d)^2=(a+b+c-d)(a+b-c+d)$. Ans. (Art. **523.**)

(20) (*a*) On removing the vinculum, we have

$2a-\{3b+[4c-4a-(2a+2b)]+[3a-b-c]\}$. (Art. **482.**)

Removing the parenthesis,

$$2a-\{3b+[4c-4a-2a-2b]+[3a-b-c]\}.$$

Removing the brackets,

$$2a-\{3b+4c-4a-2a-2b+3a-b-c\}.$$

Removing the brace,

$$2a-3b-4c+4a+2a+2b-3a+b+c.$$

Combining like terms, the result is $5a-3c$. Ans.

(*b*) Removing the parenthesis, we have

$$7a-\{3a-[2a-5a+4a]\}.$$

Removing the brackets,

$$7a-\{3a-2a+5a-4a\}.$$

Removing the brace,

$$7a-3a+2a-5a+4a.$$

Combining terms, the result is $5a$. Ans.

(*c*) Removing the parenthesis, we have

$$a - \{2b + [3c - 3a - a - b] + [2a - b - c]\}.$$

Removing the brackets,

$$a - \{2b + 3c - 3a - a - b + 2a - b - c\}.$$

Removing the brace,

$$a - 2b - 3c + 3a + a + b - 2a + b + c.$$

Combining like terms, the result is $3a - 2c$. Ans.

(21) (*a*) $6a^4b^4 + a^3b^3 - 7a^2b^2 + 2abc + 3.$

(*b*) $3 + 2abc + a^2b^2 - 7a^3b^3 + 6a^4b^4.$

(*c*) $1 + ax + a^2 + 2a^3$. Written like this, the a in the second term is understood as having 1 for an exponent; hence, if we represent the first term by a^0, in value it will be equal to 1, since $a^0 = 1$. Therefore, 1 should be written as the first term when arranged according to the increasing powers of a.

(22) (*a*) According to Art. **563,** $x^{\frac{3}{4}}$ expressed radically is $\sqrt[4]{x^3}$;

$$3x^{\frac{1}{3}}y^{-\frac{1}{3}} \text{ expressed radically is } 3\sqrt[3]{xy^{-1}};$$

$$3x^{\frac{1}{6}}y^{-\frac{5}{6}}z^{\frac{1}{3}} = 3\sqrt[6]{xy^{-5}z^2}, \text{ since } z^{\frac{1}{3}} = z^{\frac{2}{6}}. \quad \text{Ans.}$$

(*b*) (See Art. **565.**) $a^{-1}b^{\frac{1}{2}} + \dfrac{c^{-2}}{a+b} + (m-n)^{-1} - \dfrac{a^2b^{-2}c}{c^{-3}}$

$$= \frac{b^{\frac{1}{2}}}{a} + \frac{1}{c^2(a+b)} + \frac{1}{m-n} - \frac{a^2c^4}{b^2}. \quad \text{Ans.}$$

(*c*) $\sqrt[7]{x^6} = x^{\frac{6}{7}}$. Ans. $\sqrt[8]{x^{-4}} = x^{-\frac{1}{2}}$. Ans.

$$(\sqrt[4]{b^5x^2})^3 = (b^{\frac{5}{4}}x^{\frac{1}{2}})^3 = b^{\frac{15}{4}}x^{\frac{3}{2}}. \quad \text{Ans.}$$

(23) (*a*) $\dfrac{2ax + x^2}{a^3 - x^3} \div \dfrac{x}{a - x} = \dfrac{x(2a + x)}{a^3 - x^3} \times \dfrac{a - x}{x}.$

(Art. **549.**)

Canceling common factors, the result is $\dfrac{2a + x}{a^2 + ax + x^2}$. Ans.

$$
\begin{array}{r|l}
a - x & a^3 - x^3 \;(a^2 + ax + x^2 \\
 & \underline{a^3 - a^2x} \\
 & \quad a^2x - x^3 \\
 & \quad \underline{a^2x - ax^2} \\
 & \qquad ax^2 - x^3 \\
 & \qquad \underline{ax^2 - x^3}
\end{array}
$$

(*b*) Inverting the divisor and factoring, we have

$$\frac{3n(2m^2n-1)}{(2m^2n-1)(2m^2n-1)} \times \frac{(2m^2n+1)(2m^2n-1)}{3n}.$$

Canceling common factors, the result is $2m^2n+1$. Ans.

(*c*) $9 + \frac{5y^2}{x^2-y^2} \div \left(3 + \frac{5y}{x-y}\right)$ simplified gives $\frac{9x^2-4y^2}{x^2-y^2} \div \frac{3x+2y}{x-y}$.

Inverting the divisor, we have $\frac{9x^2-4y^2}{x^2-y^2} \times \frac{x-y}{3x+2y}$. Canceling common factors, the result is $\frac{3x-2y}{x+y}$. Ans.

(**24**) (*a*) $\frac{x}{x-y} + \frac{x-y}{y-x}$. If the denominator of the second fraction were written $x-y$, instead of $y-x$, then $x-y$ would be the common denominator.

By Art. **531**, the signs of the denominator and the sign before the fraction $\frac{x-y}{y-x}$ may be changed, giving $-\frac{x-y}{x-y}$. We now have

$$\frac{x}{x-y} - \frac{x-y}{x-y} = \frac{x-x+y}{x-y} = \frac{y}{x-y}. \quad \text{Ans.}$$

(*b*) $\frac{x^2}{x^2-1} + \frac{x}{x+1} - \frac{x}{1-x}$. If we write the denominator of the third fraction $x-1$ instead of $1-x$, x^2-1 will then be the common denominator.

By Art. **531**, the signs of the denominator and the sign before the fraction may be changed, thereby giving $\frac{x}{x-1}$.

We now have

$$\frac{x^2}{x^2-1}+\frac{x}{x+1}+\frac{x}{x-1}=\frac{x^2+x(x-1)+x(x+1)}{x^2-1}$$

$$=\frac{x^2+x^2-x+x^2+x}{x^2-1}=\frac{3x^2}{x^2-1}.\quad \text{Ans.}$$

(*c*) $$\frac{3a-4b}{7}-\frac{2a-b+c}{3}+\frac{13a-4c}{12}$$

when reduced to a common denominator

$$=\frac{12(3a-4b)-28(2a-b+c)+7(13a-4c)}{84}.$$

Expanding the terms and removing the parentheses, we have

$$\frac{36a-48b-56a+28b-28c+91a-28c}{84}.$$

Combining like terms in the numerator, we have as the result

$$\frac{71a-20b-56c}{84}.\quad \text{Ans.}$$

(**25**) (*a*) Factoring each expression (Art. **519**), we have

$$9x^4+12x^2y^2+4y^4=(3x^2+2y^2)(3x^2+2y^2)=(3x^2+2y^2)^2.$$ Ans.

(*b*) $49a^4-154a^2b^2+121b^4=(7a^2-11b^2)(7a^2-11b^2)$ $=(7a^2-11b^2)^2$. Ans.

(*c*) $64x^2y^2+64xy+16=16(2xy+1)^2$. Ans.

(**26**) (*a*) Arrange the dividend according to the decreasing powers of x and divide. Thus,

$$\begin{array}{l} 3x-1\,)\;9x^3+3x^2+x-1\;(\;3x^2+2x+1.\quad \text{Ans.} \\ \underline{9x^3-3x^2} \\ 6x^2+\;x \\ \underline{6x^2-2x} \\ 3x-1 \\ \underline{3x-1} \end{array}$$

(*b*)

$$
\begin{array}{l}
a - b\ )\ a^3 - 2ab^2 + b^3\ (\ a^2 + ab - b^2. \quad \text{Ans.} \\
\qquad\quad \underline{a^3 - a^2b} \\
\qquad\qquad\quad a^2b - 2ab^2 \\
\qquad\qquad\quad \underline{a^2b - \ \ ab^2} \\
\qquad\qquad\qquad\ - ab^2 + b^3 \\
\qquad\qquad\qquad\ \underline{- ab^2 + b^3}
\end{array}
$$

(*c*) Arranging the terms of the dividend according to the decreasing powers of x, we have

$$
\begin{array}{l}
7x - 3\ )\ 7x^3 - 24x^2 + 58x - 21\ (\ x^2 - 3x + 7. \quad \text{Ans.} \\
\qquad\quad\ \ \underline{7x^3 - \ \ 3x^2} \\
\qquad\qquad\quad - 21x^2 + 58x \\
\qquad\qquad\quad \underline{- 21x^2 + \ \ 9x} \\
\qquad\qquad\qquad\qquad 49x - 21 \\
\qquad\qquad\qquad\qquad \underline{49x - 21}
\end{array}
$$

(27) (*a*) $(x^4 - 1) \div (x^2 + 1) = (x^2 - 1)(x^2 + 1) \div (x^2 + 1) = x^2 - 1$. Ans. (Art. **523.**)

(*b*) $x^4 - 2x^2y^2 + y^4 = (x^2 - y^2)^2$. (Art. **519.**) $(x^2 - y^2) = (x + y)(x - y)$. (Art. **523.**)

Then $(x^2 - y^2)^2 = (x^2 - y^2)(x + y)(x - y)$. Dividing this latter quantity by $(x - y)$, we have $(x^2 - y^2)(x + y)$. Ans.

Note that $x - y$ is a factor of $(x^2 - y^2)^2$ and hence of $x^4 - 2x^2y^2 + y^4$.

(28) (*a*) $$\frac{10x + 3}{3} - \frac{6x - 7}{2} = 10(x - 1).$$

Reducing the last member to a simpler form, this becomes

$$\frac{10x + 3}{3} - \frac{6x - 7}{2} = 10x - 10.$$

Clearing of fractions by multiplying each term of both members by 6, the common denominator, and changing the sign of each term of the numerator of the second fraction, since it is preceded by the minus sign, we have

$$20x + 6 - 18x + 21 = 60x - 60.$$

Transposing terms, $20x - 18x - 60x = -60 - 21 - 6.$

Combining like terms, $-58x = -87.$

Changing signs, $58x = 87;$

hence, $x = \frac{87}{58} = 1\frac{1}{2}.$ Ans.

(*b*) $(a^2 + x)^2 = x^2 + 4a^2 + a^4.$

Performing the operation indicated in the first member, the equation becomes

$$a^4 + 2a^2x + x^2 = x^2 + 4a^2 + a^4.$$

Canceling a^4 and x^2 (Art. **576**),

$$2a^2x = 4a^2.$$

Dividing by $2a^2$, $x = 2.$ Ans.

(*c*) $$\frac{x-1}{x-2} - \frac{x+1}{x+2} = \frac{3}{x^2-4}.$$

Clearing of fractions, the equation becomes

$$(x-1)(x+2) - (x+1)(x-2) = 3.$$

Expanding, $x^2 + x - 2 - x^2 + x + 2 = 3.$

Uniting terms, $2x = 3.$

$x = \frac{3}{2} = 1\frac{1}{2}.$ Ans.

(29) (*a*) $-7my\,)\,\underline{35m^2y + 28m^2y^2 - 14my^3}$

$-5m^2 - 4my + 2y^2.$ Ans. (Art. **506.**)

(*b*) $a^4\,)\,\underline{4a^4 - 3a^5b - a^6b^2}$

$4 - 3ab - a^2b^2.$ Ans.

(*c*) $4x^2\,)\,\underline{4x^3 - 8x^5 + 12x^7 - 16x^9}$

$x - 2x^3 + 3x^5 - 4x^7.$ Ans.

(30) Let x = the length of the post.

Then, $\frac{x}{5}$ = the part in the earth.

$\frac{3x}{7}$ = the part in the water.

From the conditions of the problem, we have therefore the following statement:

$$\frac{x}{5} + \frac{3x}{7} + 13 = x;$$

from which $7x + 15x + 455 = 35x;$

$$-13x = -455;$$

and $x = 35$ feet. Ans.

(31) Let $x =$ the whole quantity.

Then, $\frac{2x}{3} + 10 =$ the quantity of niter.

$\frac{x}{6} - 4\frac{1}{2} =$ the quantity of sulphur.

$\frac{1}{7}\left(\frac{2x}{3} + 10\right) - 2 =$ the quantity of charcoal.

Hence, $x = \frac{2x}{3} + 10 + \frac{x}{6} - 4\frac{1}{2} + \frac{1}{7}\left(\frac{2x}{3} + 10\right) - 2.$

Clearing of fractions and expanding terms,

$$42x = 28x + 420 + 7x - 189 + 4x + 60 - 84.$$

Transposing,

$$42x - 28x - 7x - 4x = 420 - 189 + 60 - 84.$$

$$3x = 207.$$

$x = 69$ lb., the quantity of gunpowder. Ans.

$\frac{2x}{3} + 10 = \frac{2 \times 69}{3} + 10 = 56$ lb., the quantity of niter. Ans.

$\frac{x}{6} - 4\frac{1}{2} = \frac{69}{6} - 4\frac{1}{2} = 7$ lb., the quantity of sulphur. Ans.

56 lb. $\times \frac{1}{7} - 2$ lb. $= 6$ lb., the quantity of charcoal. Ans.

(32) (*a*) See Art. **568.** The cube root of -125 is -5. Dividing each of the exponents of the literal part by 3, the index of the root, the cube root of $x^3y^6z^9$ is $x^{\frac{3}{3}}y^{\frac{6}{3}}z^{\frac{9}{3}} = xy^2z^3$; hence, $\sqrt[3]{-125x^3y^6z^9} = -5xy^2z^3$. Ans.

(*b*) and (*c*) Proceed exactly as in (*a*). $\sqrt[4]{10,000} = \pm 10$, and $\sqrt[5]{243} = 3$; $\sqrt[4]{a^{16}b^{20}c^8} = a^{\frac{16}{4}}b^{\frac{20}{4}}c^{\frac{8}{4}} = a^4b^5c^2$, and $\sqrt[5]{m^{15}n^{20}} = m^3n^4$.

Hence, $\sqrt[4]{10,000a^{16}b^{20}c^8} = \pm 10a^4b^5c^2$. Ans.

$$\sqrt[5]{243m^{15}n^{20}} = 3m^3n^4. \quad \text{Ans.}$$

(*d*) Dividing the exponent of each letter in both numerator and denominator by 5, the index of the root,

$$\sqrt[5]{-\frac{x^5y^{10}z^{15}}{a^{20}b^{15}c^{10}d^5}} = -\frac{xy^2z^3}{a^4b^3c^2d}. \quad \text{Ans.}$$

(33) See Arts. **602** and **603.**

Sin 17° 28′ = .30015.
Sin 17° 27′ = .29987.

30015 − 29987 = 28, difference for 1′.
28 × $\frac{37}{60}$ = 17, difference for 37″.
29987 + 17 = 30004.

Hence, locating the decimal point,
Sin 17° 27′ 37″ = .30004.

Cos 17° 27′ = .95398.
Cos 17° 28′ = .95389.
95398 − 95389 = 9, difference for 1′.
9 × $\frac{37}{60}$ = 6, difference for 37″.
95398 − 6 = 95392.

Hence, Cos 17° 27′ 37″ = .95392.

Tan 17° 28′ = .31466.
Tan 17° 27′ = .31434.

31466 − 31434 = 32, difference for 1′.
32 × $\frac{37}{60}$ = 20, difference for 37″.
31434 + 20 = 31454.

Hence, Tan 17° 27′ 37″ = .31454.

$$\left.\begin{array}{l}\text{Sin } 17^\circ\ 27'\ 37'' = .30004.\\ \text{Cos } 17^\circ\ 27'\ 37'' = .95392.\\ \text{Tan } 17^\circ\ 27'\ 37'' = .31454.\end{array}\right\}\text{Ans.}$$

(34) .27038 = sin 15° 41′ 12.9″. (Art. **606.**)
.27038 = cos 74° 18′ 47.1″.
2.27038 = tan 66° 13′ 43.2″.

(35) The side $BC = \sqrt{\overline{AB}^2 - \overline{AC}^2}$, or $BC = \sqrt{17.69^2 - 9.75^2}$ $= \sqrt{217.8736}$ = 14 ft. 9 in. To find the angle BAC, we have $\cos BAC = \frac{AC}{AB}$, or $\cos BAC = \frac{9.75}{17.69} = .55115$. (Art. **595.**)

.55115 = cos 56° 33′ 15″.

Angle $A\,B\,C = 90° -$ angle $B\,A\,C$, or $90° - 56°\ 33'\ 15'' = 33°\ 26'\ 45''$.

$$\left.\begin{array}{l}\text{Side } BC = 14 \text{ ft. } 9 \text{ in.}\\ \text{Angle } BAC = 56°\ 33'\ 15''.\\ \text{Angle } ABC = 33°\ 26'\ 45''.\end{array}\right\}\text{Ans.}$$

(36) The hypotenuse $A\,B = \sqrt{\overline{A\,C}^2 + \overline{B\,C}^2}$, or

$A\,B = \sqrt{17.5^2 + 21.3^2} = \sqrt{759.94} = 27.57$, nearly. Ans.

$$\text{Tan } A = \frac{A\,C}{B\,C} = \frac{17.5}{21.3} = .82160. \quad (\text{Art. } \mathbf{596.})$$

$$.82160 = \tan 39°\ 24'\ 23''.$$

Angle $A = 39°\ 24'\ 23''$. Ans.

Angle $B = 90° -$ angle $A = 90° - 39°\ 24'\ 23'' = 50°\ 35'\ 37''$. Ans.

(37) The angle $B = 90° -$ angle $A = 90° - 65°\ 13'\ 29'' = 24°\ 46'\ 31''$. Ans.

The side $B\,C = A\,B \times \sin A = 5.5$ yd. $\times \sin 65°\ 13'\ 29'' = 5.5$ yd. $\times .90796 = 4.9938$ yd. (Art. **609.**) 4.9938 yd. $= 14$ ft. $11\frac{3}{4}$ in. Ans.

Side $A\,C = A\,B \times \cos A = 5.5$ yd. $\times .41906 = 2.3048$ yd. $= 6$ ft. 11 in., nearly. Ans.

(38)

$$\begin{array}{r} 159°\ 27'\ 34.6'' \\ 25°\ 16'\ \ 8.7'' \\ 3°\ 48'\ 53.0'' \\ \hline 188°\ 32'\ 36.3'' \end{array}$$

(39) Using the proportion of Art. **615,**

$$A\,B : B\,C = \sin C : \sin A,$$

or

$$70 : 42 = \sin C : \sin 36°\ 10'.$$

Hence, $\sin C = \frac{70}{42} \times \sin 36°\ 10' = \frac{70}{42} \times .59014 = .98357$.

The angle whose sine is .98357 is $79°\ 36'$; hence, angle $C = 79°\ 36'$. Ans.

Angle $B = 180° - (A + C) = 180° - (36°\ 10' + 79°\ 36') = 64°\ 14'$. Ans.

Using the proportion again,

$$A\,C : B\,C = \sin B : \sin A;$$

or $A\,C : 42 \text{ ft.} = \sin\ 64°\ 14' : \sin\ 36°\ 10' = .90057 : .59014.$

Hence, $A\,C = \dfrac{42 \text{ ft.} \times .90057}{.59014} = 64.1$ ft., nearly. Ans.

(40) (*a*) See Art. **591.**

$180° - 72°\ 11'\ 36'' = 107°\ 48'\ 24''.$ Ans.

(*b*) See Art. **590.**

$90° - 22°\ 34'\ 17'' = 67°\ 25'\ 43''.$ Ans.

MECHANICS.

(PART 1.)

(1) See Arts. **1800** and **1801.**

(2) (*a*) Applying formula **102,**

$$D = \frac{1\frac{1}{2} \times 50}{3.1416} = 23.87 \text{ in.} \quad \text{Ans.}$$

(*b*) See Art. **1872.** Addendum = .3 of the pitch. 1.5 in. × .3 = .45 in. .45 in. × 2 = .9 in. = difference between the diameter of the pitch circle and the outside diameter Hence, outside diameter = 23.87 + .9 = 24.77 in. Ans.

(3) Apply formula **110.**
Pitch = $\frac{1}{11}$ in.; therefore,

$$W = \frac{6.2832 \times 24 \times 11}{\frac{1}{11}} = 21{,}563.94 \text{ lb.} \quad \text{Ans.}$$

(4) The pull on the support equals the centrifugal force of the ball. Hence, applying formula **112,**

$$F = .00034 \times 5 \times \tfrac{32}{12} \times 350^2 = 555\tfrac{1}{3} \text{ lb.} \quad \text{Ans.}$$

(5) Apply formula **113.**

$$K = \frac{2 \times 600^2}{64.32} = 11{,}194 \text{ ft.-lb.} \quad \text{Ans.}$$

(6) 7 ft. = 84 in. Arc of contact $= \dfrac{84}{63 \times 3.1416} \times 360°$ = 153°. 800 + 3 (180 − 153) = 881. Applying formula **115,**

$$W = \frac{881 \times 150}{3{,}000} = 44.05 \text{ in.}$$

Using formula **117**,

$$W_1 = 44.05 \times \tfrac{2}{3} = 29.37 \text{ in., or say } 29.5 \text{ in.} \quad \text{Ans.}$$

(7) See Arts. **1802** to **1823**.

(8) See Art. **1840**.

(9) See Arts. **1860** and **1861**.

(10) 13 ft. = 156 in. Applying formula **99**,

$$N = \frac{91 \times 108}{156} = 63 \text{ rev. per min., the speed of the engine.}$$
Ans.

(11) See Art. **1899**.

(12) Arc of contact $= \dfrac{18}{14 \times 3.1416} \times 360° = 147°$. $800 + 3\,(180 - 147) = 899$. Applying formula **115**,

$$W = \frac{899 \times 2.5}{2{,}000} = 1.12 \text{ in., say 1 in.} \quad \text{Ans.}$$

(13) (*a*) See Arts. **1810** and **1835**.

(*b*) and (*c*) See Art. **1809**.

(14) $\frac{48}{12} \times 3.1416 = 12.5664$ ft. = circumference of pulley. $\dfrac{3{,}000}{12.5664} = 238.73$ revolutions in 1 minute, or 60 seconds. To make 100 revolutions will require $\dfrac{100}{238.73} \times 60 = 25.13$ sec., nearly. Ans.

(15) 4 ft. 6 in. = 54 in. $54 \times 2 \times \frac{3}{4} \times .261 = 21.141$ lb. = weight of lever. Considering the weight of the lever to be concentrated at its center of gravity, we have three weights of 47, 21.141, and 71 lb., with the smaller weight $\frac{54}{2} = 27$ in. from the other two. To find the center of gravity of the two large weights, apply formula **93**. $l_1 = \dfrac{47 \times 54}{71 + 47} = 21.508$ in. = the distance bc in Fig. I. Consider both weights to be concentrated at b; that is, imagine both weights removed and to be replaced by the dotted

weight W, equal to $71 + 47 = 118$ lb. The dotted circle w represents the weight of the bar. The distance $a\,e = 27$

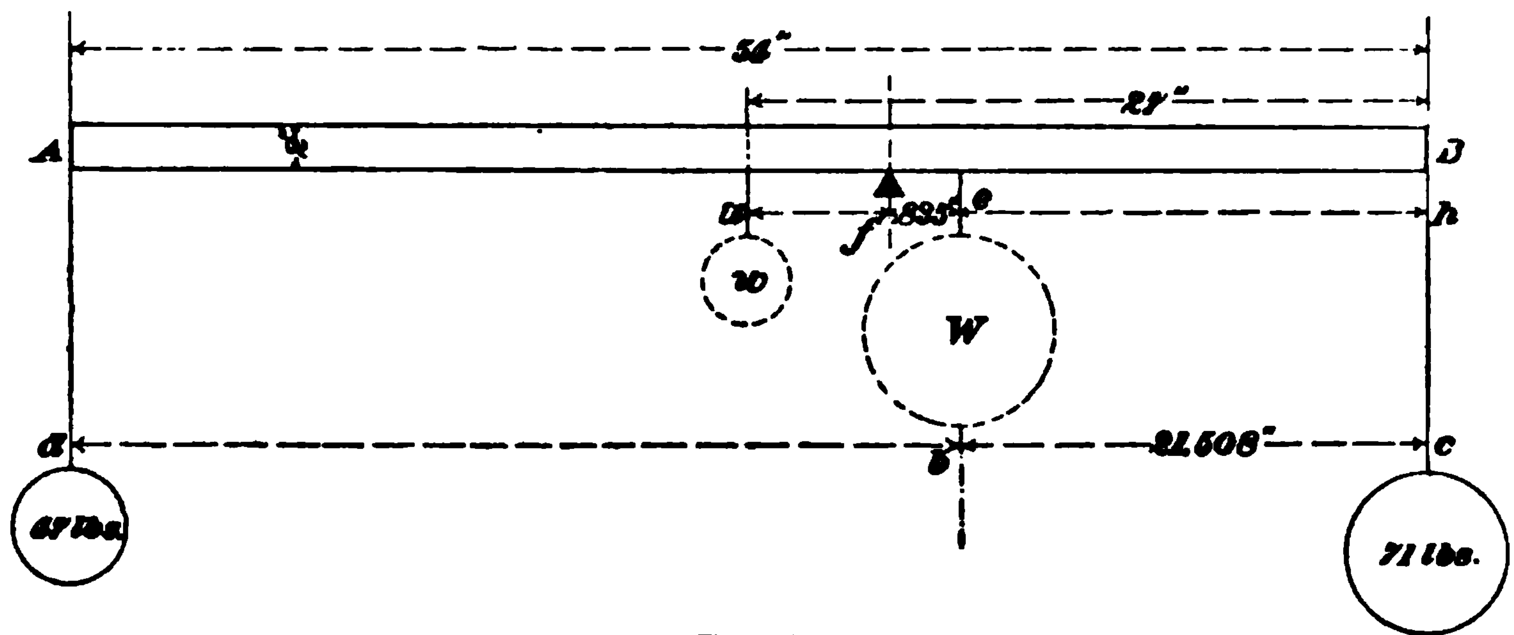

FIG. I.

$- 21.508 = 5.492$ in. Distance of balancing point f from e is found by means of formula **93** to be $\dfrac{21.141 \times 5.492}{118 + 21.141}$ $= .835$ in. Finally, $f\,h =$

$21.508 + .835 = 22.343$ in. = short arm. Ans.
$54 - 22.343 = 31.657$ in. = long arm. Ans.

(16) See Art. **1899.** $\dfrac{51}{62.5} = .816$, the specific gravity. Ans.

(17) See Arts. **1824** and **1826.**

(18) See Art. **1830.**

(19) In Fig. II, $A\,B\,C$ represents the triangle. The center of gravity is found as explained in Art. **1845.** The distance of the center of gravity from the side $A\,C = 1\frac{3}{32}$ in. Ans.

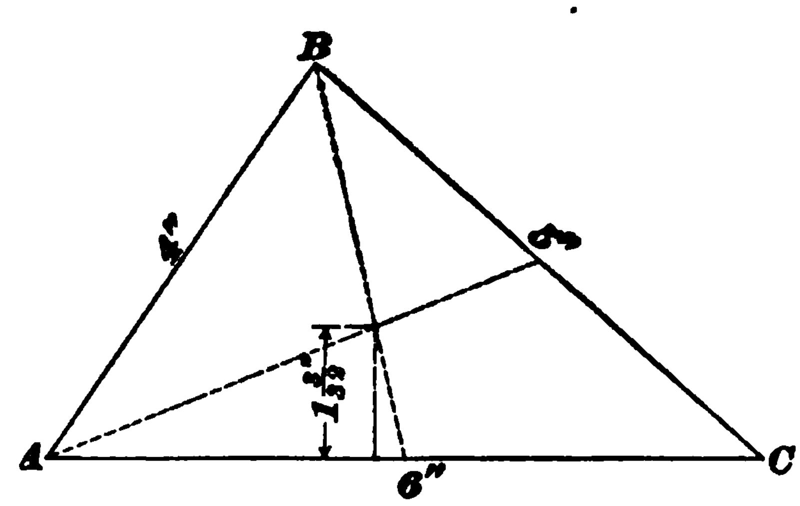

FIG. II.

(20) Speed of a point on the pitch circle in feet per minute $= \frac{30}{12} \times 3.1416 \times 100 = 785.4$ ft. per min. Apply formula **109.**

H. P. $= .01 \times 785.4 \times 1.57^2 = 19.36$. Ans.

(**21**) See Art. **1826.**

(**22**) Applying formula **100,**

$$P = \frac{6,000 \times 6 \times 5 \times 8 \times 3}{18 \times 12 \times 15 \times 12} = 111\tfrac{1}{9}\text{ lb.}$$

Since there is a loss of 20%, $111\frac{1}{9}$ represents 80% of the total force. Hence, the force actually required $= 111\frac{1}{9} \div .80 = 138\frac{8}{9}$ lb. Ans.

(**23**) Apply formula **104.**

$$P = \frac{3.1416 \times 24.16}{38} = 1.9974 \text{ in. Ans.}$$

(**24**) See Art. **1858.** Since there are eight parts of the rope, the force required $= 1,890 \div 8 = 236\frac{1}{4}$ lb. Ans.

(**25**) Volume $= (\frac{1}{2})^2 \times .7854 \times 10 = 1.963$ cu. in. One cu. in. of lead weighs .411 lb. (see table of Weights per Cubic Foot); consequently, $1.963 \times .411 = .807$ lb. $= 12.91$ oz. Ans.

(**26**) Length of power-arm = 4 ft. − 4 in. = 48 in. − 4 in. = 44 in. According to formula **94,** $P \times 44 = 1,500 \times 4 = 6,000$; hence, $P = \frac{6,000}{44} = 136\frac{4}{11}$ lb. Ans.

(**27**) Length of power-arm = 4 ft. = 48 in. Hence, as in the preceding question, $P = \frac{6,000}{48} = 125$ lb. Ans.

(**28**) See Arts. **1869** and **1870.**

(**29**) See Art. **1885.** $4,000 \times 45 = 400 \times$ the force. Hence, force $= \frac{4,000 \times 45}{400} = 450$ lb. Ans.

(**30**) 14 ft. = 168 in. Applying formula **114,**

$$B = 3\tfrac{1}{4} \times \frac{18 + 14}{2} + 2 \times 168 = 388 \text{ in.} = 32 \text{ ft. } 4 \text{ in. Ans.}$$

(**31**) See Arts. **1871** and **1872.**

(**32**) See Arts. **1876** and **1877.**

(33) The weight which comes on the block and tackle is the same as the force required to pull the body up the plane, or is equal to $\frac{50,000 \times 125}{1,200} = 5,208\frac{1}{3}$ lb. Since there are 12 parts to the rope, the force required to be exerted on the free end is $5,208 \div 12 = 434$ lb. Ans.

(34) See Art. **1898.**

(35) See Art. **1838.**

(36) Applying formula **95,** letting P represent the required force,

$$P \times 30 \times 20 \times 10 \times 15 = 1,250 \times 6 \times 5 \times 4 \times 7,$$

or
$$P = \frac{1,250 \times 6 \times 5 \times 4 \times 7}{30 \times 20 \times 10 \times 15} = 11\tfrac{2}{3} \text{ lb. Ans.}$$

(37) See Art. **1872.**

(38) One cubic foot of water weighs 62.5 lb.; hence, 20 cu. ft. weigh $62.5 \times 20 = 1,250$ lb. The work done $= 1,250 \times 50 = 62,500$ ft.-lb. Ans.

(39) $18,000 + 10,000 = 28,000$ lb. = the load which the screw must overcome.

Using formula **111,**

$$P = \frac{\frac{1}{3} \times 28,000}{6.2832 \times 15} = 99 \text{ lb., nearly. Ans.}$$

(40) $30 \times 14\frac{1}{2} \times 2 = 870$. $870 \div 5 = 174$ lb. Ans.

MECHANICS.

(PART 2.)

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(1) That force which will produce the same final effect upon a body as all the other forces acting separately or together.

(2) This example is solved by the parallelogram of forces, as in Art. **1917.** Measuring the diagonal, the total pressure on the shaft is found to be $7\frac{1}{4}$ tons, nearly. Ans.

(3) See Arts. **1932** and **1933.**

(4) Applying formula **122,**

$$W = 12{,}000 \times (\tfrac{3}{8})^2 = 1{,}687.5 \text{ lb.} \quad \text{Ans.}$$

(5) Apply formula **125,** and use 1,000 instead of 600, as the rope is of steel.

$$W = 1{,}000 \times (5\tfrac{1}{4})^2 = 27{,}562.5 \text{ lb.} \quad \text{Ans.}$$

(6) (*a*) If a 5-inch line = 20 lb., a 1-inch line = 4 lb. $1 \div 4 = \frac{1}{4}$ inch = 1 lb. Ans.

(*b*) $6\frac{1}{4} \div 4 = 1.5625$ inches $= 6\frac{1}{4}$ lb. Ans.

(7) See Art. **1964.**

(**8**) The method of obtaining the resultant is shown in Fig. I. The forces are laid off to scale to form a polygon, and the closing line gives the direction and magnitude of the resultant. See Art. **1918.**

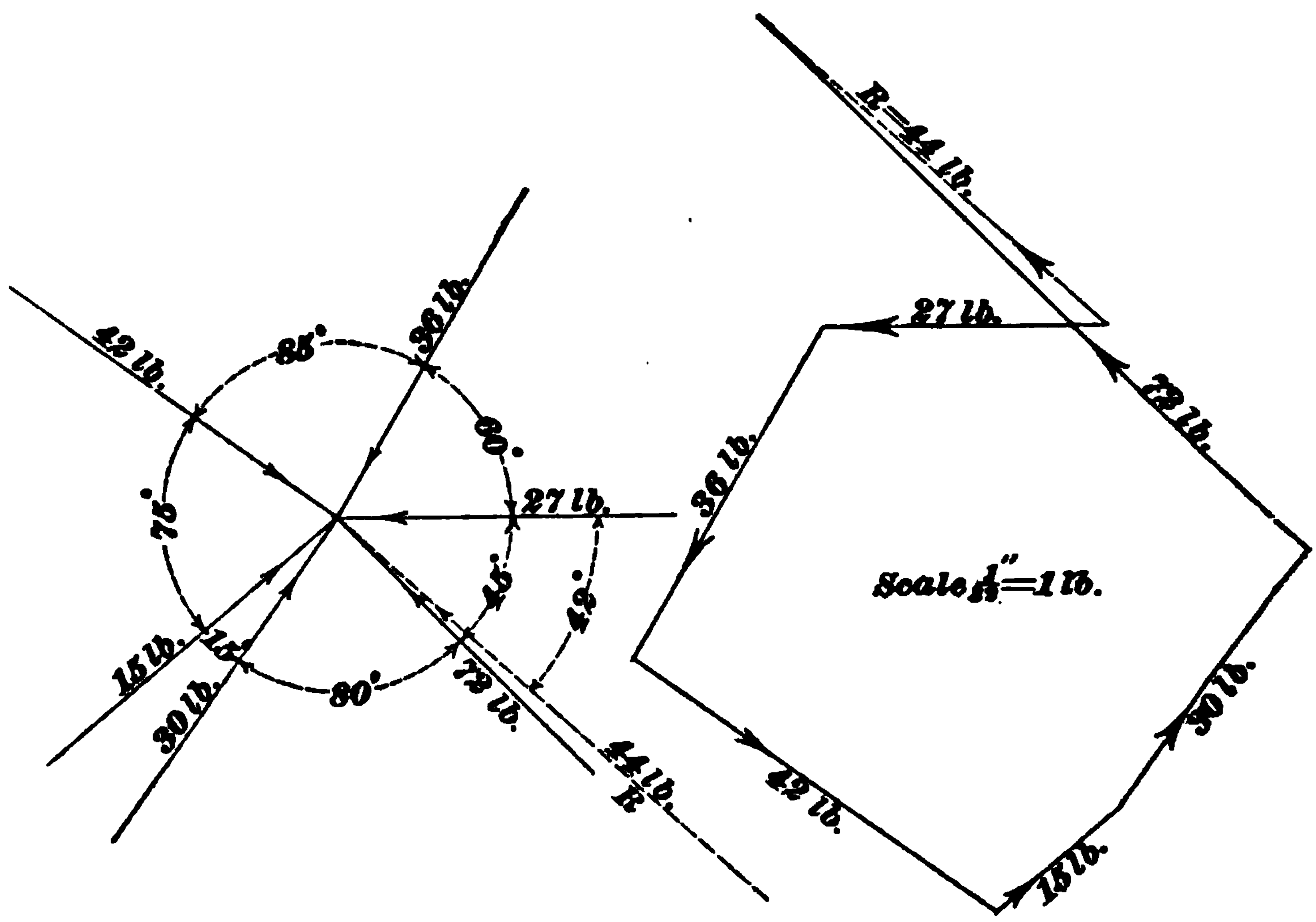

FIG. I.

(**9**) Area of cross-section $= 8^2 \times .7854 = 50.2656$ sq. in. 10 ft. $= 120$ in. $= L$. Crushing strength $= 3.5$ tons per sq. in. (see Table 33). $a = 187.5$ (see Table 36). Substituting these values in formula **127,**

$$W = \frac{3.5 \times 50.2656}{\frac{120^2}{187.5 \times 8^2} + 1} = 80 \text{ tons, very nearly.}$$

Hence, $80 \div 6 = 13\frac{1}{3}$ tons = safe load. Ans.

(**10**) Those forces by which the given force may be replaced and which will produce the same effect on a body as the given force.

(**11**) Apply formula **119.**

$$A = \frac{12{,}000}{5{,}000} = 2.4 \text{ sq. in., the area of the bolt.}$$

$$\text{Diameter} = \sqrt{\frac{2.4}{.7854}} = 1.74+ \text{ in. Ans.}$$

(12) First calculate the load it will sustain in the middle, by means of formula **130.**

$$\text{Load in middle} = \frac{4 \times 10^2 \times 8 \times 30}{28} = 3{,}428\tfrac{4}{7} \text{ lb.}$$

$$\text{Uniform load} = 3{,}428\tfrac{4}{7} \times 2 = 6{,}857\tfrac{1}{7} \text{ lb. Ans.}$$

(13) Apply formula **133.** From Table 40, the proper constant is 70.

$$\text{Horsepower} = \frac{10^3 \times 200}{70} = 2{,}857\tfrac{1}{7}. \text{ Ans.}$$

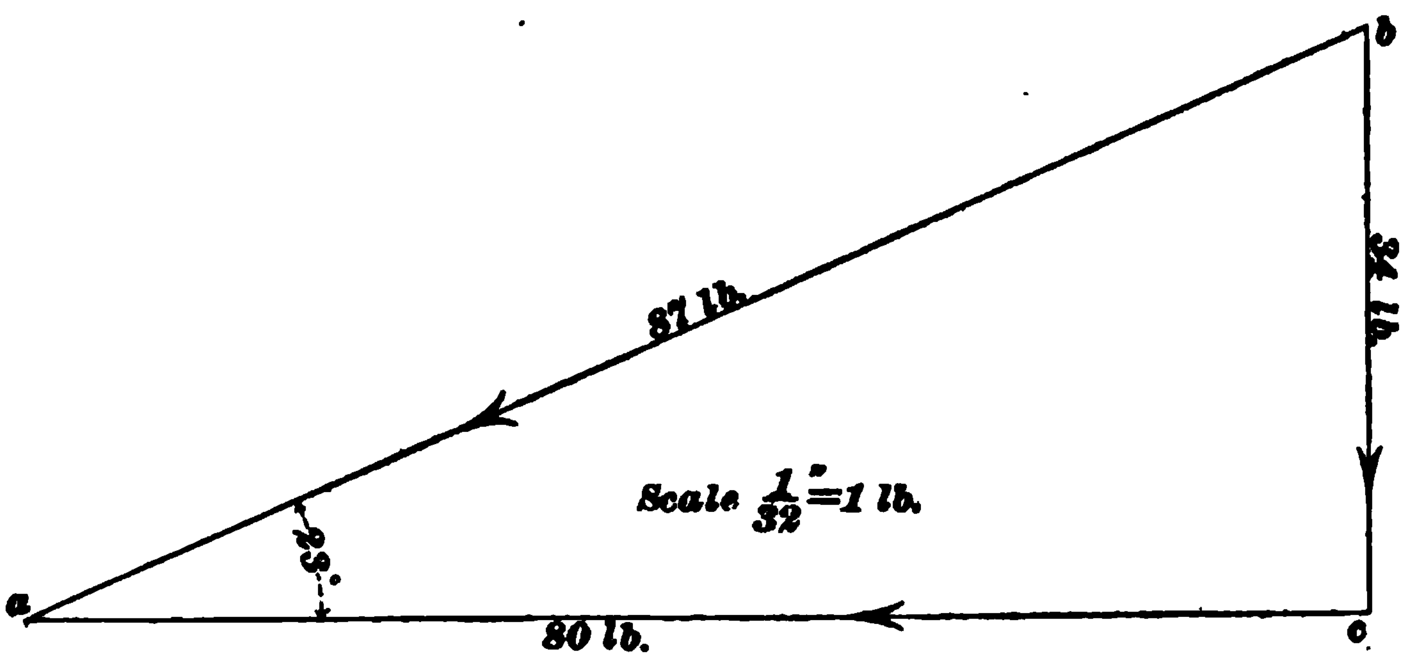

FIG. II.

(14) See Fig. II. By trigonometry, $bc = 87 \times \sin 23° = 87 \times .39073 = 33.994$ lb. $ac = 87 \times \cos 23° = 87 \times .92050 = 80.084$ lb.

(15) Apply formula **121.**

$$W = 18{,}000 \times .5^2 = 4{,}500 \text{ lb., the load. Ans.}$$

(16) Applying formula **126,**

$C = .0408\sqrt{14{,}000} = 4.83$ in., the circumference, nearly. Ans.

(17) See Fig. III.

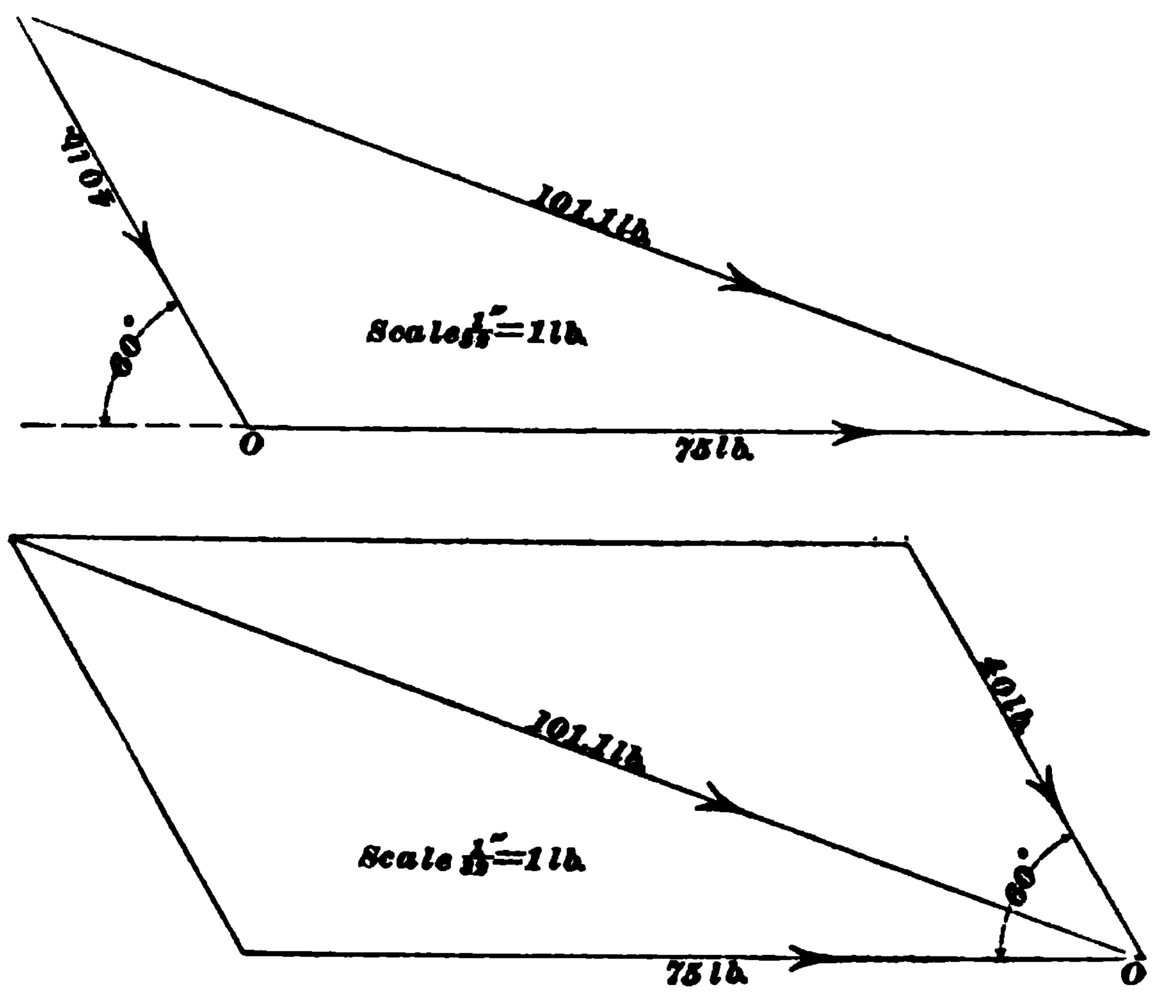

FIG. III.

(18) See Fig. IV.

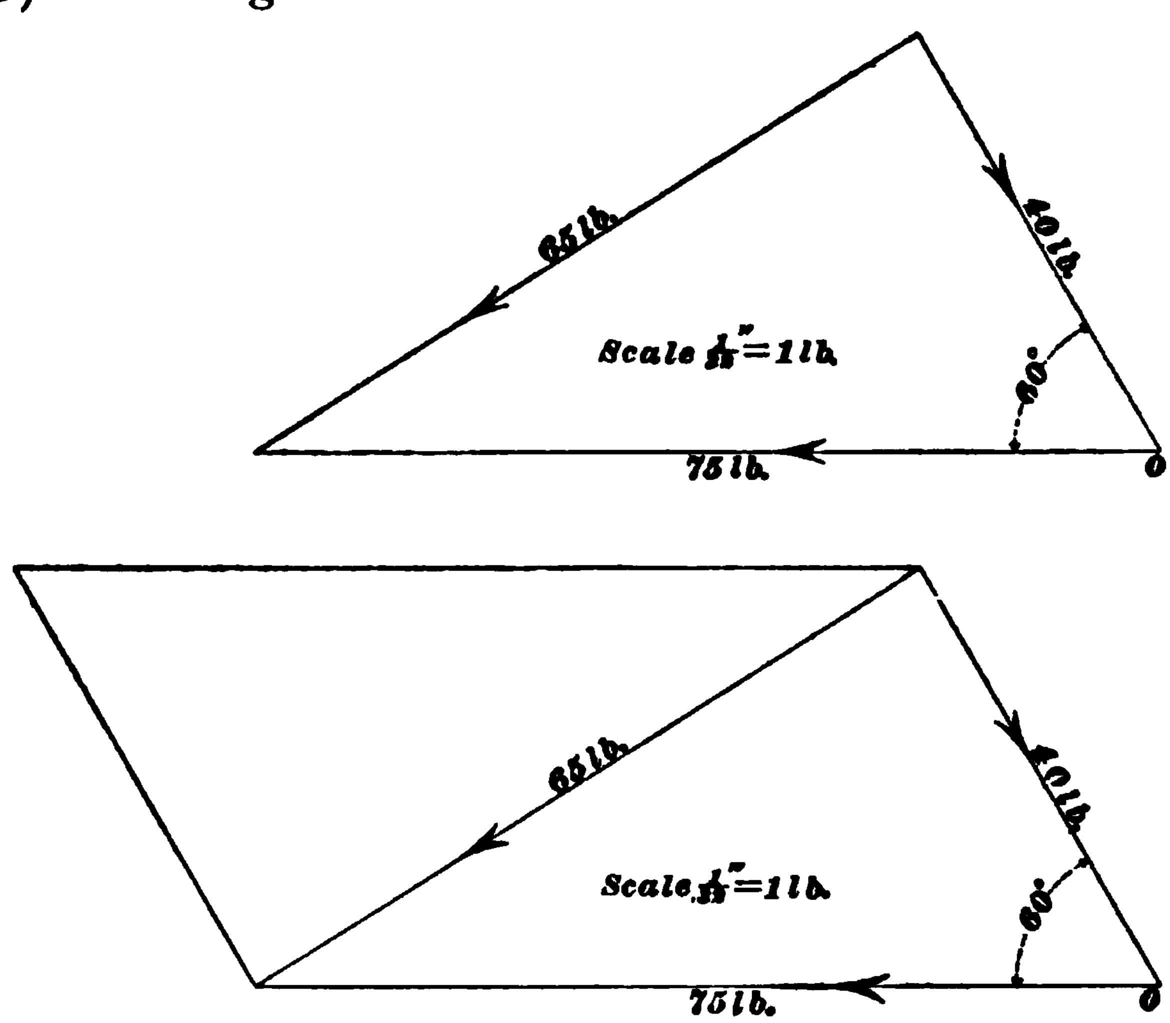

FIG. IV.

(19) $46 - 27 = 19$ lb., acting in the direction of the force of 46 lb. Ans.

(20) Area of cross-section $= 1\frac{3}{4} \times 3 = 5.25$ sq. in. Applying formula **118,**

$$W = 5.25 \times 6{,}000 = 31{,}500 \text{ lb., the safe load. Ans.}$$

(21) The graphical construction is clearly shown in Fig. V.

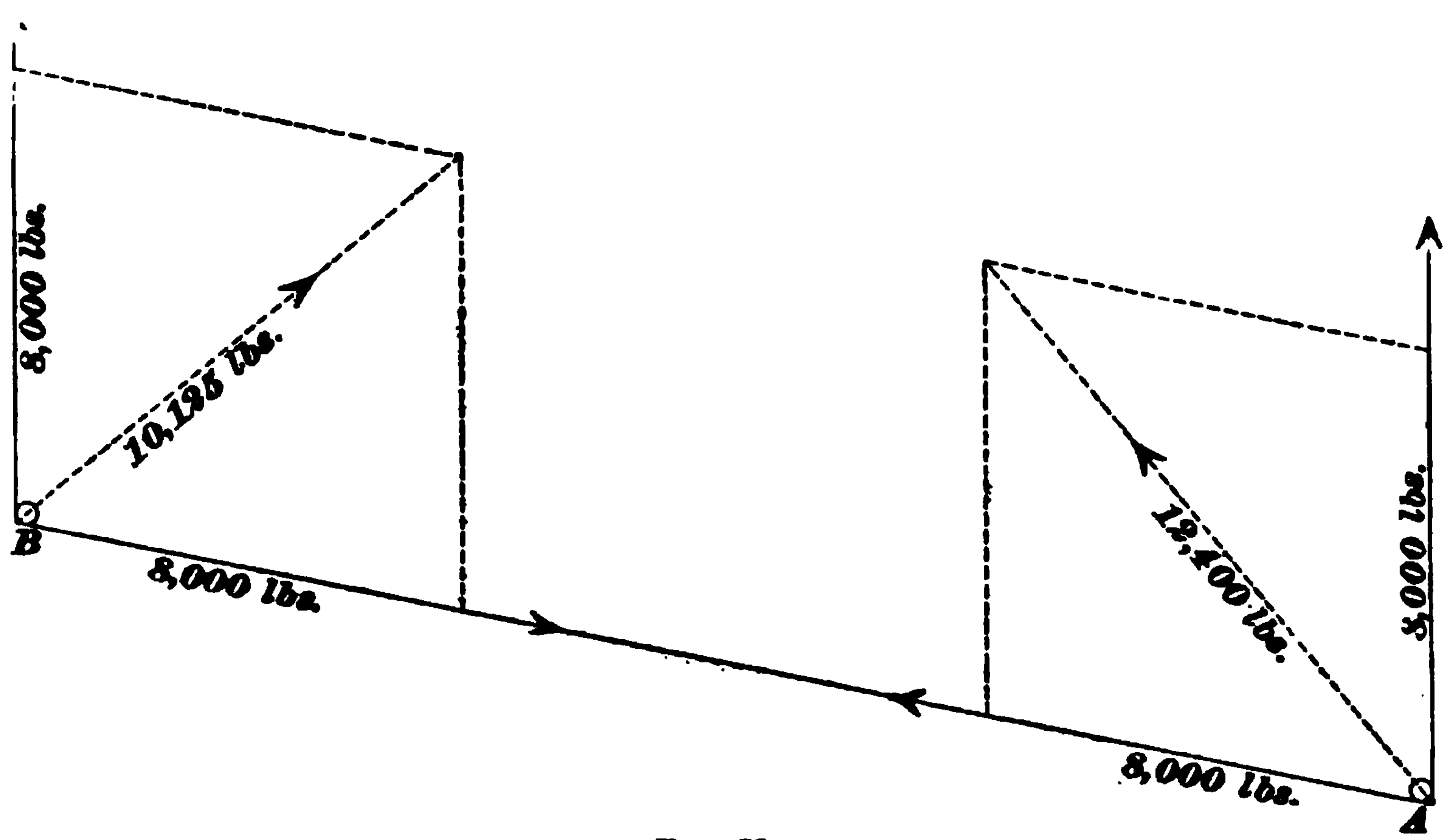

Fig. V.

(22) See Arts. **1926** to **1928.**

(23) Apply formula **126,** and use .0316 instead of .0408, since the rope is of steel.

$$C = .0316\sqrt{8{,}000} = 2.83 \text{ in. Ans.}$$

(24) Apply formula **130,** and multiply the result by 2.

$$W = \frac{4 \times 6^2 \times 2 \times 160}{20} \times 2 = 4{,}608 \text{ lb., the load. Ans.}$$

(25) See Arts. **1929** to **1931.**

(26) Apply formula **125.**

$$W = 600 \times 6^2 = 21{,}600 \text{ lb. Ans.}$$

(27) 4 ft. = 48 in. Area to be sheared = 48 × $\frac{1}{2}$ = 24 sq. in. Applying formula **132,**

$W = 24 \times 40{,}000 = 960{,}000$ lb., the force required. Ans.

(28) Applying formula **134,**

$$\frac{70 \times 200}{7^3} = 40.8 \text{ revolutions per minute, nearly. Ans.}$$

(29) Apply formula **125.**

$$\text{Load} = 600 \times 4^2 = 9{,}600 \text{ lb. Ans.}$$

(30) Apply formula **128.**

$$\text{Load} = \frac{2.5^2 \times 1.5 \times 100}{4\frac{8}{12}} = 201 \text{ lb., nearly. Ans.}$$

(31) Apply formula **133.**

$$\text{Horsepower} = \frac{(2\frac{7}{16})^3 \times 120}{85} = 20.445. \text{ Ans.}$$

(32) Area of cross-section = $(1\frac{1}{2})^2 \times .7854 = 1.7671$ sq. in. Apply formula **118.**

Safe steady load = $12{,}000 \times 1.7671 = 21{,}205.2$ lb. Ans.

(33) Substituting the values of $C = 40$, $S = 14^2 \times .7854 - 11.5^2 \times .7854 = 50.0693$, $L = 20 \times 12 = 240$, $a = 562.5$, and $d = 14$ in formula **127,** we have

$$W = \frac{40 \times 50.0693}{\frac{240^2}{562.5 \times 14^2} + 1} = \frac{2{,}002.772}{1.5225} = 1{,}315.45 \text{ tons.}$$

$$\frac{1{,}315.45}{6} = 219.24 \text{ tons. Ans.}$$

(34) See Art. **1963.** Area punched = $1\frac{1}{2} \times 3.1416 \times \frac{3}{4} = 3.5343$ sq. in. Force = $3.5343 \times 60{,}000 = 212{,}058$ lb. Ans.

(35) See Fig. VI

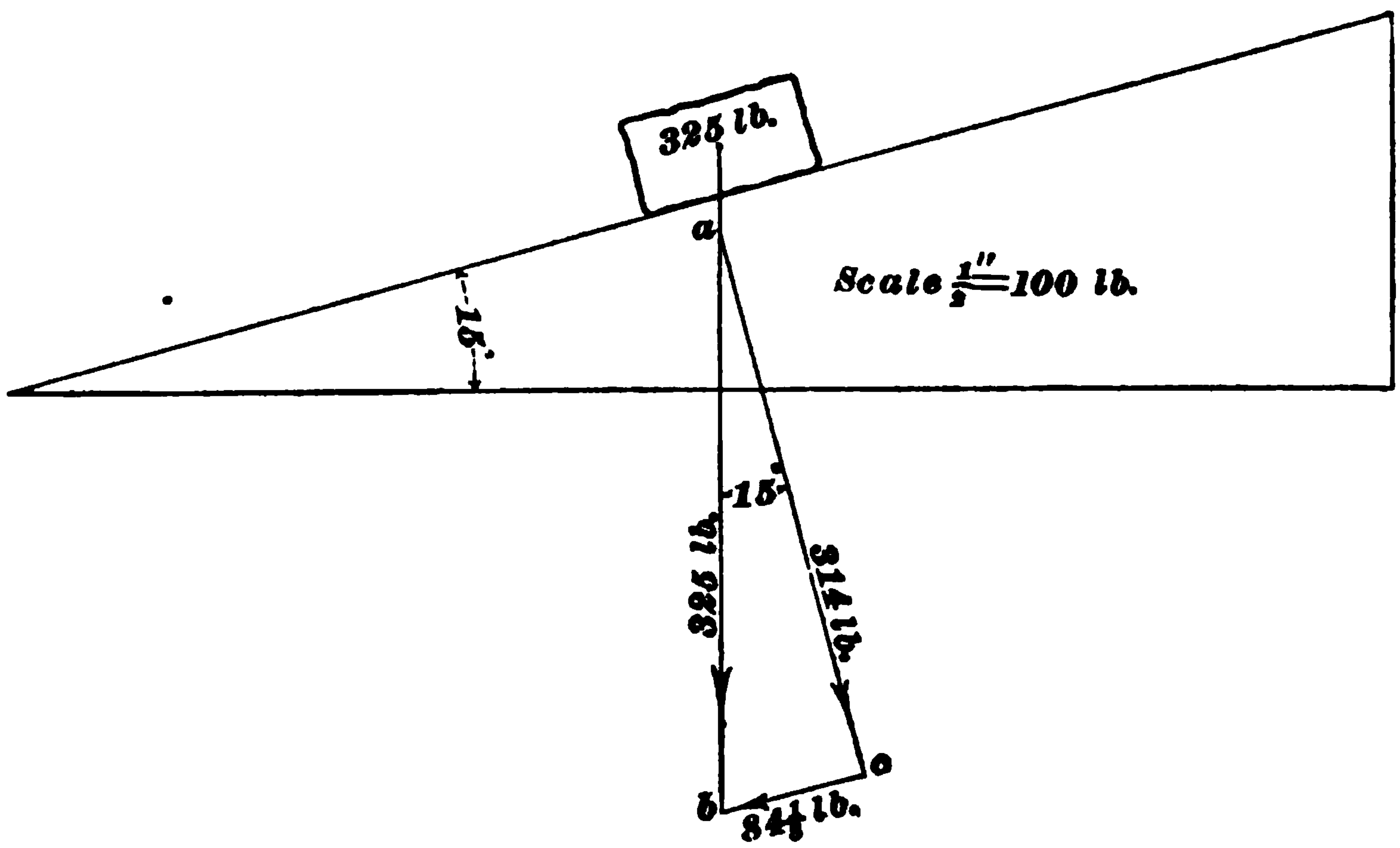

FIG. VI.

(a) $a\,c = 325 \times \cos 15° = 325 \times .96593 = 313.93$ lb. Ans.
(b) $b\,c = 325 \times \sin 15° = 325 \times .25882 = 84.12$ lb. Ans.

HYDRAULICS AND HYDRAULIC MACHINERY

(PART 1)

(**1**) *a*, 363.6 lb.; *b*, 127.26 lb.; *c*, 18.18 lb.; *d*, 109.08 lb.; and *f*, 72.72 lb. Ans. See Art. **3.**

(**2**) (*a*) $3.1416 \times 48'' \times 2'' = 301.59$ sq. in., area of the strip. $20' - 1.5' = 18.5' \times 12 = 222$ inches depth, and a column of water 1 cu. in. base and 222 inches high weighs $222 \times .036 = 7.99$; therefore, the total pressure on the strip is $301.59 \times 7.99 = 2{,}409$ lb. per sq. in. Ans.

(*b*) $20 - 1.5 = 18.5$, and $18.5 \times 12 \times .036 = 7.99$ lb. per sq. in. Ans. See Art. **5.**

(**3**) (*a*) See Art. **16.**

(*b*) See Art. **1.**

(**4**) See Art. **17.**

(**5**) It is about .8 the diameter of the orifice. See Art. **22.**

(**6**) The hydraulic grade line is a straight line extending from the reservoir to the end of the pipe line. It is not measured from the top of the reservoir, a certain head being necessary to overcome friction in a pipe line, and this loss of head will decrease the flow. See Arts. **26, 27,** and **28.**

(**7**) $250 \times 12 \times .036 = 108$ lb. Ans. See Art. **16.**

(8) Friction of water in flowing through the pipes. See Art. **17.**

(9) $h = \frac{175^2}{2 \times 32.16} = \frac{30625}{64.32} = 476$ ft. Ans. See Art. **17.**

(10) $.1435 \times 60 = 8.616$, and $8.616 \div .615 = 14$ cubic feet per minute. Ans. See Art. **22.**

(11) See Arts. **30** and **31.**

(12) Water hammering is due to suddenly stopping a flow of water through pipes, and is reaction. It may be prevented by turning off or on the water slowly. See Art. **53.**

(13) Flumes can be used to cross gullies and run along the face of cliffs, or in places where the soil is porous and not water-tight. They can also be given steeper grades than ditches. See Art. **54.**

(14) See Art. **52.**

HYDRAULICS AND HYDRAULIC MACHINERY

(PART 2)

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(1) He will make a preliminary survey and follow the method given in Art. **1.**

(2) The extent of the surface over which the water flows; the grade of the canal; the length of the canal, etc. See Arts. **3** and **4.**

(3) Depth and side slope.

(4) The limit is reached when the width of the canal is equal to twice its depth. See Art. **69.**

(5) They wash, and must be built in excavation with proper side slopes. See general remarks in Art. **11.**

(6) From watershed in the vicinity. See Art. **20.**

(7) See Arts. **22, 23,** and **26.**

(8) To prevent water trickling through and forming channels which may eventually undermine the dam. See Arts. **30-33.**

(9) 1 to $2\frac{1}{2}$ on down-stream side; 1 to $3\frac{1}{2}$ on the water side. See Art. **34.**

(10) The spillway is one of the important features of a dam, since it permits surplus water to run to waste. The dimensions must be proportioned to the amount of water liable to go over in times of freshets. See Art. **35.**

(11) Wing dams are constructed to divert the water out of its natural channel. See Art. **42.**

(12) (*a*) $MR = \frac{130 \times 18}{3}(4 \times 9 - \frac{16}{2} + 81) = 85{,}020$ ft.-lb. Ans.

(*b*) $MT = 10.42 \times 5{,}832 = 60{,}769$, and $F = \frac{85{,}020}{60{,}769} = 1.4$. Ans. See Arts. **44-48.**

(13) $B = \sqrt{\frac{62.5 \times 2.5 \times 1{,}600 + 140 \times 64 + 16}{280}} - 4 = 26$ ft. Ans. See Arts. **48** and **49.**

(14) A miner's inch is variously described, because it differs in localities. See Art. **67.**

(15) See Art. **50.**

(16) Current meters, floats on the surface, and floats submerged. See Arts. **70-73.**

HYDRAULICS AND HYDRAULIC MACHINERY

(PART 3)

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(1) The theoretical work that water can do is equal to its energy. See Arts. **1** and **3.**

(2) By foot-pounds of work per second, and this may be converted into horsepower. See Art. **2.**

(3) See Art. **5.**

(4) When the watei leaves the surface of the bucket with the least absolute velocity. See Art. **7.**

(5) Using formula **7,**

$$P = 2\,c\,w\,a\,h = \frac{2 \times .98 \times 62.5 \times 3.1416 \times 75}{144} = 200.44 \text{ lb.}$$

Ans.

(6) See Art. **8.**

(7) The theoretical power is $\frac{1,600 \times 30}{550} = 87.27$ horsepower; and the efficiency is, therefore, $\frac{40}{87.27} = 45$ per cent. Ans.

(8) $\frac{1,600 \times 30}{550} = 87.27$, and $87.27 \times 60 = 52.36$ per cent. Ans.

(9) Circumference ; number and breadth of buckets; form of buckets; height of fall.

(10) Take the circumferential velocity of the wheel as $v = 8$ feet per second and the velocity of entry $v_e = 2\,v$. The head required to produce the velocity of entry is, from formula **12,** $h = 1.1 \times \frac{16^2}{64.32} = 4.38$ feet. Since this corresponds to the maximum value of v_e for the assumed velocity v, a value h somewhat less, say 4 feet, may be taken, and the diameter of the wheel $D = H - h = 15 - 4 = 11$ feet. The number of buckets $Z = 10 \times 11 = 110$, and the pitch of the buckets at the crown $t = \frac{3.1416 \times 11}{110} = .314$ foot. If the buckets are made 12 inches deep, the breadth of the wheel may be found by formula **14.** $\frac{3\,Q}{d\,v} = \frac{3 \times 15}{8 \times 1} = \frac{45}{8}$ $= 5.625$ feet. The trough should be a trifle less than this, say 5.5 feet. The number of revolutions with the assumed velocity v is $N = 19.1 \times \frac{8}{11} = 13.9$ per minute.

(11) The water acts somewhat by impulse; they may be used for low falls; their efficiency is fairly good.

(12) Assume the velocity of the circumference of the wheel as 6 feet per second and the velocity of entry as $v_e = 1\frac{1}{2}\,v = 6 \times 1\frac{1}{2} = 9$ feet per second. The depth of the floats may be made 12 inches and the pitch will be approximately 12 inches. The diameter of the wheel $= 2 \times H = 2 \times 7\frac{1}{2} = 15$ feet. With this diameter and assumed pitch, the number of buckets is $\frac{3.1416 \times 15}{1} = 47.124$, or 50 even. Make the breadth of the wheel $b = 1\frac{1}{4}\frac{Q}{d\,v} = 1\frac{1}{4} \times \frac{15}{1 \times 6}$ $= 3.75$ feet. The number of revolutions with the assumed velocity v is $N = \frac{3.1416 \times 15}{6} = 7.854$ per minute.

(13) When the number of revolutions is such that the actual velocity of the cups corresponds nearly to the theoretical velocity. See Art. **24.**

(14) The velocity of the jet is $.98 \times 8.02\sqrt{100} = 78.6$ feet per second. The circumferential velocity of the wheel is

$78.6 \div 2 = 39.3$ feet per second, and the diameter required for 90 revolutions per minute is $d = \frac{39.3 \times 60}{90 \times 3.1416} = 8.34$ feet. Ans.

(15) The difference between the pressures on its two sides. See Art. **41.**

(16) Yes. See Arts. **42** and **43.**

HYDRAULICS AND HYDRAULIC MACHINERY

(PART 4)

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(1) By gates, valves, and waterwheel governors. See Arts. **9, 10,** and **11.**

(2) Pumps and waterwheels. See Arts. **12** and **13.**

(3) Suction or lift pumps, plunger or force pumps, and centrifugal pumps. See Arts. **14, 20, 28,** and **50.**

(4) (*a*) Raises water.

(*b*) Theoretical height, 34 feet; actual height, between 26 and 32 feet.

(5) (*a*) One that both draws and forces water on each stroke of the piston.

(*b*) Two pumps driven by cranks on a single shaft. See Arts. **21** and **34.**

(6) The air chamber is supposed to catch air from the water; it is also supposed to have the air alternately compressed and expanded, thus assisting in keeping up a steady flow of water between strokes, and thereby relieving the apparatus from shocks. In mining, these air chambers are often more of a nuisance than assistance, especially where lifts are high. See Art. **19.**

(7) They are geared pumps. See Arts. **20-24.**

(8) See Art. **21.**

(**9**) See Arts. **30** and **31**.

(**10**) The movement of one piston rod throws the steam valves on the opposite pump. See Arts. **34-36**.

(**11**) It has high- and low-pressure steam cylinders and high- and low-pressure steam chests. They are explained in Arts. **37** and **38**.

(**12**) They must be strong, durable, easily repaired, and make an air-tight joint with the valve seat. Besides these suggestions, there are eight important details named in Art. **41**.

(**13**) Cold water may have soft rubber or leather valves. Hot water must have valves which are hard and not softened by heat. Acid liquors require valves not corroded by the acids; sometimes they are composition metal, as in the case of bronze for acid mine water, and again the valves must be lead.

(**14**) Rotary pumps cut out sections of water, while centrifugal pumps whirl the water so as to give it a motion which causes it to leave the center of the pump and discharge at the circumference. See Arts. **49** and **50**.

(**15**) Yes. They are largely used in harbor dredging, in gold dredging, and in mill work.

(**16**) See Art. **60**.

(**17**) See Arts. **58** and **59**.

(**18**) $112 \times .434 = 48.6$ lb. per sq. in. Ans.

(**19**) $48.6 \times 2.304 = 111.97$, or 112 ft. Ans. See Art. **62**.

(**20**) $h = \frac{H}{.00038\,G} = \frac{60}{.00038 \times 500} = 316$ ft. Ans.

TABLES

OF

NATURAL SINES, COSINES, TANGENTS, AND COTANGENTS

GIVING THE VALUES OF THE FUNCTIONS FOR ALL DEGREES AND MINUTES FROM

0° TO 90°

′	0°		1°		2°		3°		4°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	.00000	1.	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	60
1	.00029	1.	.01774	.99984	.03519	.99938	.05263	.99861	.07005	.99754	59
2	.00058	1.	.01803	.99984	.03548	.99937	.05292	.99860	.07034	.99752	58
3	.00087	1.	.01832	.99983	.03577	.99936	.05321	.99858	.07063	.99750	57
4	.00116	1.	.01862	.99983	.03606	.99935	.05350	.99857	.07092	.99748	56
5	.00145	1.	.01891	.99982	.03635	.99934	.05379	.99855	.07121	.99746	55
6	.00175	1.	.01920	.99982	.03664	.99933	.05408	.99854	.07150	.99744	54
7	.00204	1.	.01949	.99981	.03693	.99932	.05437	.99852	.07179	.99742	53
8	.00233	1.	.01978	.99980	.03723	.99931	.05466	.99851	.07208	.99740	52
9	.00262	1.	.02007	.99980	.03752	.99930	.05495	.99849	.07237	.99738	51
10	.00291	1.	.02036	.99979	.03781	.99929	.05524	.99847	.07266	.99736	50
11	.00320	.99999	.02065	.99979	.03810	.99927	.05553	.99846	.07295	.99734	49
12	.00349	.99999	.02094	.99978	.03839	.99926	.05582	.99844	.07324	.99731	48
13	.00378	.99999	.02123	.99977	.03868	.99925	.05611	.99842	.07353	.99729	47
14	.00407	.99999	.02152	.99977	.03897	.99924	.05640	.99841	.07382	.99727	46
15	.00436	.99999	.02181	.99976	.03926	.99923	.05669	.99839	.07411	.99725	45
16	.00465	.99999	.02211	.99976	.03955	.99922	.05698	.99838	.07440	.99723	44
17	.00495	.99999	.02240	.99975	.03984	.99921	.05727	.99836	.07469	.99721	43
18	.00524	.99999	.02269	.99974	.04013	.99919	.05756	.99834	.07498	.99719	42
19	.00553	.99998	.02298	.99974	.04042	.99918	.05785	.99833	.07527	.99716	41
20	.00582	.99998	.02327	.99973	.04071	.99917	.05814	.99831	.07556	.99714	40
21	.00611	.99998	.02356	.99972	.04100	.99916	.05844	.99829	.07585	.99712	39
22	.00640	.99998	.02385	.99972	.04129	.99915	.05873	.99827	.07614	.99710	38
23	.00669	.99998	.02414	.99971	.04159	.99913	.05902	.99826	.07643	.99708	37
24	.00698	.99998	.02443	.99970	.04188	.99912	.05931	.99824	.07672	.99705	36
25	.00727	.99997	.02472	.99969	.04217	.99911	.05960	.99822	.07701	.99703	35
26	.00756	.99997	.02501	.99969	.04246	.99910	.05989	.99821	.07730	.99701	34
27	.00785	.99997	.02530	.99968	.04275	.99909	.06018	.99819	.07759	.99699	33
28	.00814	.99997	.02560	.99967	.04304	.99907	.06047	.99817	.07788	.99696	32
29	.00844	.99996	.02589	.99966	.04333	.99906	.06076	.99815	.07817	.99694	31
30	.00873	.99996	.02618	.99966	.04362	.99905	.06105	.99813	.07846	.99692	30
31	.00902	.99996	.02647	.99965	.04391	.99904	.06134	.99812	.07875	.99689	29
32	.00931	.99996	.02676	.99964	.04420	.99902	.06163	.99810	.07904	.99687	28
33	.00960	.99995	.02705	.99963	.04449	.99901	.06192	.99808	.07933	.99685	27
34	.00989	.99995	.02734	.99963	.04478	.99900	.06221	.99806	.07962	.99683	26
35	.01018	.99995	.02763	.99962	.04507	.99898	.06250	.99804	.07991	.99680	25
36	.01047	.99995	.02792	.99961	.04536	.99897	.06279	.99803	.08020	.99678	24
37	.01076	.99994	.02821	.99960	.04565	.99896	.06308	.99801	.08049	.99676	23
38	.01105	.99994	.02850	.99959	.04594	.99894	.06337	.99799	.08078	.99673	22
39	.01134	.99994	.02879	.99959	.04623	.99893	.06366	.99797	.08107	.99671	21
40	.01164	.99993	.02908	.99958	.04653	.99892	.06395	.99795	.08136	.99668	20
41	.01193	.99993	.02938	.99957	.04682	.99890	.06424	.99793	.08165	.99666	19
42	.01222	.99993	.02967	.99956	.04711	.99889	.06453	.99792	.08194	.99664	18
43	.01251	.99992	.02996	.99955	.04740	.99888	.06482	.99790	.08223	.99661	17
44	.01280	.99992	.03025	.99954	.04769	.99886	.06511	.99788	.08252	.99659	16
45	.01309	.99991	.03054	.99953	.04798	.99885	.06540	.99786	.08281	.99657	15
46	.01338	.99991	.03083	.99952	.04827	.99883	.06569	.99784	.08310	.99654	14
47	.01367	.99991	.03112	.99952	.04856	.99882	.06598	.99782	.08339	.99652	13
48	.01396	.99990	.03141	.99951	.04885	.99881	.06627	.99780	.08368	.99649	12
49	.01425	.99990	.03170	.99950	.04914	.99879	.06656	.99778	.08397	.99647	11
50	.01454	.99989	.03199	.99949	.04943	.99878	.06685	.99776	.08426	.99644	10
51	.01483	.99989	.03228	.99948	.04972	.99876	.06714	.99774	.08455	.99642	9
52	.01513	.99989	.03257	.99947	.05001	.99875	.06743	.99772	.08484	.99639	8
53	.01542	.99988	.03286	.99946	.05030	.99873	.06773	.99770	.08513	.99637	7
54	.01571	.99988	.03316	.99945	.05059	.99872	.06802	.99768	.08542	.99635	6
55	.01600	.99987	.03345	.99944	.05088	.99870	.06831	.99766	.08571	.99632	5
56	.01629	.99987	.03374	.99943	.05117	.99869	.06860	.99764	.08600	.99630	4
57	.01658	.99986	.03403	.99942	.05146	.99867	.06889	.99762	.08629	.99627	3
58	.01687	.99986	.03432	.99941	.05175	.99866	.06918	.99760	.08658	.99625	2
59	.01716	.99985	.03461	.99940	.05205	.99864	.06947	.99758	.08687	.99622	1
60	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	.08716	.99619	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	89°		88°		87°		86°		85°		

′	5°		6°		7°		8°		9°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	.08716	.99619	.10453	.99452	.12187	.99255	.13917	.99027	.15643	.98769	60
1	.08745	.99617	.10482	.99449	.12216	.99251	.13946	.99023	.15672	.98764	59
2	.08774	.99614	.10511	.99446	.12245	.99248	.13975	.99019	.15701	.98760	58
3	.08803	.99612	.10540	.99443	.12274	.99244	.14004	.99015	.15730	.98755	57
4	.08831	.99609	.10569	.99440	.12302	.99240	.14033	.99011	.15758	.98751	56
5	.08860	.99607	.10597	.99437	.12331	.99237	.14061	.99006	.15787	.98746	55
6	.08889	.99604	.10626	.99434	.12360	.99233	.14090	.99002	.15816	.98741	54
7	.08918	.99602	.10655	.99431	.12389	.99230	.14119	.98998	.15845	.98737	53
8	.08947	.99599	.10684	.99428	.12418	.99226	.14148	.98994	.15873	.98732	52
9	.08976	.99596	.10713	.99424	.12447	.99222	.14177	.98990	.15902	.98728	51
10	.09005	.99594	.10742	.99421	.12476	.99219	.14205	.98986	.15931	.98723	50
11	.09034	.99591	.10771	.99418	.12504	.99215	.14234	.98982	.15959	.98718	49
12	.09063	.99588	.10800	.99415	.12533	.99211	.14263	.98978	.15988	.98714	48
13	.09092	.99586	.10829	.99412	.12562	.99208	.14292	.98973	.16017	.98709	47
14	.09121	.99583	.10858	.99409	.12591	.99204	.14320	.98969	.16046	.98704	46
15	.09150	.99580	.10887	.99406	.12620	.99200	.14349	.98965	.16074	.98700	45
16	.09179	.99578	.10916	.99402	.12649	.99197	.14378	.98961	.16103	.98695	44
17	.09208	.99575	.10945	.99399	.12678	.99193	.14407	.98957	.16132	.98690	43
18	.09237	.99572	.10973	.99396	.12706	.99189	.14436	.98953	.16160	.98686	42
19	.09266	.99570	.11002	.99393	.12735	.99186	.14464	.98948	.16189	.98681	41
20	.09295	.99567	.11031	.99390	.12764	.99182	.14493	.98944	.16218	.98676	40
21	.09324	.99564	.11060	.99386	.12793	.99178	.14522	.98940	.16246	.98671	39
22	.09353	.99562	.11089	.99383	.12822	.99175	.14551	.98936	.16275	.98667	38
23	.09382	.99559	.11118	.99380	.12851	.99171	.14580	.98931	.16304	.98662	37
24	.09411	.99556	.11147	.99377	.12880	.99167	.14608	.98927	.16333	.98657	36
25	.09440	.99553	.11176	.99374	.12908	.99163	.14637	.98923	.16361	.98652	35
26	.09469	.99551	.11205	.99370	.12937	.99160	.14666	.98919	.16390	.98648	34
27	.09498	.99548	.11234	.99367	.12966	.99156	.14695	.98914	.16419	.98643	33
28	.09527	.99545	.11263	.99364	.12995	.99152	.14723	.98910	.16447	.98638	32
29	.09556	.99542	.11291	.99360	.13024	.99148	.14752	.98906	.16476	.98633	31
30	.09585	.99540	.11320	.99357	.13053	.99144	.14781	.98902	.16505	.98629	30
31	.09614	.99537	.11349	.99354	.13081	.99141	.14810	.98897	.16533	.98624	29
32	.09642	.99534	.11378	.99351	.13110	.99137	.14838	.98893	.16562	.98619	28
33	.09671	.99531	.11407	.99347	.13139	.99133	.14867	.98889	.16591	.98614	27
34	.09700	.99528	.11436	.99344	.13168	.99129	.14896	.98884	.16620	.98609	26
35	.09729	.99526	.11465	.99341	.13197	.99125	.14925	.98880	.16648	.98604	25
36	.09758	.99523	.11494	.99337	.13226	.99122	.14954	.98876	.16677	.98600	24
37	.09787	.99520	.11523	.99334	.13254	.99118	.14982	.98871	.16706	.98595	23
38	.09816	.99517	.11552	.99331	.13283	.99114	.15011	.98867	.16734	.98590	22
39	.09845	.99514	.11580	.99327	.13312	.99110	.15040	.98863	.16763	.98585	21
40	.09874	.99511	.11609	.99324	.13341	.99106	.15069	.98858	.16792	.98580	20
41	.09903	.99508	.11638	.99320	.13370	.99102	.15097	.98854	.16820	.98575	19
42	.09932	.99506	.11667	.99317	.13399	.99098	.15126	.98849	.16849	.98570	18
43	.09961	.99503	.11696	.99314	.13427	.99094	.15155	.98845	.16878	.98565	17
44	.09990	.99500	.11725	.99310	.13456	.99091	.15184	.98841	.16906	.98561	16
45	.10019	.99497	.11754	.99307	.13485	.99087	.15212	.98836	.16935	.98556	15
46	.10048	.99494	.11783	.99303	.13514	.99083	.15241	.98832	.16964	.98551	14
47	.10077	.99491	.11812	.99300	.13543	.99079	.15270	.98827	.16992	.98546	13
48	.10106	.99488	.11840	.99297	.13572	.99075	.15299	.98823	.17021	.98541	12
49	.10135	.99485	.11869	.99293	.13600	.99071	.15327	.98818	.17050	.98536	11
50	.10164	.99482	.11898	.99290	.13629	.99067	.15356	.98814	.17078	.98531	10
51	.10192	.99479	.11927	.99286	.13658	.99063	.15385	.98809	.17107	.98526	9
52	.10221	.99476	.11956	.99283	.13687	.99059	.15414	.98805	.17136	.98521	8
53	.10250	.99473	.11985	.99279	.13716	.99055	.15442	.98800	.17164	.98516	7
54	.10279	.99470	.12014	.99276	.13744	.99051	.15471	.98796	.17193	.98511	6
55	.10308	.99467	.12043	.99272	.13773	.99047	.15500	.98791	.17222	.98506	5
56	.10337	.99464	.12071	.99269	.13802	.99043	.15529	.98787	.17250	.98501	4
57	.10366	.99461	.12100	.99265	.13831	.99039	.15557	.98782	.17279	.98496	3
58	.10395	.99458	.12129	.99262	.13860	.99035	.15586	.98778	.17308	.98491	2
59	.10424	.99455	.12158	.99258	.13889	.99031	.15615	.98773	.17336	.98486	1
60	.10453	.99452	.12187	.99255	.13917	.99027	.15643	.98769	.17365	.98481	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	84°		83°		82°		81°		80°		

′	10°		11°		12°		13°		14°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	.17365	.98481	.19081	.98163	.20791	.97815	.22495	.97437	.24192	.97030	60
1	.17393	.98476	.19109	.98157	.20820	.97809	.22523	.97430	.24220	.97023	59
2	.17422	.98471	.19138	.98152	.20848	.97803	.22552	.97424	.24249	.97015	58
3	.17451	.98466	.19167	.98146	.20877	.97797	.22580	.97417	.24277	.97008	57
4	.17479	.98461	.19195	.98140	.20905	.97791	.22608	.97411	.24305	.97001	56
5	.17508	.98455	.19224	.98135	.20933	.97784	.22637	.97404	.24333	.96994	55
6	.17537	.98450	.19252	.98129	.20962	.97778	.22665	.97398	.24362	.96987	54
7	.17565	.98445	.19281	.98124	.20990	.97772	.22693	.97391	.24390	.96980	53
8	.17594	.98440	.19309	.98118	.21019	.97766	.22722	.97384	.24418	.96973	52
9	.17623	.98435	.19338	.98112	.21047	.97760	.22750	.97378	.24446	.96966	51
10	.17651	.98430	.19366	.98107	.21076	.97754	.22778	.97371	.24474	.96959	50
11	.17680	.98425	.19395	.98101	.21104	.97748	.22807	.97365	.24503	.96952	49
12	.17708	.98420	.19423	.98096	.21132	.97742	.22835	.97358	.24531	.96945	48
13	.17737	.98414	.19452	.98090	.21161	.97735	.22863	.97351	.24559	.96937	47
14	.17766	.98409	.19481	.98084	.21189	.97729	.22892	.97345	.24587	.96930	46
15	.17794	.98404	.19509	.98079	.21218	.97723	.22920	.97338	.24615	.96923	45
16	.17823	.98399	.19538	.98073	.21246	.97717	.22948	.97331	.24644	.96916	44
17	.17852	.98304	.19566	.98067	.21275	.97711	.22977	.97325	.24672	.96909	43
18	.17880	.98389	.19595	.98061	.21303	.97705	.23005	.97318	.24700	.96902	42
19	.17909	.98383	.19623	.98056	.21331	.97698	.23033	.97311	.24728	.96894	41
20	.17937	.98378	.19652	.98050	.21360	.97692	.23062	.97304	.24756	.96887	40
21	.17966	.98373	.19680	.98044	.21388	.97686	.23090	.97298	.24784	.96880	39
22	.17995	.98368	.19709	.98039	.21417	.97680	.23118	.97291	.24813	.96873	38
23	.18023	.98362	.19737	.98033	.21445	.97673	.23146	.97284	.24841	.96866	37
24	.18052	.98357	.19766	.98027	.21474	.97667	.23175	.97278	.24869	.96858	36
25	.18081	.98352	.19794	.98021	.21502	.97661	.23203	.97271	.24897	.96851	35
26	.18109	.98347	.19823	.98016	.21530	.97655	.23231	.97264	.24925	.96844	34
27	.18138	.98341	.19851	.98010	.21559	.97648	.23260	.97257	.24954	.96837	33
28	.18166	.98336	.19880	.98004	.21587	.97642	.23288	.97251	.24982	.96829	32
29	.18195	.98331	.19908	.97998	.21616	.97636	.23316	.97244	.25010	.96822	31
30	.18224	.98325	.19937	.97992	.21644	.97630	.23345	.97237	.25038	.96815	30
31	.18252	.98320	.19965	.97987	.21672	.97623	.23373	.97230	.25066	.96807	29
32	.18281	.98315	.19994	.97981	.21701	.97617	.23401	.97223	.25094	.96800	28
33	.18309	.98310	.20022	.97975	.21729	.97611	.23429	.97217	.25122	.96793	27
34	.18338	.98304	.20051	.97969	.21758	.97604	.23458	.97210	.25151	.96786	26
35	.18367	.98299	.20079	.97963	.21786	.97598	.23486	.97203	.25179	.96778	25
36	.18395	.98294	.20108	.97958	.21814	.97592	.23514	.97196	.25207	.96771	24
37	.18424	.98288	.20136	.97952	.21843	.97585	.23542	.97189	.25235	.96764	23
38	.18452	.98283	.20165	.97946	.21871	.97579	.23571	.97182	.25263	.96756	22
39	.18481	.98277	.20193	.97940	.21899	.97573	.23599	.97176	.25291	.96749	21
40	.18509	.98272	.20222	.97934	.21928	.97566	.23627	.97169	.25320	.96742	20
41	.18538	.98267	.20250	.97928	.21956	.97560	.23656	.97162	.25348	.96734	19
42	.18567	.98261	.20279	.97922	.21985	.97553	.23684	.97155	.25376	.96727	18
43	.18595	.98256	.20307	.97916	.22013	.97547	.23712	.97148	.25404	.96719	17
44	.18624	.98250	.20336	.97910	.22041	.97541	.23740	.97141	.25432	.96712	16
45	.18652	.98245	.20364	.97905	.22070	.97534	.23769	.97134	.25460	.96705	15
46	.18681	.98240	.20393	.97899	.22098	.97528	.23797	.97127	.25488	.96697	14
47	.18710	.98234	.20421	.97893	.22126	.97521	.23825	.97120	.25516	.96690	13
48	.18738	.98229	.20450	.97887	.22155	.97515	.23853	.97113	.25545	.96682	12
49	.18767	.98223	.20478	.97881	.22183	.97508	.23882	.97106	.25573	.96675	11
50	.18795	.98218	.20507	.97875	.22212	.97502	.23910	.97100	.25601	.96667	10
51	.18824	.98212	.20535	.97869	.22240	.97496	.23938	.97093	.25629	.96660	9
52	.18852	.98207	.20563	.97863	.22268	.97489	.23966	.97086	.25657	.96653	8
53	.18881	.98201	.20592	.97857	.22297	.97483	.23995	.97079	.25685	.96645	7
54	.18910	.98196	.20620	.97851	.22325	.97476	.24023	.97072	.25713	.96638	6
55	.18938	.98190	.20649	.97845	.22353	.97470	.24051	.97065	.25741	.96630	5
56	.18967	.98185	.20677	.97839	.22382	.97463	.24079	.97058	.25769	.96623	4
57	.18995	.98179	.20706	.97833	.22410	.97457	.24108	.97051	.25798	.96615	3
58	19024	.98174	.20734	.97827	.22438	.97450	.24136	.97044	.25826	.96608	2
59	.19052	.98168	.20763	.97821	.22467	.97444	.24164	.97037	.25854	.96600	1
60	.19081	.98163	.20791	.97815	.22495	.97437	.24192	.97030	.25882	.96593	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	79°		78°		77°		76°		75°		

′	15° Sine	15° Cosine	16° Sine	16° Cosine	17° Sine	17° Cosine	18° Sine	18° Cosine	19° Sine	19° Cosine	′
0	.25882	.96593	.27564	.96126	.29237	.95630	.30902	.95106	.32557	.94552	60
1	.25910	.96585	.27592	.96118	.29265	.95622	.30929	.95097	.32584	.94542	59
2	.25938	.96578	.27620	.96110	.29293	.95613	.30957	.95088	.32612	.94533	58
3	.25966	.96570	.27648	.96102	.29321	.95605	.30985	.95079	.32639	.94523	57
4	.25994	.96562	.27676	.96094	.29348	.95596	.31012	.95070	.32667	.94514	56
5	.26022	.96555	.27704	.96086	.29376	.95588	.31040	.95061	.32694	.94504	55
6	.26050	.96547	.27731	.96078	.29404	.95579	.31068	.95052	.32722	.94495	54
7	.26079	.96540	.27759	.96070	.29432	.95571	.31095	.95043	.32749	.94485	53
8	.26107	.96532	.27787	.96062	.29460	.95562	.31123	.95033	.32777	.94476	52
9	.26135	.96524	.27815	.96054	.29487	.95554	.31151	.95024	.32804	.94466	51
10	.26163	.96517	.27843	.96046	.29515	.95545	.31178	.95015	.32832	.94457	50
11	.26191	.96509	.27871	.96037	.29543	.95536	.31206	.95006	.32859	.94447	49
12	.26219	.96502	.27899	.96029	.29571	.95528	.31233	.94997	.32887	.94438	48
13	.26247	.96494	.27927	.96021	.29599	.95519	.31261	.94988	.32914	.94428	47
14	.26275	.96486	.27955	.96013	.29626	.95511	.31289	.94979	.32942	.94418	46
15	.26303	.96479	.27983	.96005	.29654	.95502	.31316	.94970	.32969	.94409	45
16	.26331	.96471	.28011	.95997	.29682	.95493	.31344	.94961	.32997	.94399	44
17	.26359	.96463	.28039	.95989	.29710	.95485	.31372	.94952	.33024	.94390	43
18	.26387	.96456	.28067	.95981	.29737	.95476	.31399	.94943	.33051	.94380	42
19	.26415	.96448	.28095	.95972	.29765	.95467	.31427	.94933	.33079	.94370	41
20	.26443	.96440	.28123	.95964	.29793	.95459	.31454	.94924	.33106	.94361	40
21	.26471	.96433	.28150	.95956	.29821	.95450	.31482	.94915	.33134	.94351	39
22	.26500	.96425	.28178	.95948	.29849	.95441	.31510	.94906	.33161	.94342	38
23	.26528	.96417	.28206	.95940	.29876	.95433	.31537	.94897	.33189	.94332	37
24	.26556	.96410	.28234	.95931	.29904	.95424	.31565	.94888	.33216	.94322	36
25	.26584	.96402	.28262	.95923	.29932	.95415	.31593	.94878	.33244	.94313	35
26	.26612	.96394	.28290	.95915	.29960	.95407	.31620	.94869	.33271	.94303	34
27	.26640	.96386	.28318	.95907	.29987	.95398	.31648	.94860	.33298	.94293	33
28	.26668	.96379	.28346	.95898	.30015	.95389	.31675	.94851	.33326	.94284	32
29	.26696	.96371	.28374	.95890	.30043	.95380	.31703	.94842	.33353	.94274	31
30	.26724	.96363	.28402	.95882	.30071	.95372	.31730	.94832	.33381	.94264	30
31	.26752	.96355	.28429	.95874	.30098	.95363	.31758	.94823	.33408	.94254	29
32	.26780	.96347	.28457	.95865	.30126	.95354	.31786	.94814	.33436	.94245	28
33	.26808	.96340	.28485	.95857	.30154	.95345	.31813	.94805	.33463	.94235	27
34	.26836	.96332	.28513	.95849	.30182	.95337	.31841	.94795	.33490	.94225	26
35	.26864	.96324	.28541	.95841	.30209	.95328	.31868	.94786	.33518	.94215	25
36	.26892	.96316	.28569	.95832	.30237	.95319	.31896	.94777	.33545	.94206	24
37	.26920	.96308	.28597	.95824	.30265	.95310	.31923	.94768	.33573	.94196	23
38	.26948	.96301	.28625	.95816	.30292	.95301	.31951	.94758	.33600	.94186	22
39	.26976	.96293	.28652	.95807	.30320	.95293	.31979	.94749	.33627	.94176	21
40	.27004	.96285	.28680	.95799	.30348	.95284	.32006	.94740	.33655	.94167	20
41	.27032	.96277	.28708	.95791	.30376	.95275	.32034	.94730	.33682	.94157	19
42	.27060	.96269	.28736	.95782	.30403	.95266	.32061	.94721	.33710	.94147	18
43	.27088	.96261	.28764	.95774	.30431	.95257	.32089	.94712	.33737	.94137	17
44	.27116	.96253	.28792	.95766	.30459	.95248	.32116	.94702	.33764	.94127	16
45	.27144	.96246	.28820	.95757	.30486	.95240	.32144	.94693	.33792	.94118	15
46	.27172	.96238	.28847	.95749	.30514	.95231	.32171	.94684	.33819	.94108	14
47	.27200	.96230	.28875	.95740	.30542	.95222	.32199	.94674	.33846	.94098	13
48	.27228	.96222	.28903	.95732	.30570	.95213	.32227	.94665	.33874	.94088	12
49	.27256	.96214	.28931	.95724	.30597	.95204	.32254	.94656	.33901	.94078	11
50	.27284	.96206	.28959	.95715	.30625	.95195	.32282	.94646	.33929	.94068	10
51	.27312	.96198	.28987	.95707	.30653	.95186	.32309	.94637	.33956	.94058	9
52	.27340	.96190	.29015	.95698	.30680	.95177	.32337	.94627	.33983	.94049	8
53	.27368	.96182	.29042	.95690	.30708	.95168	.32364	.94618	.34011	.94039	7
54	.27396	.96174	.29070	.95681	.30736	.95159	.32392	.94609	.34038	.94029	6
55	.27424	.96166	.29098	.95673	.30763	.95150	.32419	.94599	.34065	.94019	5
56	.27452	.96158	.29126	.95664	.30791	.95142	.32447	.94590	.34093	.94009	4
57	.27480	.96150	.29154	.95656	.30819	.95133	.32474	.94580	.34120	.93999	3
58	.27508	.96142	.29182	.95647	.30846	.95124	.32502	.94571	.34147	.93989	2
59	.27536	.96134	.29209	.95639	.30874	.95115	.32529	.94561	.34175	.93979	1
60	.27564	.96126	.29237	.95630	.30902	.95106	.32557	.94552	.34202	.93969	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	74°		73°		72°		71°		70°		

′	20°		21°		22°		23°		24°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	.34202	.93969	.35837	.93358	.37461	.92718	.39073	.92050	.40674	.91355	60
1	.34229	.93959	.35864	.93348	.37488	.92707	.39100	.92039	.40700	.91343	59
2	.34257	.93949	.35891	.93337	.37515	.92697	.39127	.92028	.40727	.91331	58
3	.34284	.93939	.35918	.93327	.37542	.92686	.39153	.92016	.40753	.91319	57
4	.34311	.93929	.35945	.93316	.37569	.92675	.39180	.92005	.40780	.91307	56
5	.34339	.93919	.35973	.93306	.37595	.92664	.39207	.91994	.40806	.91295	55
6	.34366	.93909	.36000	.93295	.37622	.92653	.39234	.91982	.40833	.91283	54
7	.34393	.93899	.36027	.93285	.37649	.92642	.39260	.91971	.40860	.91272	53
8	.34421	.93889	.36054	.93274	.37676	.92631	.39287	.91959	.40886	.91260	52
9	.34448	.93879	.36081	.93264	.37703	.92620	.39314	.91948	.40913	.91248	51
10	.34475	.93869	.36108	.93253	.37730	.92609	.39341	.91936	.40939	.91236	50
11	.34503	.93859	.36135	.93243	.37757	.92598	.39367	.91925	.40966	.91224	49
12	.34530	.93849	.36162	.93232	.37784	.92587	.39394	.91914	.40992	.91212	48
13	.34557	.93839	.36190	.93222	.37811	.92576	.39421	.91902	.41019	.91200	47
14	.34584	.93829	.36217	.93211	.37838	.92565	.39448	.91891	.41045	.91188	46
15	.34612	.93819	.36244	.93201	.37865	.92554	.39474	.91879	.41072	.91176	45
16	.34639	.93809	.36271	.93190	.37892	.92543	.39501	.91868	.41098	.91164	44
17	.34666	.93799	.36298	.93180	.37919	.92532	.39528	.91856	.41125	.91152	43
18	.34694	.93789	.36325	.93169	.37946	.92521	.39555	.91845	.41151	.91140	42
19	.34721	.93779	.36352	.93159	.37973	.92510	.39581	.91833	.41178	.91128	41
20	.34748	.93769	.36379	.93148	.37999	.92499	.39608	.91822	.41204	.91116	40
21	.34775	.93759	.36406	.93137	.38026	.92488	.39635	.91810	.41231	.91104	39
22	.34803	.93748	.36434	.93127	.38053	.92477	.39661	.91799	.41257	.91092	38
23	.34830	.93738	.36461	.93116	.38080	.92466	.39688	.91787	.41284	.91080	37
24	.34857	.93728	.36488	.93106	.38107	.92455	.39715	.91775	.41310	.91068	36
25	.34884	.93718	.36515	.93095	.38134	.92444	.39741	.91764	.41337	.91056	35
26	.34912	.93708	.36542	.93084	.38161	.92432	.39768	.91752	.41363	.91044	34
27	.34939	.93698	.36569	.93074	.38188	.92421	.39795	.91741	.41390	.91032	33
28	.34966	.93688	.36596	.93063	.38215	.92410	.39822	.91729	.41416	.91020	32
29	.34993	.93677	.36623	.93052	.38241	.92399	.39848	.91718	.41443	.91008	31
30	.35021	.93667	.36650	.93042	.38268	.92388	.39875	.91706	.41469	.90996	30
31	.35048	.93657	.36677	.93031	.38295	.92377	.39902	.91694	.41496	.90984	29
32	.35075	.93647	.36704	.93020	.38322	.92366	.39928	.91683	.41522	.90972	28
33	.35102	.93637	.36731	.93010	.38349	.92355	.39955	.91671	.41549	.90960	27
34	.35130	.93626	.36758	.92999	.38376	.92343	.39982	.91660	.41575	.90948	26
35	.35157	.93616	.36785	.92988	.38403	.92332	.40008	.91648	.41602	.90936	25
36	.35184	.93606	.36812	.92978	.38430	.92321	.40035	.91636	.41628	.90924	24
37	.35211	.93596	.36839	.92967	.38456	.92310	.40062	.91625	.41655	.90911	23
38	.35239	.93585	.36867	.92956	.38483	.92299	.40088	.91613	.41681	.90899	22
39	.35266	.93575	.36894	.92945	.38510	.92287	.40115	.91601	.41707	.90887	21
40	.35293	.93565	.36921	.92935	.38537	.92276	.40141	.91590	.41734	.90875	20
41	.35320	.93555	.36948	.92924	.38564	.92265	.40168	.91578	.41760	.90863	19
42	.35347	.93544	.36975	.92913	.38591	.92254	.40195	.91566	.41787	.90851	18
43	.35375	.93534	.37002	.92902	.38617	.92243	.40221	.91555	.41813	.90839	17
44	.35402	.93524	.37029	.92892	.38644	.92231	.40248	.91543	.41840	.90826	16
45	.35429	.93514	.37056	.92881	.38671	.92220	.40275	.91531	.41866	.90814	15
46	.35456	.93503	.37083	.92870	.38698	.92209	.40301	.91519	.41892	.90802	14
47	.35484	.93493	.37110	.92859	.38725	.92198	.40328	.91508	.41919	.90790	13
48	.35511	.93483	.37137	.92849	.38752	.92186	.40355	.91496	.41945	.90778	12
49	.35538	.93472	.37164	.92838	.38778	.92175	.40381	.91484	.41972	.90766	11
50	.35565	.93462	.37191	.92827	.38805	.92164	.40408	.91472	.41998	.90753	10
51	.35592	.93452	.37218	.92816	.38832	.92152	.40434	.91461	.42024	.90741	9
52	.35619	.93441	.37245	.92805	.38859	.92141	.40461	.91449	.42051	.90729	8
53	.35647	.93431	.37272	.92794	.38886	.92130	.40488	.91437	.42077	.90717	7
54	.35674	.93420	.37299	.92784	.38912	.92119	.40514	.91425	.42104	.90704	6
55	.35701	.93410	.37326	.92773	.38939	.92107	.40541	.91414	.42130	.90692	5
56	.35728	.93400	.37353	.92762	.38966	.92096	.40567	.91402	.42156	.90680	4
57	.35755	.93389	.37380	.92751	.38993	.92085	.40594	.91390	.42183	.90668	3
58	.35782	.93379	.37407	.92740	.39020	.92073	.40621	.91378	.42209	.90655	2
59	.35810	.93368	.37434	.92729	.39046	.92062	.40647	.91366	.42235	.90643	1
60	.35837	.93358	.37461	.92718	.39073	.92050	.40674	.91355	.42262	.90631	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	69°		68°		67°		66°		65°		

′	25° Sine	25° Cosine	26° Sine	26° Cosine	27° Sine	27° Cosine	28° Sine	28° Cosine	29° Sine	29° Cosine	′
0	.42262	.90631	.43837	.89879	.45399	.89101	.46947	.88295	.48481	.87462	60
1	.42288	.90618	.43863	.89867	.45425	.89087	.46973	.88281	.48506	.87448	59
2	.42315	.90606	.43889	.89854	.45451	.89074	.46999	.88267	.48532	.87434	58
3	.42341	.90594	.43916	.89841	.45477	.89061	.47024	.88254	.48557	.87420	57
4	.42367	.90582	.43942	.89828	.45503	.89048	.47050	.88240	.48583	.87406	56
5	.42394	.90569	.43968	.89816	.45529	.89035	.47076	.88226	.48608	.87391	55
6	.42420	.90557	.43994	.89803	.45554	.89021	.47101	.88213	.48634	.87377	54
7	.42446	.90545	.44020	.89790	.45580	.89008	.47127	.88199	.48659	.87363	53
8	.42473	.90532	.44046	.89777	.45606	.88995	.47153	.88185	.48684	.87349	52
9	.42499	.90520	.44072	.89764	.45632	.88981	.47178	.88172	.48710	.87335	51
10	.42525	.90507	.44098	.89752	.45658	.88968	.47204	.88158	.48735	.87321	50
11	.42552	.90495	.44124	.89739	.45684	.88955	.47229	.88144	.48761	.87306	49
12	.42578	.90483	.44151	.89726	.45710	.88942	.47255	.88130	.48786	.87292	48
13	.42604	.90470	.44177	.89713	.45736	.88928	.47281	.88117	.48811	.87278	47
14	.42631	.90458	.44203	.89700	.45762	.88915	.47306	.88103	.48837	.87264	46
15	.42657	.90446	.44229	.89687	.45787	.88902	.47332	.88089	.48862	.87250	45
16	.42683	.90433	.44255	.89674	.45813	.88888	.47358	.88075	.48888	.87235	44
17	.42709	.90421	.44281	.89662	.45839	.88875	.47383	.88062	.48913	.87221	43
18	.42736	.90408	.44307	.89649	.45865	.88862	.47409	.88048	.48938	.87207	42
19	.42762	.90396	.44333	.89636	.45891	.88848	.47434	.88034	.48964	.87193	41
20	.42788	.90383	.44359	.89623	.45917	.88835	.47460	.88020	.48989	.87178	40
21	.42815	.90371	.44385	.89610	.45942	.88822	.47486	.88006	.49014	.87164	39
22	.42841	.90358	.44411	.89597	.45968	.88808	.47511	.87993	.49040	.87150	38
23	.42867	.90346	.44437	.89584	.45994	.88795	.47537	.87979	.49065	.87136	37
24	.42894	.90334	.44464	.89571	.46020	.88782	.47562	.87965	.49090	.87121	36
25	.42920	.90321	.44490	.89558	.46046	.88768	.47588	.87951	.49116	.87107	35
26	.42946	.90309	.44516	.89545	.46072	.88755	.47614	.87937	.49141	.87093	34
27	.42972	.90296	.44542	.89532	.46097	.88741	.47639	.87923	.49166	.87079	33
28	.42999	.90284	.44568	.89519	.46123	.88728	.47665	.87909	.49192	.87064	32
29	.43025	.90271	.44594	.89506	.46149	.88715	.47690	.87896	.49217	.87050	31
30	.43051	.90259	.44620	.89493	.46175	.88701	.47716	.87882	.49242	.87036	30
31	.43077	.90246	.44646	.89480	.46201	.88688	.47741	.87868	.49268	.87021	29
32	.43104	.90233	.44672	.89467	.46226	.88674	.47767	.87854	.49293	.87007	28
33	.43130	.90221	.44698	.89454	.46252	.88661	.47793	.87840	.49318	.86993	27
34	.43156	.90208	.44724	.89441	.46278	.88647	.47818	.87826	.49344	.86978	26
35	.43182	.90196	.44750	.89428	.46304	.88634	.47844	.87812	.49369	.86964	25
36	.43209	.90183	.44776	.89415	.46330	.88620	.47869	.87798	.49394	.86949	24
37	.43235	.90171	.44802	.89402	.46355	.88607	.47895	.87784	.49419	.86935	23
38	.43261	.90158	.44828	.89389	.46381	.88593	.47920	.87770	.49445	.86921	22
39	.43287	.90146	.44854	.89376	.46407	.88580	.47946	.87756	.49470	.86906	21
40	.43313	.90133	.44880	.89363	.46433	.88566	.47971	.87743	.49495	.86892	20
41	.43340	.90120	.44906	.89350	.46458	.88553	.47997	.87729	.49521	.86878	19
42	.43366	.90108	.44932	.89337	.46484	.88539	.48022	.87715	.49546	.86863	18
43	.43392	.90095	.44958	.89324	.46510	.88526	.48048	.87701	.49571	.86849	17
44	.43418	.90082	.44984	.89311	.46536	.88512	.48073	.87687	.49596	.86834	16
45	.43445	.90070	.45010	.89298	.46561	.88499	.48099	.87673	.49622	.86820	15
46	.43471	.90057	.45036	.89285	.46587	.88485	.48124	.87659	.49647	.86805	14
47	.43497	.90045	.45062	.89272	.46613	.88472	.48150	.87645	.49672	.86791	13
48	.43523	.90032	.45088	.89259	.46639	.88458	.48175	.87631	.49697	.86777	12
49	.43549	.90019	.45114	.89245	.46664	.88445	.48201	.87617	.49723	.86762	11
50	.43575	.90007	.45140	.89232	.46690	.88431	.48226	.87603	.49748	.86748	10
51	.43602	.89994	.45166	.89219	.46716	.88417	.48252	.87589	.49773	.86733	9
52	.43628	.89981	.45192	.89206	.46742	.88404	.48277	.87575	.49798	.86719	8
53	.43654	.89968	.45218	.89193	.46767	.88390	.48303	.87561	.49824	.86704	7
54	.43680	.89956	.45243	.89180	.46793	.88377	.48328	.87546	.49849	.86690	6
55	.43706	.89943	.45269	.89167	.46819	.88363	.48354	.87532	.49874	.86675	5
56	.43733	.89930	.45295	.89153	.46844	.88349	.48379	.87518	.49899	.86661	4
57	.43759	.89918	.45321	.89140	.46870	.88336	.48405	.87504	.49924	.86646	3
58	.43785	.89905	.45347	.89127	.46896	.88322	.48430	.87490	.49950	.86632	2
59	.43811	.89892	.45373	.89114	.46921	.88308	.48456	.87476	.49975	.86617	1
60	.43837	.89879	.45399	.89101	.46947	.88295	.48481	.87462	.50000	.86603	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	64°		63°		62°		61°		60°		

′	30° Sine	30° Cosine	31° Sine	31° Cosine	32° Sine	32° Cosine	33° Sine	33° Cosine	34° Sine	34° Cosine	′
0	.50000	.86603	.51504	.85717	.52992	.84805	.54464	.83867	.55919	.82904	60
1	.50025	.86588	.51529	.85702	.53017	.84789	.54488	.83851	.55943	.82887	59
2	.50050	.86573	.51554	.85687	.53041	.84774	.54513	.83835	.55968	.82871	58
3	.50076	.86559	.51579	.85672	.53066	.84759	.54537	.83819	.55992	.82855	57
4	.50101	.86544	.51604	.85657	.53091	.84743	.54561	.83804	.56016	.82839	56
5	.50126	.86530	.51628	.85642	.53115	.84728	.54586	.83788	.56040	.82822	55
6	.50151	.86515	.51653	.85627	.53140	.84712	.54610	.83772	.56064	.82806	54
7	.50176	.86501	.51678	.85612	.53164	.84697	.54635	.83756	.56088	.82790	53
8	.50201	.86486	.51703	.85597	.53189	.84681	.54659	.83740	.56112	.82773	52
9	.50227	.86471	.51728	.85582	.53214	.84666	.54683	.83724	.56136	.82757	51
10	.50252	.86457	.51753	.85567	.53238	.84650	.54708	.83708	.56160	.82741	50
11	.50277	.86442	.51778	.85551	.53263	.84635	.54732	.83692	.56184	.82724	49
12	.50302	.86427	.51803	.85536	.53288	.84619	.54756	.83676	.56208	.82708	48
13	.50327	.86413	.51828	.85521	.53312	.84604	.54781	.83660	.56232	.82692	47
14	.50352	.86398	.51852	.85506	.53337	.84588	.54805	.83645	.56256	.82675	46
15	.50377	.86384	.51877	.85491	.53361	.84573	.54829	.83629	.56280	.82659	45
16	.50403	.86369	.51902	.85476	.53386	.84557	.54854	.83613	.56305	.82643	44
17	.50428	.86354	.51927	.85461	.53411	.84542	.54878	.83597	.56329	.82626	43
18	.50453	.86340	.51952	.85446	.53435	.84526	.54902	.83581	.56353	.82610	42
19	.50478	.86325	.51977	.85431	.53460	.84511	.54927	.83565	.56377	.82593	41
20	.50503	.86310	.52002	.85416	.53484	.84495	.54951	.83549	.56401	.82577	40
21	.50528	.86295	.52026	.85401	.53509	.84480	.54975	.83533	.56425	.82561	39
22	.50553	.86281	.52051	.85385	.53534	.84464	.54999	.83517	.56449	.82544	38
23	.50578	.86266	.52076	.85370	.53558	.84448	.55024	.83501	.56473	.82528	37
24	.50603	.86251	.52101	.85355	.53583	.84433	.55048	.83485	.56497	.82511	36
25	.50628	.86237	.52126	.85340	.53607	.84417	.55072	.83469	.56521	.82495	35
26	.50654	.86222	.52151	.85325	.53632	.84402	.55097	.83453	.56545	.82478	34
27	.50679	.86207	.52175	.85310	.53656	.84386	.55121	.83437	.56569	.82462	33
28	.50704	.86192	.52200	.85294	.53681	.84370	.55145	.83421	.56593	.82446	32
29	.50729	.86178	.52225	.85279	.53705	.84355	.55169	.83405	.56617	.82429	31
30	.50754	.86163	.52250	.85264	.53730	.84339	.55194	.83389	.56641	.82413	30
31	.50779	.86148	.52275	.85249	.53754	.84324	.55218	.83373	.56665	.82396	29
32	.50804	.86133	.52299	.85234	.53779	.84308	.55242	.83356	.56689	.82380	28
33	.50829	.86119	.52324	.85218	.53804	.84292	.55266	.83340	.56713	.82363	27
34	.50854	.86104	.52349	.85203	.53828	.84277	.55291	.83324	.56736	.82347	26
35	.50879	.86089	.52374	.85188	.53853	.84261	.55315	.83308	.56760	.82330	25
36	.50904	.86074	.52399	.85173	.53877	.84245	.55339	.83292	.56784	.82314	24
37	.50929	.86059	.52423	.85157	.53902	.84230	.55363	.83276	.56808	.82297	23
38	.50954	.86045	.52448	.85142	.53926	.84214	.55388	.83260	.56832	.82281	22
39	.50979	.86030	.52473	.85127	.53951	.84198	.55412	.83244	.56856	.82264	21
40	.51004	.86015	.52498	.85112	.53975	.84182	.55436	.83228	.56880	.82248	20
41	.51029	.86000	.52522	.85096	.54000	.84167	.55460	.83212	.56904	.82231	19
42	.51054	.85985	.52547	.85081	.54024	.84151	.55484	.83195	.56928	.82214	18
43	.51079	.85970	.52572	.85066	.54049	.84135	.55509	.83179	.56952	.82198	17
44	.51104	.85956	.52597	.85051	.54073	.84120	.55533	.83163	.56976	.82181	16
45	.51129	.85941	.52621	.85035	.54097	.84104	.55557	.83147	.57000	.82165	15
46	.51154	.85926	.52646	.85020	.54122	.84088	.55581	.83131	.57024	.82148	14
47	.51179	.85911	.52671	.85005	.54146	.84072	.55605	.83115	.57047	.82132	13
48	.51204	.85896	.52696	.84989	.54171	.84057	.55630	.83098	.57071	.82115	12
49	.51229	.85881	.52720	.84974	.54195	.84041	.55654	.83082	.57095	.82098	11
50	.51254	.85866	.52745	.84959	.54220	.84025	.55678	.83066	.57119	.82082	10
51	.51279	.85851	.52770	.84943	.54244	.84009	.55702	.83050	.57143	.82065	9
52	.51304	.85836	.52794	.84928	.54269	.83994	.55726	.83034	.57167	.82048	8
53	.51329	.85821	.52819	.84913	.54293	.83978	.55750	.83017	.57191	.82032	7
54	.51354	.85806	.52844	.84897	.54317	.83962	.55775	.83001	.57215	.82015	6
55	.51379	.85792	.52869	.84882	.54342	.83946	.55799	.82985	.57238	.81999	5
56	.51404	.85777	.52893	.84866	.54366	.83930	.55823	.82969	.57262	.81982	4
57	.51429	.85762	.52918	.84851	.54391	.83915	.55847	.82953	.57286	.81965	3
58	.51454	.85747	.52943	.84836	.54415	.83899	.55871	.82936	.57310	.81949	2
59	.51479	.85732	.52967	.84820	.54440	.83883	.55895	.82920	.57334	.81932	1
60	.51504	.85717	.52992	.84805	.54464	.83867	.55919	.82904	.57358	.81915	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	59°		58°		57°		56°		55°		

′	35° Sine	35° Cosine	36° Sine	36° Cosine	37° Sine	37° Cosine	38° Sine	38° Cosine	39° Sine	39° Cosine	′
0	.57358	.81915	.58779	.80902	.60182	.79864	.61566	.78801	.62932	.77715	60
1	.57381	.81899	.58802	.80885	.60205	.79846	.61589	.78783	.62955	.77696	59
2	.57405	.81882	.58826	.80867	.60228	.79829	.61612	.78765	.62977	.77678	58
3	.57429	.81865	.58849	.80850	.60251	.79811	.61635	.78747	.63000	.77660	57
4	.57453	.81848	.58873	.80833	.60274	.79793	.61658	.78729	.63022	.77641	56
5	.57477	.81832	.58896	.80816	.60298	.79776	.61681	.78711	.63045	.77623	55
6	.57501	.81815	.58920	.80799	.60321	.79758	.61704	.78694	.63068	.77605	54
7	.57524	.81798	.58943	.80782	.60344	.79741	.61726	.78676	.63090	.77586	53
8	.57548	.81782	.58967	.80765	.60367	.79723	.61749	.78658	.63113	.77568	52
9	.57572	.81765	.58990	.80748	.60390	.79706	.61772	.78640	.63135	.77550	51
10	.57596	.81748	.59014	.80730	.60414	.79688	.61795	.78622	.63158	.77531	50
11	.57619	.81731	.59037	.80713	.60437	.79671	.61818	.78604	.63180	.77513	49
12	.57643	.81714	.59061	.80696	.60460	.79653	.61841	.78586	.63203	.77494	48
13	.57667	.81698	.59084	.80679	.60483	.79635	.61864	.78568	.63225	.77476	47
14	.57691	.81681	.59108	.80662	.60506	.79618	.61887	.78550	.63248	.77458	46
15	.57715	.81664	.59131	.80644	.60529	.79600	.61909	.78532	.63271	.77439	45
16	.57738	.81647	.59154	.80627	.60553	.79583	.61932	.78514	.63293	.77421	44
17	.57762	.81631	.59178	.80610	.60576	.79565	.61955	.78496	.63316	.77402	43
18	.57786	.81614	.59201	.80593	.60599	.79547	.61978	.78478	.63338	.77384	42
19	.57810	.81597	.59225	.80576	.60622	.79530	.62001	.78460	.63361	.77366	41
20	.57833	.81580	.59248	.80558	.60645	.79512	.62024	.78442	.63383	.77347	40
21	.57857	.81563	.59272	.80541	.60668	.79494	.62046	.78424	.63406	.77329	39
22	.57881	.81546	.59295	.80524	.60691	.79477	.62069	.78405	.63428	.77310	38
23	.57904	.81530	.59318	.80507	.60714	.79459	.62092	.78387	.63451	.77292	37
24	.57928	.81513	.59342	.80489	.60738	.79441	.62115	.78369	.63473	.77273	36
25	.57952	.81496	.59365	.80472	.60761	.79424	.62138	.78351	.63496	.77255	35
26	.57976	.81479	.59389	.80455	.60784	.79406	.62160	.78333	.63518	.77236	34
27	.57999	.81462	.59412	.80438	.60807	.79388	.62183	.78315	.63540	.77218	33
28	.58023	.81445	.59436	.80420	.60830	.79371	.62206	.78297	.63563	.77199	32
29	.58047	.81428	.59459	.80403	.60853	.79353	.62229	.78279	.63585	.77181	31
30	.58070	.81412	.59482	.80386	.60876	.79335	.62251	.78261	.63608	.77162	30
31	.58094	.81395	.59506	.80368	.60899	.79318	.62274	.78243	.63630	.77144	29
32	.58118	.81378	.59529	.80351	.60922	.79300	.62297	.78225	.63653	.77125	28
33	.58141	.81361	.59552	.80334	.60945	.79282	.62320	.78206	.63675	.77107	27
34	.58165	.81344	.59576	.80316	.60968	.79264	.62342	.78188	.63698	.77088	26
35	.58189	.81327	.59599	.80299	.60991	.79247	.62365	.78170	.63720	.77070	25
36	.58212	.81310	.59622	.80282	.61015	.79229	.62388	.78152	.63742	.77051	24
37	.58236	.81293	.59646	.80264	.61038	.79211	.62411	.78134	.63765	.77033	23
38	.58260	.81276	.59669	.80247	.61061	.79193	.62433	.78116	.63787	.77014	22
39	.58283	.81259	.59693	.80230	.61084	.79176	.62456	.78098	.63810	.76996	21
40	.58307	.81242	.59716	.80212	.61107	.79158	.62479	.78079	.63832	.76977	20
41	.58330	.81225	.59739	.80195	.61130	.79140	.62502	.78061	.63854	.76959	19
42	.58354	.81208	.59763	.80178	.61153	.79122	.62524	.78043	.63877	.76940	18
43	.58378	.81191	.59786	.80160	.61176	.79105	.62547	.78025	.63899	.76921	17
44	.58401	.81174	.59809	.80143	.61199	.79087	.62570	.78007	.63922	.76903	16
45	.58425	.81157	.59832	.80125	.61222	.79069	.62592	.77988	.63944	.76884	15
46	.58449	.81140	.59856	.80108	.61245	.79051	.62615	.77970	.63966	.76866	14
47	.58472	.81123	.59879	.80091	.61268	.79033	.62638	.77952	.63989	.76847	13
48	.58496	.81106	.59902	.80073	.61291	.79016	.62660	.77934	.64011	.76828	12
49	.58519	.81089	.59926	.80056	.61314	.78998	.62683	.77916	.64033	.76810	11
50	.58543	.81072	.59949	.80038	.61337	.78980	.62706	.77897	.64056	.76791	10
51	.58567	.81055	.59972	.80021	.61360	.78962	.62728	.77879	.64078	.76772	9
52	.58590	.81038	.59995	.80003	.61383	.78944	.62751	.77861	.64100	.76754	8
53	.58614	.81021	.60019	.79986	.61406	.78926	.62774	.77843	.64123	.76735	7
54	.58637	.81004	.60042	.79968	.61429	.78908	.62796	.77824	.64145	.76717	6
55	.58661	.80987	.60065	.79951	.61451	.78891	.62819	.77806	.64167	.76698	5
56	.58684	.80970	.60089	.79934	.61474	.78873	.62842	.77788	.64190	.76679	4
57	.58708	.80953	.60112	.79916	.61497	.78855	.62864	.77769	.64212	.76661	3
58	.58731	.80936	.60135	.79899	.61520	.78837	.62887	.77751	.64234	.76642	2
59	.58755	.80919	.60158	.79881	.61543	.78819	.62909	.77733	.64256	.76623	1
60	.58779	.80902	.60182	.79864	.61566	.78801	.62932	.77715	.64279	.76604	0
′	54° Cosine	54° Sine	53° Cosine	53° Sine	52° Cosine	52° Sine	51° Cosine	51° Sine	50° Cosine	50° Sine	′

′	40°		41°		42°		43°		44°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	.64279	.76604	.65606	.75471	.66913	.74314	.68200	.73135	.69466	.71934	60
1	.64301	.76586	.65628	.75452	.66935	.74295	.68221	.73116	.69487	.71914	59
2	.64323	.76567	.65650	.75433	.66956	.74276	.68242	.73096	.69508	.71894	58
3	.64346	.76548	.65672	.75414	.66978	.74256	.68264	.73076	.69529	.71873	57
4	.64368	.76530	.65694	.75395	.66999	.74237	.68285	.73056	.69549	.71853	56
5	.64390	.76511	.65716	.75375	.67021	.74217	.68306	.73036	.69570	.71833	55
6	.64412	.76492	.65738	.75356	.67043	.74198	.68327	.73016	.69591	.71813	54
7	.64435	.76473	.65759	.75337	.67064	.74178	.68349	.72996	.69612	.71792	53
8	.64457	.76455	.65781	.75318	.67086	.74159	.68370	.72976	.69633	.71772	52
9	.64479	.76436	.65803	.75299	.67107	.74139	.68391	.72957	.69654	.71752	51
10	.64501	.76417	.65825	.75280	.67129	.74120	.68412	.72937	.69675	.71732	50
11	.64524	.76398	.65847	.75261	.67151	.74100	.68434	.72917	.69696	.71711	49
12	.64546	.76380	.65869	.75241	.67172	.74080	.68455	.72897	.69717	.71691	48
13	.64568	.76361	.65891	.75222	.67194	.74061	.68476	.72877	.69737	.71671	47
14	.64590	.76342	.65913	.75203	.67215	.74041	.68497	.72857	.69758	.71650	46
15	.64612	.76323	.65935	.75184	.67237	.74022	.68518	.72837	.69779	.71630	45
16	.64635	.76304	.65956	.75165	.67258	.74002	.68539	.72817	.69800	.71610	44
17	.64657	.76286	.65978	.75146	.67280	.73983	.68561	.72797	.69821	.71590	43
18	.64679	.76267	.66000	.75126	.67301	.73963	.68582	.72777	.69842	.71569	42
19	.64701	.76248	.66022	.75107	.67323	.73944	.68603	.72757	.69862	.71549	41
20	.64723	.76229	.66044	.75088	.67344	.73924	.68624	.72737	.69883	.71529	40
21	.64746	.76210	.66066	.75069	.67366	.73904	.68645	.72717	.69904	.71508	39
22	.64768	.76192	.66088	.75050	.67387	.73885	.68666	.72697	.69925	.71488	38
23	.64790	.76173	.66109	.75030	.67409	.73865	.68688	.72677	.69946	.71468	37
24	.64812	.76154	.66131	.75011	.67430	.73846	.68709	.72657	.69966	.71447	36
25	.64834	.76135	.66153	.74992	.67452	.73826	.68730	.72637	.69987	.71427	35
26	.64856	.76116	.66175	.74973	.67473	.73806	.68751	.72617	.70008	.71407	34
27	.64878	.76097	.66197	.74953	.67495	.73787	.68772	.72597	.70029	.71386	33
28	.64901	.76078	.66218	.74934	.67516	.73767	.68793	.72577	.70049	.71366	32
29	.64923	.76059	.66240	.74915	.67538	.73747	.68814	.72557	.70070	.71345	31
30	.64945	.76041	.66262	.74896	.67559	.73728	.68835	.72537	.70091	.71325	30
31	.64967	.76022	.66284	.74876	.67580	.73708	.68857	.72517	.70112	.71305	29
32	.64989	.76003	.66306	.74857	.67602	.73688	.68878	.72497	.70132	.71284	28
33	.65011	.75984	.66327	.74838	.67623	.73669	.68899	.72477	.70153	.71264	27
34	.65033	.75965	.66349	.74818	.67645	.73649	.68920	.72457	.70174	.71243	26
35	.65055	.75946	.66371	.74799	.67666	.73629	.68941	.72437	.70195	.71223	25
36	.65077	.75927	.66393	.74780	.67688	.73610	.68962	.72417	.70215	.71203	24
37	.65100	.75908	.66414	.74760	.67709	.73590	.68983	.72397	.70236	.71182	23
38	.65122	.75889	.66436	.74741	.67730	.73570	.69004	.72377	.70257	.71162	22
39	.65144	.75870	.66458	.74722	.67752	.73551	.69025	.72357	.70277	.71141	21
40	.65166	.75851	.66480	.74703	.67773	.73531	.69046	.72337	.70298	.71121	20
41	.65188	.75832	.66501	.74683	.67795	.73511	.69067	.72317	.70319	.71100	19
42	.65210	.75813	.66523	.74664	.67816	.73491	.69088	.72297	.70339	.71080	18
43	.65232	.75794	.66545	.74644	.67837	.73472	.69109	.72277	.70360	.71059	17
44	.65254	.75775	.66566	.74625	.67859	.73452	.69130	.72257	.70381	.71039	16
45	.65276	.75756	.66588	.74606	.67880	.73432	.69151	.72236	.70401	.71019	15
46	.65298	.75738	.66610	.74586	.67901	.73413	.69172	.72216	.70422	.70998	14
47	.65320	.75719	.66632	.74567	.67923	.73393	.69193	.72196	.70443	.70978	13
48	.65342	.75700	.66653	.74548	.67944	.73373	.69214	.72176	.70463	.70957	12
49	.65364	.75680	.66675	.74528	.67965	.73353	.69235	.72156	.70484	.70937	11
50	.65386	.75661	.66697	.74509	.67987	.73333	.69256	.72136	.70505	.70916	10
51	.65408	.75642	.66718	.74489	.68008	.73314	.69277	.72116	.70525	.70896	9
52	.65430	.75623	.66740	.74470	.68029	.73294	.69298	.72095	.70546	.70875	8
53	.65452	.75604	.66762	.74451	.68051	.73274	.69319	.72075	.70567	.70855	7
54	.65474	.75585	.66783	.74431	.68072	.73254	.69340	.72055	.70587	.70834	6
55	.65496	.75566	.66805	.74412	.68093	.73234	.69361	.72035	.70608	.70813	5
56	.65518	.75547	.66827	.74392	.68115	.73215	.69382	.72015	.70628	.70793	4
57	.65540	.75528	.66848	.74373	.68136	.73195	.69403	.71995	.70649	.70772	3
58	.65562	.75509	.66870	.74353	.68157	.73175	.69424	.71974	.70670	.70752	2
59	.65584	.75490	.66891	.74334	.68179	.73155	.69445	.71954	.70690	.70731	1
60	.65606	.75471	.66913	.74314	.68200	.73135	.69466	.71934	.70711	.70711	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	49°		48°		47°		46°		45°		

′	0°		1°		2°		3°		4°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.00000	Infin.	.01746	57.2900	.03492	28.6363	.05241	19.0811	.06993	14.3007	60
1	.00029	3437.75	.01775	56.3506	.03521	28.3994	.05270	18.9755	.07022	14.2411	59
2	.00058	1718.87	.01804	55.4415	.03550	28.1664	.05299	18.8711	.07051	14.1821	58
3	.00087	1145.92	.01833	54.5613	.03579	27.9372	.05328	18.7678	.07080	14.1235	57
4	.00116	859.436	.01862	53.7086	.03609	27.7117	.05357	18.6656	.07110	14.0655	56
5	.00145	687.549	.01891	52.8821	.03638	27.4899	.05387	18.5645	.07139	14.0079	55
6	.00175	572.957	.01920	52.0807	.03667	27.2715	.05416	18.4645	.07168	13.9507	54
7	.00204	491.106	.01949	51.3032	.03696	27.0566	.05445	18.3655	.07197	13.8940	53
8	.00233	429.718	.01978	50.5485	.03725	26.8450	.05474	18.2677	.07227	13.8378	52
9	.00262	381.971	.02007	49.8157	.03754	26.6367	.05503	18.1708	.07256	13.7821	51
10	.00291	343.774	.02036	49.1039	.03783	26.4316	.05533	18.0750	.07285	13.7267	50
11	.00320	312.521	.02066	48.4121	.03812	26.2296	.05562	17.9802	.07314	13.6719	49
12	.00349	286.478	.02095	47.7395	.03842	26.0307	.05591	17.8863	.07344	13.6174	48
13	.00378	264.441	.02124	47.0853	.03871	25.8348	.05620	17.7934	.07373	13.5634	47
14	.00407	245.552	.02153	46.4489	.03900	25.6418	.05649	17.7015	.07402	13.5098	46
15	.00436	229.182	.02182	45.8294	.03929	25.4517	.05678	17.6106	.07431	13.4566	45
16	.00465	214.858	.02211	45.2261	.03958	25.2644	.05708	17.5205	.07461	13.4039	44
17	.00495	202.219	.02240	44.6386	.03987	25.0798	.05737	17.4314	.07490	13.3515	43
18	.00524	190.984	.02269	44.0661	.04016	24.8978	.05766	17.3432	.07519	13.2996	42
19	.00553	180.932	.02298	43.5081	.04046	24.7185	.05795	17.2558	.07548	13.2480	41
20	.00582	171.885	.02328	42.9641	.04075	24.5418	.05824	17.1693	.07578	13.1969	40
21	.00611	163.700	.02357	42.4335	.04104	24.3675	.05854	17.0837	.07607	13.1461	39
22	.00640	156.259	.02386	41.9158	.04133	24.1957	.05883	16.9990	.07636	13.0958	38
23	.00669	149.465	.02415	41.4106	.04162	24.0263	.05912	16.9150	.07665	13.0458	37
24	.00698	143.237	.02444	40.9174	.04191	23.8593	.05941	16.8319	.07695	12.9962	36
25	.00727	137.507	.02473	40.4358	.04220	23.6945	.05970	16.7496	.07724	12.9469	35
26	.00756	132.219	.02502	39.9655	.04250	23.5321	.05999	16.6681	.07753	12.8981	34
27	.00785	127.321	.02531	39.5059	.04279	23.3718	.06029	16.5874	.07782	12.8496	33
28	.00815	122.774	.02560	39.0568	.04308	23.2137	.06058	16.5075	.07812	12.8014	32
29	.00844	118.540	.02589	38.6177	.04337	23.0577	.06087	16.4283	.07841	12.7536	31
30	.00873	114.589	.02619	38.1885	.04366	22.9038	.06116	16.3499	.07870	12.7062	30
31	.00902	110.892	.02648	37.7686	.04395	22.7519	.06145	16.2722	.07899	12.6591	29
32	.00931	107.426	.02677	37.3579	.04424	22.6020	.06175	16.1952	.07929	12.6124	28
33	.00960	104.171	.02706	36.9560	.04454	22.4541	.06204	16.1190	.07958	12.5660	27
34	.00989	101.107	.02735	36.5627	.04483	22.3081	.06233	16.0435	.07987	12.5199	26
35	.01018	98.2179	.02764	36.1776	.04512	22.1640	.06262	15.9687	.08017	12.4742	25
36	.01047	95.4895	.02793	35.8006	.04541	22.0217	.06291	15.8945	.08046	12.4288	24
37	.01076	92.9085	.02822	35.4313	.04570	21.8813	.06321	15.8211	.08075	12.3838	23
38	.01105	90.4633	.02851	35.0695	.04599	21.7426	.06350	15.7483	.08104	12.3390	22
39	.01135	88.1436	.02881	34.7151	.04628	21.6056	.06379	15.6762	.08134	12.2946	21
40	.01164	85.9398	.02910	34.3678	.04658	21.4704	.06408	15.6048	.08163	12.2505	20
41	.01193	83.8435	.02939	34.0273	.04687	21.3369	.06437	15.5340	.08192	12.2067	19
42	.01222	81.8470	.02968	33.6935	.04716	21.2049	.06467	15.4638	.08221	12.1632	18
43	.01251	79.9434	.02997	33.3662	.04745	21.0747	.06496	15.3943	.08251	12.1201	17
44	.01280	78.1263	.03026	33.0452	.04774	20.9460	.06525	15.3254	.08280	12.0772	16
45	.01309	76.3900	.03055	32.7303	.04803	20.8188	.06554	15.2571	.08309	12.0346	15
46	.01338	74.7292	.03084	32.4213	.04833	20.6932	.06584	15.1893	.08339	11.9923	14
47	.01367	73.1390	.03114	32.1181	.04862	20.5691	.06613	15.1222	.08368	11.9504	13
48	.01396	71.6151	.03143	31.8205	.04891	20.4465	.06642	15.0557	.08397	11.9087	12
49	.01425	70.1533	.03172	31.5284	.04920	20.3253	.06671	14.9898	.08427	11.8673	11
50	.01455	68.7501	.03201	31.2416	.04949	20.2056	.06700	14.9244	.08456	11.8262	10
51	.01484	67.4019	.03230	30.9599	.04978	20.0872	.06730	14.8596	.08485	11.7853	9
52	.01513	66.1055	.03259	30.6833	.05007	19.9702	.06759	14.7954	.08514	11.7448	8
53	.01542	64.8580	.03288	30.4116	.05037	19.8546	.06788	14.7317	.08544	11.7045	7
54	.01571	63.6567	.03317	30.1446	.05066	19.7403	.06817	14.6685	.08573	11.6645	6
55	.01600	62.4992	.03346	29.8823	.05095	19.6273	.06847	14.6059	.08602	11.6248	5
56	.01629	61.3829	.03376	29.6245	.05124	19.5156	.06876	14.5438	.08632	11.5853	4
57	.01658	60.3058	.03405	29.3711	.05153	19.4051	.06905	14.4823	.08661	11.5461	3
58	.01687	59.2659	.03434	29.1220	.05182	19.2959	.06934	14.4212	.08690	11.5072	2
59	.01716	58.2612	.03463	28.8771	.05212	19.1879	.06963	14.3607	.08720	11.4685	1
60	.01746	57.2900	.03492	28.6363	.05241	19.0811	.06993	14.3007	.08749	11.4301	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	89°		88°		87°		86°		85°		

′	5°		6°		7°		8°		9°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.08749	11.4301	.10510	9.51436	.12278	8.14435	.14054	7.11537	.15838	6.31375	60
1	.08778	11.3919	.10540	9.48781	.12308	8.12481	.14084	7.10038	.15868	6.30189	59
2	.08807	11.3540	.10569	9.46141	.12338	8.10536	.14113	7.08546	.15898	6.29007	58
3	.08837	11.3163	.10599	9.43515	.12367	8.08600	.14143	7.07059	.15928	6.27829	57
4	.08866	11.2789	.10628	9.40904	.12397	8.06674	14173	7.05579	.15958	6.26655	56
5	.08895	11.2417	.10657	9.38307	.12426	8.04756	.14202	7.04105	.15988	6.25486	55
6	.08925	11.2048	.10687	9.35724	.12456	8.02848	.14232	7.02637	.16017	6.24321	54
7	.08954	11.1681	.10716	9.33155	.12485	8.00948	.14262	7.01174	.16047	6.23160	53
8	.08983	11.1316	.10746	9.30599	.12515	7.99058	.14291	6.99718	.16077	6.22003	52
9	.09013	11.0954	.10775	9.28058	.12544	7.97176	.14321	6.98268	.16107	6.20851	51
10	.09042	11.0594	.10805	9.25530	.12574	7.95302	.14351	6.96823	.16137	6.19703	50
11	.09071	11.0237	.10834	9.23016	.12603	7.93438	.14381	6.95385	.16167	6.18559	49
12	.09101	10.9882	.10863	9.20516	.12633	7.91582	.14410	6.93952	.16196	6.17419	48
13	.09130	10.9529	.10893	9.18028	.12662	7.89734	.14440	6.92525	.16226	6.16283	47
14	.09159	10.9178	.10922	9.15554	.12692	7.87895	.14470	6.91104	.16256	6.15151	46
15	.09189	10.8829	.10952	9.13093	.12722	7.86064	.14499	6.89688	.16286	6.14023	45
16	.09218	10.8483	.10981	9.10646	.12751	7.84242	.14529	6.88278	.16316	6.12899	44
17	.09247	10.8139	.11011	9.08211	.12781	7.82428	.14559	6.86874	.16346	6.11779	43
18	.09277	10.7797	.11040	9.05789	.12810	7.80622	.14588	6.85475	.16376	6.10664	42
19	.09306	10.7457	.11070	9.03379	.12840	7.78825	.14618	6.84082	.16405	6.09552	41
20	.09335	10.7119	.11099	9.00983	.12869	7.77035	.14648	6.82694	.16435	6.08444	40
21	.09365	10.6783	.11128	8.98598	.12899	7.75254	.14678	6.81812	.16465	6.07340	39
22	.09394	10.6450	.11158	8.96227	.12929	7.73480	.14707	6.79936	.16495	6.06240	38
23	.09423	10.6118	.11187	8.93867	.12958	7.71715	.14737	6.78564	.16525	6.05143	37
24	.09453	10.5789	.11217	8.91520	.12988	7.69957	.14767	6.77199	.16555	6.04051	36
25	.09482	10.5462	.11246	8.89185	.13017	7.68208	.14796	6.75838	.16585	6.02962	35
26	.09511	10.5136	.11276	8.86862	.13047	7.66466	.14826	6.74483	.16615	6.01878	34
27	.09541	10.4813	.11305	8.84551	.13076	7.64732	.14856	6.73133	.16645	6.00797	33
28	.09570	10.4491	.11335	8.82252	.13106	7.63005	.14886	6.71789	.16674	5.99720	32
29	.09600	10.4172	.11364	8.79964	.13136	7.61287	.14915	6.70450	.16704	5.98646	31
30	.09629	10.3854	.11394	8.77689	.13165	7.59575	.14945	6.69116	.16734	5.97576	30
31	.09658	10.3538	.11423	8.75425	.13195	7.57872	.14975	6.67787	.16764	5.96510	29
32	.09688	10.3224	.11452	8.73172	.13224	7.56176	.15005	6.66463	.16794	5.95448	28
33	.09717	10.2913	.11482	8.70931	.13254	7.54487	.15034	6.65144	.16824	5.94390	27
34	.09746	10.2602	.11511	8.68701	.13284	7.52806	.15064	6.63831	.16854	5.93335	26
35	.09776	10.2294	.11541	8.66482	.13313	7.51132	.15094	6.62523	.16884	5.92283	25
36	.09805	10.1988	.11570	8.64275	.13343	7.49465	.15124	6.61219	.16914	5.91236	24
37	.09834	10.1683	.11600	8.62078	.13372	7.47806	.15153	6.59921	.16944	5.90191	23
38	.09864	10.1381	.11629	8.59893	.13402	7.46154	.15183	6.58627	.16974	5.89151	22
39	.09893	10.1080	.11659	8.57718	.13432	7.44509	.15213	6.57339	.17004	5.88114	21
40	.09923	10.0780	.11688	8.55555	.13461	7.42871	.15243	6.56055	.17033	5.87080	20
41	.09952	10.0483	.11718	8.53402	.13491	7.41240	.15272	6.54777	.17063	5.86051	19
42	.09981	10.0187	.11747	8.51259	.13521	7.39616	.15302	6.53503	.17093	5.85024	18
43	.10011	9.98931	.11777	8.49128	.13550	7.37999	.15332	6.52234	.17123	5.84001	17
44	.10040	9.96007	.11806	8.47007	.13580	7.36389	.15362	6.50970	.17153	5.82982	16
45	.10069	9.93101	.11836	8.44896	.13609	7.34786	.15391	6.49710	.17183	5.81966	15
46	.10099	9.90211	.11865	8.42795	.13639	7.33190	.15421	6.48456	.17213	5.80953	14
47	.10128	9.87338	.11895	8.40705	.13669	7.31600	.15451	6.47206	.17243	5.79944	13
48	.10158	9.84482	.11924	8.38625	.13698	7.30018	.15481	6.45961	.17273	5.78938	12
49	.10187	9.81641	.11954	8.36555	.13728	7.28442	.15511	6.44720	.17303	5.77936	11
50	.10216	9.78817	.11983	8.34496	.13758	7.26873	.15540	6.43484	.17333	5.76937	10
51	.10246	9.76009	.12013	8.32446	.13787	7.25310	.15570	6.42253	.17363	5.75941	9
52	.10275	9.73217	.12042	8.30406	.13817	7.23754	.15600	6.41026	.17393	5.74949	8
53	.10305	9.70441	.12072	8.28376	.13846	7.22204	.15630	6.39804	.17423	5.73960	7
54	.10334	9.67680	.12101	8.26355	.13876	7.20661	.15660	6.38587	.17453	5.72974	6
55	.10363	9.64935	.12131	8.24345	.13906	7.19125	.15689	6.37374	.17483	5.71992	5
56	.10393	9.62205	.12160	8.22344	.13935	7.17594	.15719	6.36165	.17513	5.71013	4
57	.10422	9.59490	.12190	8.20352	.13965	7.16071	.15749	6.34961	.17543	5.70037	3
58	.10452	9.56791	.12219	8.18370	.13995	7.14553	.15779	6.33761	.17573	5.69064	2
59	.10481	9.54106	.12249	8.16398	.14024	7.13042	.15809	6.32566	.17603	5.68094	1
60	.10510	9.51436	.12278	8.14435	.14054	7.11537	.15838	6.31375	.17633	5.67128	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	84°		83°		82°		81°		80°		

′	10° Tang	10° Cotang	11° Tang	11° Cotang	12° Tang	12° Cotang	13° Tang	13° Cotang	14° Tang	14° Cotang	′
0	.17633	5.67128	.19438	5.14455	.21256	4.70463	.23087	4.33148	.24933	4.01078	60
1	.17663	5.66165	.19468	5.13658	.21286	4.69791	.23117	4.32573	.24964	4.00582	59
2	.17693	5.65205	.19498	5.12862	.21316	4.69121	.23148	4.32001	.24995	4.00086	58
3	.17723	5.64248	.19529	5.12069	.21347	4.68452	.23179	4.31430	.25026	3.99592	57
4	.17753	5.63295	.19559	5.11279	.21377	4.67786	.23209	4.30860	.25056	3.99099	56
5	.17783	5.62344	.19589	5.10490	.21408	4.67121	.23240	4.30291	.25087	3.98607	55
6	.17813	5.61397	.19619	5.09704	.21438	4.66458	.23271	4.29724	.25118	3.98117	54
7	.17843	5.60452	.19649	5.08921	.21469	4.65797	.23301	4.29159	.25149	3.97627	53
8	.17873	5.59511	.19680	5.08139	.21499	4.65138	.23332	4.28595	.25180	3.97139	52
9	.17903	5.58573	.19710	5.07360	.21529	4.64480	.23363	4.28032	.25211	3.96651	51
10	.17933	5.57638	.19740	5.06584	.21560	4.63825	.23393	4.27471	.25242	3.96165	50
11	.17963	5.56706	.19770	5.05809	.21590	4.63171	.23424	4.26911	.25273	3.95680	49
12	.17993	5.55777	.19801	5.05037	.21621	4.62518	.23455	4.26352	.25304	3.95196	48
13	.18023	5.54851	.19831	5.04267	.21651	4.61868	.23485	4.25795	.25335	3.94713	47
14	.18053	5.53927	.19861	5.03499	.21682	4.61219	.23516	4.25239	.25366	3.94232	46
15	.18083	5.53007	.19891	5.02734	.21712	4.60572	.23547	4.24685	.25397	3.93751	45
16	.18113	5.52090	.19921	5.01971	.21743	4.59927	.23578	4.24132	.25428	3.93271	44
17	.18143	5.51176	.19952	5.01210	.21773	4.59283	.23608	4.23580	.25459	3.92793	43
18	.18173	5.50264	.19982	5.00451	.21804	4.58641	.23639	4.23030	.25490	3.92316	42
19	.18203	5.49356	.20012	4.99695	.21834	4.58001	.23670	4.22481	.25521	3.91839	41
20	.18233	5.48451	.20042	4.98940	.21864	4.57363	.23700	4.21933	.25552	3.91364	40
21	.18263	5.47548	.20073	4.98188	.21895	4.56726	.23731	4.21387	.25583	3.90890	39
22	.18293	5.46648	.20103	4.97438	.21925	4.56091	.23762	4.20842	.25614	3.90417	38
23	.18323	5.45751	.20133	4.96690	.21956	4.55458	.23793	4.20298	.25645	3.89945	37
24	.18353	5.44857	.20164	4.95945	.21986	4.54826	.23823	4.19756	.25676	3.89474	36
25	.18384	5.43966	.20194	4.95201	.22017	4.54196	.23854	4.19215	.25707	3.89004	35
26	.18414	5.43077	.20224	4.94460	.22047	4.53568	.23885	4.18675	.25738	3.88536	34
27	.18444	5.42192	.20254	4.93721	.22078	4.52941	.23916	4.18137	.25769	3.88068	33
28	.18474	5.41309	.20285	4.92984	.22108	4.52316	.23946	4.17600	.25800	3.87601	32
29	.18504	5.40429	.20315	4.92249	.22139	4.51693	.23977	4.17064	.25831	3.87136	31
30	.18534	5.39552	.20345	4.91516	.22169	4.51071	.24008	4.16530	.25862	3.86671	30
31	.18564	5.38677	.20376	4.90785	.22200	4.50451	.24039	4.15997	.25893	3.86208	29
32	.18594	5.37805	.20406	4.90056	.22231	4.49832	.24069	4.15465	.25924	3.85745	28
33	.18624	5.36936	.20436	4.89330	.22261	4.49215	.24100	4.14934	.25955	3.85284	27
34	.18654	5.36070	.20466	4.88605	.22292	4.48600	.24131	4.14405	.25986	3.84824	26
35	.18684	5.35206	.20497	4.87882	.22322	4.47986	.24162	4.13877	.26017	3.84364	25
36	.18714	5.34345	.20527	4.87162	.22353	4.47374	.24193	4.13350	.26048	3.83906	24
37	.18745	5.33487	.20557	4.86444	.22383	4.46764	.24223	4.12825	.26079	3.83449	23
38	.18775	5.32631	.20588	4.85727	.22414	4.46155	.24254	4.12301	.26110	3.82992	22
39	.18805	5.31778	.20618	4.85013	.22444	4.45548	.24285	4.11778	.26141	3.82537	21
40	.18835	5.30928	.20648	4.84300	.22475	4.44942	.24316	4.11256	.26172	3.82083	20
41	.18865	5.30080	.20679	4.83590	.22505	4.44338	.24347	4.10736	.26203	3.81630	19
42	.18895	5.29235	.20709	4.82882	.22536	4.43735	.24377	4.10216	.26235	3.81177	18
43	.18925	5.28393	.20739	4.82175	.22567	4.43134	.24408	4.09699	.26266	3.80726	17
44	.18955	5.27553	.20770	4.81471	.22597	4.42534	.24439	4.09182	.26297	3.80276	16
45	.18986	5.26715	.20800	4.80769	.22628	4.41936	.24470	4.08666	.26328	3.79827	15
46	.19016	5.25880	.20830	4.80068	.22658	4.41340	.24501	4.08152	.26359	3.79378	14
47	.19046	5.25048	.20861	4.79370	.22689	4.40745	.24532	4.07639	.26390	3.78931	13
48	.19076	5.24218	.20891	4.78673	.22719	4.40152	.24562	4.07127	.26421	3.78485	12
49	.19106	5.23391	.20921	4.77978	.22750	4.39560	.24593	4.06616	.26452	3.78040	11
50	.19136	5.22566	.20952	4.77286	.22781	4.38969	.24624	4.06107	.26483	3.77595	10
51	.19166	5.21744	.20982	4.76595	.22811	4.38381	.24655	4.05599	.26515	3.77152	9
52	.19197	5.20925	.21013	4.75906	.22842	4.37793	.24686	4.05092	.26546	3.76709	8
53	.19227	5.20107	.21043	4.75219	.22872	4.37207	.24717	4.04586	.26577	3.76268	7
54	.19257	5.19293	.21073	4.74534	.22903	4.36623	.24747	4.04081	.26608	3.75828	6
55	.19287	5.18480	.21104	4.73851	.22934	4.36040	.24778	4.03578	.26639	3.75388	5
56	.19317	5.17671	.21134	4.73170	.22964	4.35459	.24809	4.03076	.26670	3.74950	4
57	.19347	5.16863	.21164	4.72490	.22995	4.34879	.24840	4.02574	.26701	3.74512	3
58	.19378	5.16058	.21195	4.71813	.23026	4.34300	.24871	4.02074	.26733	3.74075	2
59	.19408	5.15256	.21225	4.71137	.23056	4.33723	.24902	4.01576	.26764	3.73640	1
60	.19438	5.14455	.21256	4.70463	.23087	4.33148	.24933	4.01078	.26795	3.73205	0
′	79° Cotang	79° Tang	78° Cotang	78° Tang	77° Cotang	77° Tang	76° Cotang	76° Tang	75° Cotang	75° Tang	′

′	15°		16°		17°		18°		19°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.26795	3.73205	.28075	3.48741	.30573	3.27085	.32492	3.07768	.34433	2.90421	60
1	.26826	3.72771	.28706	3.48359	.30605	3.26745	.32524	3.07464	.34465	2.90147	59
2	.26857	3.72338	.28738	3.47977	.30637	3.26406	.32556	3.07160	.34498	2.89873	58
3	.26888	3.71907	.28769	3.47596	.30669	3.26067	.32588	3.06857	.34530	2.89600	57
4	.26920	3.71476	.28800	3.47216	.30700	3.25729	.32621	3.06554	.34563	2.89327	56
5	.26951	3.71046	.28832	3.46837	.30732	3.25392	.32653	3.06252	.34596	2.89055	55
6	.26982	3.70616	.28864	3.46458	.30764	3.25055	.32685	3.05950	.34628	2.88783	54
7	.27013	3.70188	.28895	3.46080	.30796	3.24719	.32717	3.05649	.34661	2.88511	53
8	.27044	3.69761	.28927	3.45703	.30828	3.24383	.32749	3.05349	.34693	2.88240	52
9	.27076	3.69335	.28958	3.45327	.30860	3.24049	.32782	3.05049	.34726	2.87970	51
10	.27107	3.68909	.28990	3.44951	.30891	3.23714	.32814	3.04749	.34758	2.87700	50
11	.27138	3.68485	.29021	3.44576	.30923	3.23381	.32846	3.04450	.34791	2.87430	49
12	.27169	3.68061	.29053	3.44202	.30955	3.23048	.32878	3.04152	.34824	2.87161	48
13	.27201	3.67638	.29084	3.43829	.30987	3.22715	.32911	3.03854	.34856	2.86892	47
14	.27232	3.67217	.29116	3.43456	.31019	3.22384	.32943	3.03556	.34889	2.86624	46
15	.27263	3.66796	.29147	3.43084	.31051	3.22053	.32975	3.03260	.34922	2.86356	45
16	.27294	3.66376	.29179	3.42713	.31083	3.21722	.33007	3.02963	.34954	2.86089	44
17	.27326	3.65957	.29210	3.42343	.31115	3.21392	.33040	3.02667	.34987	2.85822	43
18	.27357	3.65538	.29242	3.41973	.31147	3.21063	.33072	3.02372	.35020	2.85555	42
19	.27388	3.65121	.29274	3.41604	.31178	3.20734	.33104	3.02077	.35052	2.85289	41
20	.27419	3.64705	.29305	3.41236	.31210	3.20406	.33136	3.01783	.35085	2.85023	40
21	.27451	3.64289	.29337	3.40869	.31242	3.20079	.33169	3.01489	.35118	2.84758	39
22	.27482	3.63874	.29368	3.40502	.31274	3.19752	.33201	3.01196	.35150	2.84494	38
23	.27513	3.63461	.29400	3.40136	.31306	3.19426	.33233	3.00903	.35183	2.84229	37
24	.27545	3.63048	.29432	3.39771	.31338	3.19100	.33266	3.00611	.35216	2.83965	36
25	.27576	3.62636	.29463	3.39406	.31370	3.18775	.33298	3.00319	.35248	2.83702	35
26	.27607	3.62224	.29495	3.39042	.31402	3.18451	.33330	3.00028	.35281	2.83439	34
27	.27638	3.61814	.29526	3.38679	.31434	3.18127	.33363	2.99738	.35314	2.83176	33
28	.27670	3.61405	.29558	3.38317	.31466	3.17804	.33395	2.99447	.35346	2.82914	32
29	.27701	3.60996	.29590	3.37955	.31498	3.17481	.33427	2.99158	.35379	2.82653	31
30	.27732	3.60588	.29621	3.37594	.31530	3.17159	.33460	2.98868	.35412	2.82391	30
31	.27764	3.60181	.29653	3.37234	.31562	3.16838	.33492	2.98580	.35445	2.82130	29
32	.27795	3.59775	.29685	3.36875	.31594	3.16517	.33524	2.98292	.35477	2.81870	28
33	.27826	3.59370	.29716	3.36516	.31626	3.16197	.33557	2.98004	.35510	2.81610	27
34	.27858	3.58966	.29748	3.36158	.31658	3.15877	.33589	2.97717	.35543	2.81350	26
35	.27889	3.58562	.29780	3.35800	.31690	3.15558	.33621	2.97430	.35576	2.81091	25
36	.27921	3.58160	.29811	3.35443	.31722	3.15240	.33654	2.97144	.35608	2.80833	24
37	.27952	3.57758	.29843	3.35087	.31754	3.14922	.33686	2.96858	.35641	2.80574	23
38	.27983	3.57357	.29875	3.34732	.31786	3.14605	.33718	2.96573	.35674	2.80316	22
39	.28015	3.56957	.29006	3.34377	.31818	3.14288	.33751	2.96288	.35707	2.80059	21
40	.28046	3.56557	.29938	3.34023	.31850	3.13972	.33783	2.96004	.35740	2.79802	20
41	.28077	3.56159	.29970	3.33670	.31882	3.13656	.33816	2.95721	.35772	2.79545	19
42	.28109	3.55761	.30001	3.33317	.31914	3.13341	.33848	2.95437	.35805	2.79289	18
43	.28140	3.55364	.30033	3.32965	.31946	3.13027	.33881	2.95155	.35838	2.79033	17
44	.28172	3.54968	.30065	3.32614	.31978	3.12713	.33913	2.94872	.35871	2.78778	16
45	.28203	3.54573	.30097	3.32264	.32010	3.12400	.33945	2.94591	.35904	2 78523	15
46	.28234	3.54179	.30128	3.31914	.32042	3.12087	.33978	2.94309	.35937	2.78269	14
47	.28266	3.53785	.30160	3.31565	.32074	3.11775	.34010	2.94028	.35969	2.78014	13
48	.28297	3.53393	.30192	3.31216	.32106	3.11464	.34043	2.93748	.36002	2.77761	12
49	.28329	3.53001	.30224	3.30868	.32139	3.11153	.34075	2.93468	.36035	2.77507	11
50	.28360	3.52609	.30255	3.30521	.32171	3.10842	.34108	2.93189	.36068	2.77254	10
51	.28391	3.52219	.30287	3.30174	.32203	3.10532	.34140	2.92910	.36101	2.77002	9
52	.28423	3.51829	.30319	3.29829	.32235	3.10223	.34173	2.92632	.36134	2.76750	8
53	.28454	3.51441	.30351	3.29483	.32267	3.09914	.34205	2.92354	.36167	2.76498	7
54	.28486	3.51053	.30382	3.29139	.32299	3.09606	.34238	2.92076	.36199	2.76247	6
55	.28517	3.50666	.30414	3.28795	.32331	3.09298	.34270	2.91799	.36232	2.75996	5
56	.28549	3.50279	.30446	3.28452	.32363	3.08991	.34303	2.91523	.36265	2.75746	4
57	.28580	3.49894	.30478	3.28109	.32396	3.08685	.34335	2.91246	.36298	2.75496	3
58	.28612	3.49509	.30509	3.27767	.32428	3.08379	.34368	2.90971	.36331	2.75246	2
59	.28643	3.49125	.30541	3.27426	.32460	3.08073	.34400	2.90696	.36364	2.74997	1
60	.28675	3.48741	.30573	3.27085	.32492	3.07768	.34433	2.90421	.36397	2.74748	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	74°		73°		72°		71°		70°		

′	20° Tang	20° Cotang	21° Tang	21° Cotang	22° Tang	22° Cotang	23° Tang	23° Cotang	24° Tang	24° Cotang	′
0	.36397	2.74748	.38386	2.60509	.40403	2.47509	.42447	2.35585	.44523	2.24604	60
1	.36430	2.74499	.38420	2.60283	.40436	2.47302	.42482	2.35395	.44558	2.24428	59
2	.36463	2.74251	.38453	2.60057	.40470	2.47095	.42516	2.35205	.44593	2.24252	58
3	.36496	2.74004	.38487	2.59831	.40504	2.46888	.42551	2.35015	.44627	2.24077	57
4	.36529	2.73756	.38520	2.59606	.40538	2.46682	.42585	2.34825	.44662	2.23902	56
5	.36562	2.73509	.38553	2.59381	.40572	2.46476	.42619	2.34636	.44697	2.23727	55
6	.36595	2.73263	.38587	2.59156	.40606	2.46270	.42654	2.34447	.44732	2.23553	54
7	.36628	2.73017	.38620	2.58932	.40640	2.46065	.42688	2.34258	.44767	2.23378	53
8	.36661	2.72771	.38654	2.58708	.40674	2.45860	.42722	2.34069	.44802	2.23204	52
9	.36694	2.72526	.38687	2.58484	.40707	2.45655	.42757	2.33881	.44837	2.23030	51
10	.36727	2.72281	.38721	2.58261	.40741	2.45451	.42791	2.33693	.44872	2.22857	50
11	.36760	2.72036	.38754	2.58038	.40775	2.45246	.42826	2.33505	.44907	2.22683	49
12	.36793	2.71792	.38787	2.57815	.40809	2.45043	.42860	2.33317	.44942	2.22510	48
13	.36826	2.71548	.38821	2.57593	.40843	2.44839	.42894	2.33130	.44977	2.22337	47
14	.36859	2.71305	.38854	2.57371	.40877	2.44636	.42929	2.32943	.45012	2.22164	46
15	.36892	2.71062	.38888	2.57150	.40911	2.44433	.42963	2.32756	.45047	2.21992	45
16	.36925	2.70819	.38921	2.56928	.40945	2.44230	.42998	2.32570	.45082	2.21819	44
17	.36958	2.70577	.38955	2.56707	.40979	2.44027	.43032	2.32383	.45117	2.21647	43
18	.36991	2.70335	.38988	2.56487	.41013	2.43825	.43067	2.32197	.45152	2.21475	42
19	.37024	2.70094	.39022	2.56266	.41047	2.43623	.43101	2.32012	.45187	2.21304	41
20	.37057	2.69853	.39055	2.56046	.41081	2.43422	.43136	2.31826	.45222	2.21132	40
21	.37090	2.69612	.39089	2.55827	.41115	2.43220	.43170	2.31641	.45257	2.20961	39
22	.37123	2.69371	.39122	2.55608	.41149	2.43019	.43205	2.31456	.45292	2.20790	38
23	.37157	2.69131	.39156	2.55389	.41183	2.42819	.43230	2.31271	.45327	2.20619	37
24	.37190	2.68892	.39190	2.55170	.41217	2.42618	.43274	2.31086	.45362	2.20449	36
25	.37223	2.68653	.39223	2.54952	.41251	2.42418	.43308	2.30902	.45397	2.20278	35
26	.37256	2.68414	.39257	2.54734	.41285	2.42218	.43343	2.30718	.45432	2.20108	34
27	.37289	2.68175	.39290	2.54516	.41319	2.42019	.43378	2.30534	.45467	2.19938	33
28	.37322	2.67937	.39324	2.54299	.41353	2.41819	.43412	2.30351	.45502	2.19769	32
29	.37355	2.67700	.39357	2.54082	.41387	2.41620	.43447	2.30167	.45538	2.19599	31
30	.37388	2.67462	.39391	2.53865	.41421	2.41421	.43481	2.29984	.45573	2.19430	30
31	.37422	2.67225	.39425	2.53648	.41455	2.41223	.43516	2.29801	.45608	2.19261	29
32	.37455	2.66989	.39458	2.53432	.41490	2.41025	.43550	2.29619	.45643	2.19092	28
33	.37488	2.66752	.39492	2.53217	.41524	2.40827	.43585	2.29437	.45678	2.18923	27
34	.37521	2.66516	.39526	2.53001	.41558	2.40629	.43620	2.29254	.45713	2.18755	26
35	.37554	2.66281	.39559	2.52786	.41592	2.40432	.43654	2.29073	.45748	2.18587	25
36	.37588	2.66046	.39593	2.52571	.41626	2.40235	.43689	2.28891	.45784	2.18419	24
37	.37621	2.65811	.39626	2.52357	.41660	2.40038	.43724	2.28710	.45819	2.18251	23
38	.37654	2.65576	.39660	2.52142	.41694	2.39841	.43758	2.28528	.45854	2.18084	22
39	.37687	2.65342	.39694	2.51929	.41728	2.39645	.43793	2.28348	.45889	2.17916	21
40	.37720	2.65109	.39727	2.51715	.41763	2.39449	.43828	2.28167	.45924	2.17749	20
41	.37754	2.64875	.39761	2.51502	.41797	2.39253	.43862	2.27987	.45960	2.17582	19
42	.37787	2.64642	.39795	2.51289	.41831	2.39058	.43897	2.27806	.45995	2.17416	18
43	.37820	2.64410	.39829	2.51076	.41865	2.38863	.43932	2.27626	.46030	2.17249	17
44	.37853	2.64177	.39862	2.50864	.41899	2.38668	.43966	2.27447	.46065	2.17083	16
45	.37887	2.63945	.39896	2.50652	.41933	2.38473	.44001	2.27267	.46101	2.16917	15
46	.37920	2.63714	.39930	2.50440	.41968	2.38279	.44036	2.27088	.46136	2.16751	14
47	.37953	2.63483	.39963	2.50229	.42002	2.38084	.44071	2.26909	.46171	2.16585	13
48	.37986	2.63252	.39997	2.50018	.42036	2.37891	.44105	2.26730	.46206	2.16420	12
49	.38020	2.63021	.40031	2.49807	.42070	2.37697	.44140	2.26552	.46242	2.16255	11
50	.38053	2.62791	.40065	2.49597	.42105	2.37504	.44175	2.26374	.46277	2.16090	10
51	.38086	2.62561	.40098	2.49386	.42139	2.37311	.44210	2.26196	.46312	2.15925	9
52	.38120	2.62332	.40132	2.49177	.42173	2.37118	.44244	2.26018	.46348	2.15760	8
53	.38153	2.62103	.40166	2.48967	.42207	2.36925	.44279	2.25840	.46383	2.15596	7
54	.38186	2.61874	.40200	2.48758	.42242	2.36733	.44314	2.25663	.46418	2.15432	6
55	.38220	2.61646	.40234	2.48549	.42276	2.36541	.44349	2.25486	.46454	2.15268	5
56	.38253	2.61418	.40267	2.48340	.42310	2.36349	.44384	2.25309	.46489	2.15104	4
57	.38286	2.61190	.40301	2.48132	.42345	2.36158	.44418	2.25132	.46525	2.14940	3
58	.38320	2.60963	.40335	2.47924	.42379	2.35967	.44453	2.24956	.46560	2.14777	2
59	.38353	2.60736	.40369	2.47716	.42413	2.35776	.44488	2.24780	.46595	2.14614	1
60	.38386	2.60509	.40403	2.47509	.42447	2.35585	.44523	2.24604	.46631	2.14451	0
′	69° Cotang	69° Tang	68° Cotang	68° Tang	67° Cotang	67° Tang	66° Cotang	66° Tang	65° Cotang	65° Tang	′

′	25°		26°		27°		28°		29°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.46631	2.14451	.48773	2.05030	.50953	1.96261	.53171	1.88073	.55431	1.80405	60
1	.46666	2.14288	.48809	2.04879	.50989	1.96120	.53208	1.87941	.55469	1.80281	59
2	.46702	2.14125	.48845	2.04728	.51026	1.95979	.53246	1.87809	.55507	1.80158	58
3	.46737	2.13963	.48881	2.04577	.51063	1.95838	.53283	1.87677	.55545	1.80034	57
4	.46772	2.13801	.48917	2.04426	.51099	1.95698	.53320	1.87546	.55583	1.79911	56
5	.46808	2.13639	.48953	2.04276	.51136	1.95557	.53358	1.87415	.55621	1.79788	55
6	.46843	2.13477	.48989	2.04125	.51173	1.95417	.53395	1.87283	.55659	1.79665	54
7	.46879	2.13316	.49026	2.03975	.51209	1.95277	.53432	1.87152	.55697	1.79542	53
8	.46914	2.13154	.49062	2.03825	.51246	1.95137	.53470	1.87021	.55736	1.79419	52
9	.46950	2.12993	.49098	2.03675	.51283	1.94997	.53507	1.86891	.55774	1.79296	51
10	.46985	2.12832	.49134	2.03526	.51319	1.94858	.53545	1.86760	.55812	1.79174	50
11	.47021	2.12671	.49170	2.03376	.51356	1.94718	.53582	1.86630	.55850	1.79051	49
12	.47056	2.12511	.49206	2.03227	.51393	1.94579	.53620	1.86499	.55888	1.78929	48
13	.47092	2.12350	.49242	2.03078	.51430	1.94440	.53657	1.86369	.55926	1.78807	47
14	.47128	2.12190	.49278	2.02929	.51467	1.94301	.53694	1.86239	.55964	1.78685	46
15	.47163	2.12030	.49315	2.02780	.51503	1.94162	.53732	1.86109	.56003	1.78563	45
16	.47199	2.11871	.49351	2.02631	.51540	1.94023	.53769	1.85979	.56041	1.78441	44
17	.47234	2.11711	.49387	2.02483	.51577	1.93885	.53807	1.85850	.56079	1.78319	43
18	.47270	2.11552	.49423	2.02335	.51614	1.93746	.53844	1.85720	.56117	1.78198	42
19	.47305	2.11392	.49459	2.02187	.51651	1.93608	.53882	1.85591	.56156	1.78077	41
20	.47341	2.11233	.49495	2.02039	.51688	1.93470	.53920	1.85462	.56194	1.77955	40
21	.47377	2.11075	.49532	2.01891	.51724	1.93332	.53957	1.85333	.56232	1.77834	39
22	.47412	2.10916	.49568	2.01743	.51761	1.93195	.53995	1.85204	.56270	1.77713	38
23	.47448	2.10758	.49604	2.01596	.51798	1.93057	.54032	1.85075	.56309	1.77592	37
24	.47483	2.10600	.49640	2.01449	.51835	1.92920	.54070	1.84946	.56347	1.77471	36
25	.47519	2.10442	.49677	2.01302	.51872	1.92782	.54107	1.84818	.56385	1.77351	35
26	.47555	2.10284	.49713	2.01155	.51909	1.92645	.54145	1.84689	.56424	1.77230	34
27	.47590	2.10126	.49749	2.01008	.51946	1.92508	.54183	1.84561	.56462	1.77110	33
28	.47626	2.09969	.49786	2.00862	.51983	1.92371	.54220	1.84433	.56501	1.76990	32
29	.47662	2.09811	.49822	2.00715	.52020	1.92235	.54258	1.84305	.56539	1.76869	31
30	.47698	2.09654	.49858	2.00569	.52057	1.92098	.54296	1.84177	.56577	1.76749	30
31	.47733	2.09498	.49894	2.00423	.52094	1.91962	.54333	1.84049	.56616	1.76629	29
32	.47769	2.09341	.49931	2.00277	.52131	1.91826	.54371	1.83922	.56654	1.76510	28
33	.47805	2.09184	.49967	2.00131	.52168	1.91690	.54409	1.83794	.56693	1.76390	27
34	.47840	2.09028	.50004	1.99986	.52205	1.91554	.54446	1.83667	.56731	1.76271	26
35	.47876	2.08872	.50040	1.99841	.52242	1.91418	.54484	1.83540	.56769	1.76151	25
36	.47912	2 08716	.50076	1.99695	.52279	1.91282	.54522	1.83413	.56808	1.76032	24
37	.47948	2.08560	.50113	1.99550	.52316	1.91147	.54560	1.83286	.56846	1.75913	23
38	.47984	2.08405	.50149	1.99406	.52353	1.91012	.54597	1.83159	.56885	1.75794	22
39	48019	2.08250	.50185	1.99261	.52390	1.90876	.54635	1.83033	.56923	1.75675	21
40	.48055	2.08094	.50222	1.99116	.52427	1.90741	.54673	1.82906	.56962	1.75556	20
41	.48091	2.07939	.50258	1.98972	.52464	1.90607	.54711	1.82780	.57000	1.75437	19
42	.48127	2.07785	.50295	1.98828	.52501	1.90472	.54748	1.82654	.57039	1.75319	18
43	.48163	2.07630	.50331	1.98684	.52538	1.90337	.54786	1.82528	.57078	1.75200	17
44	.48198	2.07476	.50368	1.98540	.52575	1.90203	.54824	1.82402	.57116	1.75082	16
45	.48234	2.07321	.50404	1.98396	.52613	1.90069	.54862	1.82276	.57155	1.74964	15
46	.48270	2.07167	.50441	1.98253	.52650	1.89935	.54900	1.82150	.57193	1.74846	14
47	48306	2.07014	.50477	1.98110	.52687	1.89801	.54938	1.82025	.57232	1.74728	13
48	48342	2.06860	.50514	1.97966	.52724	1.89667	.54975	1.81899	.57271	1.74610	12
49	.48378	2.06706	.50550	1.97823	.52761	1.89533	.55013	1.81774	.57309	1.74492	11
50	.48414	2.06553	.50587	1.97681	.52798	1.89400	.55051	1.81649	.57348	1.74375	10
51	.48450	2.06400	.50623	1.97538	.52836	1.89266	.55089	1.81524	.57386	1.74257	9
52	.48486	2.06247	.50660	1.97395	.52873	1.89133	.55127	1.81399	.57425	1.74140	8
53	.48521	2.06094	.50696	1.97253	.52910	1.89000	.55165	1.81274	.57464	1.74022	7
54	.48557	2.05942	.50733	1.97111	.52947	1.88867	.55203	1.81150	.57503	1.73905	6
55	.48593	2.05790	.50769	1.96969	.52985	1.88734	.55241	1.81025	.57541	1.73788	5
56	.48629	2.05637	.50806	1.96827	.53022	1.88602	.55279	1.80901	.57580	1.73671	4
57	.48665	2.05485	.50843	1.96685	.53059	1.88469	.55317	1.80777	.57619	1.73555	3
58	.48701	2.05333	.50879	1.96544	.53096	1.88337	.55355	1.80653	.57657	1.73438	2
59	.48737	2.05182	.50916	1.96402	.53134	1.88205	.55393	1.80529	.57696	1.73321	1
60	.48773	2.05030	.50953	1.96261	.53171	1.88073	.55431	1.80405	.57735	1.73205	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	64°		63°		62°		61°		60°		

′	30°		31°		32°		33°		34°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.57735	1.73205	.60086	1.66428	.62487	1.60033	.64941	1.53986	.67451	1.48256	60
1	.57774	1.73089	.60126	1.66318	.62527	1.59930	.64982	1.53888	.67493	1.48163	59
2	.57813	1.72973	.60165	1.66209	.62568	1.59826	.65024	1.53791	.67536	1.48070	58
3	.57851	1.72857	.60205	1.66099	.62608	1.59723	.65065	1.53693	.67578	1.47977	57
4	.57890	1.72741	.60245	1.65990	.62649	1.59620	.65106	1.53595	.67620	1.47885	56
5	.57929	1.72625	.60284	1.65881	.62689	1.59517	.65148	1.53497	.67663	1.47792	55
6	.57968	1.72509	.60324	1.65772	.62730	1.59414	.65189	1.53400	.67705	1.47699	54
7	.58007	1.72393	.60364	1.65663	.62770	1.59311	.65231	1.53302	.67748	1.47607	53
8	.58046	1.72278	.60403	1.65554	.62811	1.59208	.65272	1.53205	.67790	1.47514	52
9	.58085	1.72163	.60443	1.65445	.62852	1.59105	.65314	1.53107	.67832	1.47422	51
10	.58124	1.72047	.60483	1.65337	.62892	1.59002	.65355	1.53010	.67875	1.47330	50
11	.58162	1.71932	.60522	1.65228	.62933	1.58900	.65397	1.52913	.67917	1.47238	49
12	.58201	1.71817	.60562	1.65120	.62973	1.58797	.65438	1.52816	.67960	1.47146	48
13	.58240	1.71702	.60602	1.65011	.63014	1.58695	.65480	1.52719	.68002	1.47053	47
14	.58279	1.71588	.60642	1.64903	.63055	1.58593	.65521	1.52622	.68045	1.46962	46
15	.58318	1.71473	.60681	1.64795	.63095	1.58490	.65563	1.52525	.68088	1.46870	45
16	.58357	1.71358	.60721	1.64687	.63136	1.58388	.65604	1.52429	.68130	1.46778	44
17	.58396	1.71244	.60761	1.64579	.63177	1.58286	.65646	1.52332	.68173	1.46686	43
18	.58435	1.71129	.60801	1.64471	.63217	1.58184	.65688	1.52235	.68215	1.46595	42
19	.58474	1.71015	.60841	1.64363	.63258	1.58083	.65729	1.52139	.68258	1.46503	41
20	.58513	1.70901	.60881	1.64256	.63299	1.57981	.65771	1.52043	.68301	1.46411	40
21	.58552	1.70787	.60921	1.64148	.63340	1.57879	.65813	1.51946	.68343	1.46320	39
22	.58591	1.70673	.60960	1.64041	.63380	1.57778	.65854	1.51850	.68386	1.46229	38
23	.58631	1.70560	.61000	1.63934	.63421	1.57676	.65896	1.51754	.68429	1.46137	37
24	.58670	1.70446	.61040	1.63826	.63462	1.57575	.65938	1.51658	.68471	1.46046	36
25	.58709	1.70332	.61080	1.63719	.63503	1.57474	.65980	1.51562	.68514	1.45955	35
26	.58748	1.70219	.61120	1.63612	.63544	1.57372	.66021	1.51466	.68557	1.45864	34
27	.58787	1.70106	.61160	1.63505	.63584	1.57271	.66063	1.51370	.68600	1.45773	33
28	.58826	1.69992	.61200	1.63398	.63625	1.57170	.66105	1.51275	.68642	1.45682	32
29	.58865	1.69879	.61240	1.63292	.63666	1.57069	.66147	1.51179	.68685	1.45592	31
30	.58905	1.69766	.61280	1.63185	.63707	1.56969	.66189	1.51084	.68728	1.45501	30
31	.58944	1.69653	.61320	1.63079	.63748	1.56868	.66230	1.50088	.68771	1.45410	29
32	.58983	1.69541	.61360	1.62972	.63789	1.56767	.66272	1.50893	.68814	1.45320	28
33	.59022	1.69428	.61400	1.62866	.63830	1.56667	.66314	1.50797	.68857	1.45220	27
34	.59061	1.69316	.61440	1.62760	.63871	1.56566	.66356	1.50702	.68900	1.45139	26
35	.59101	1.69203	.61480	1.62654	.63912	1.56466	.66398	1.50607	.68942	1.45049	25
36	.59140	1.69091	.61520	1.62548	.63953	1.56366	.66440	1.50512	.68985	1.44958	24
37	.59179	1.68979	.61561	1.62442	.63994	1.56265	.66482	1.50417	.69028	1.44868	23
38	.59218	1.68866	.61601	1.62336	.64035	1.56165	.66524	1.50322	.69071	1.44778	22
39	.59258	1.68754	.61641	1.62230	.64076	1.56065	.66566	1.50228	.69114	1.44688	21
40	.59297	1.68643	.61681	1.62125	.64117	1.55966	.66608	1.50133	.69157	1.44598	20
41	.59336	1.68531	.61721	1.62019	.64158	1.55866	.66650	1.50038	.69200	1.44508	19
42	.59376	1.68419	.61761	1.61914	.64199	1.55766	.66692	1.49944	.69243	1.44418	18
43	.59415	1.68308	.61801	1.61808	.64240	1.55666	.66734	1.49849	.69286	1.44329	17
44	.59454	1.68196	.61842	1.61703	.64281	1.55567	.66776	1.49755	.69329	1.44239	16
45	.59494	1.68085	.61882	1.61598	.64322	1.55467	.66818	1.49661	.69372	1.44149	15
46	.59533	1.67974	.61922	1.61493	.64363	1.55368	.66860	1.49566	.69416	1.44060	14
47	.59573	1.67863	.61962	1.61388	.64404	1.55269	.66902	1.49472	.69459	1.43970	13
48	.59612	1.67752	.62003	1.61283	.64446	1.55170	.66044	1.49378	.69502	1.43881	12
49	.59651	1.67641	.62043	1.61179	.64487	1.55071	.66986	1.49284	.69545	1.43792	11
50	.59691	1.67530	.62083	1.61074	.64528	1.54972	.67028	1.49190	.69588	1.43703	10
51	.59730	1.67419	.62124	1.60970	.64569	1.54873	.67071	1.49097	.69631	1.43614	9
52	.59770	1.67309	.62164	1.60865	.64610	1.54774	.67113	1.49003	.69675	1.43525	8
53	.59809	1.67198	.62204	1.60761	.64652	1.54675	.67155	1.48909	.69718	1.43436	7
54	.59849	1.67088	.62245	1.60657	.64693	1.54576	.67197	1.48816	.69761	1.43347	6
55	.59888	1.66978	.62285	1.60553	.64734	1.54478	.67239	1.48722	.69804	1.43258	5
56	.59928	1.66867	.62325	1.60449	.64775	1.54379	.67282	1.48629	.69847	1.43169	4
57	.59967	1.66757	.62366	1.60345	.64817	1.54281	.67324	1.48536	.69891	1.43080	3
58	.60007	1.66647	.62406	1.60241	.64858	1.54183	.67366	1.48442	.69934	1.42992	2
59	.60046	1.66538	.62446	1.60137	.64899	1.54085	.67409	1.48349	.69977	1.42903	1
60	.60086	1.66428	.62487	1.60033	.64941	1.53986	.67451	1.48256	.70021	1.42815	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	59°		58°		57°		56°		55°		

′	35°		36°		37°		38°		39°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.70021	1.42815	.72654	1.37638	.75355	1.32704	.78129	1.27994	.80978	1.23490	60
1	.70064	1.42726	.72699	1.37554	.75401	1.32624	.78175	1.27917	.81027	1.23416	59
2	.70107	1.42638	.72743	1.37470	.75447	1.32544	.78222	1.27841	.81075	1.23343	58
3	.70151	1.42550	.72788	1.37386	.75492	1.32464	.78269	1.27764	.81123	1.23270	57
4	.70194	1.42462	.72832	1.37302	.75538	1.32384	.78316	1.27688	.81171	1.23196	56
5	.70238	1.42374	.72877	1.37218	.75584	1.32304	.78363	1.27611	.81220	1.23123	55
6	.70281	1.42286	.72921	1.37134	.75629	1.32224	.78410	1.27535	.81268	1.23050	54
7	.70325	1.42198	.72966	1.37050	.75675	1.32144	.78457	1.27458	.81316	1.22977	53
8	.70368	1.42110	.73010	1.36967	.75721	1.32064	.78504	1.27382	.81364	1.22904	52
9	.70412	1.42022	.73055	1.36883	.75767	1.31984	.78551	1.27306	.81413	1.22831	51
10	.70455	1.41934	.73100	1.36800	.75812	1.31904	.78598	1.27230	.81461	1.22758	50
11	.70499	1.41847	.73144	1.36716	.75858	1.31825	.78645	1.27153	.81510	1.22685	49
12	.70542	1.41759	.73189	1.36633	.75904	1.31745	.78692	1.27077	.81558	1.22612	48
13	.70586	1.41672	.73234	1.36549	.75950	1.31666	.78739	1.27001	.81606	1.22539	47
14	.70629	1.41584	.73278	1.36466	.75996	1.31586	.78786	1.26925	.81655	1.22467	46
15	.70673	1.41497	.73323	1.36383	.76042	1.31507	.78834	1.26849	.81703	1.22394	45
16	.70717	1.41409	.73368	1.36300	.76088	1.31427	.78881	1.26774	.81752	1.22321	44
17	.70760	1.41322	.73413	1.36217	.76134	1.31348	.78928	1.26698	.81800	1.22249	43
18	.70804	1.41235	.73457	1.36134	.76180	1.31269	.78975	1.26622	.81849	1.22176	42
19	.70848	1.41148	.73502	1.36051	.76226	1.31190	.79022	1.26546	.81898	1.22104	41
20	.70891	1.41061	.73547	1.35968	.76272	1.31110	.79070	1.26471	.81946	1.22031	40
21	.70935	1.40974	.73592	1.35885	.76318	1.31031	.79117	1.26395	.81995	1.21959	39
22	.70979	1.40887	.73637	1.35802	.76364	1.30952	.79164	1.26319	.82044	1.21886	38
23	.71023	1.40800	.73681	1.35719	.76410	1.30873	.79212	1.26244	.82092	1.21814	37
24	.71066	1.40714	.73726	1.35637	.76456	1.30795	.79259	1.26169	.82141	1.21742	36
25	.71110	1.40627	.73771	1.35554	.76502	1.30716	.79306	1.26093	.82190	1.21670	35
26	.71154	1.40540	.73816	1.35472	.76548	1.30637	.79354	1.26018	.82238	1.21598	34
27	.71198	1.40454	.73861	1.35389	.76594	1.30558	.79401	1.25943	.82287	1.21526	33
28	.71242	1.40367	.73906	1.35307	.76640	1.30480	.79449	1.25867	.82336	1.21454	32
29	.71285	1.40281	.73951	1.35224	.76686	1.30401	.79496	1.25792	.82385	1.21382	31
30	.71329	1.40195	.73996	1.35142	.76733	1.30323	.79544	1.25717	.82434	1.21310	30
31	.71373	1.40109	.74041	1.35060	.76779	1.30244	.79591	1.25642	.82483	1.21238	29
32	.71417	1.40022	.74086	1.34978	.76825	1.30166	.79639	1.25567	.82531	1.21166	28
33	.71461	1.39936	.74131	1.34896	.76871	1.30087	.79686	1.25492	.82580	1.21094	27
34	.71505	1.39850	.74176	1.34814	.76918	1.30009	.79734	1.25417	.82629	1.21023	26
35	.71549	1.39764	.74221	1.34732	.76964	1.29931	.79781	1.25343	.82678	1.20951	25
36	.71593	1.39679	.74267	1.34650	.77010	1.29853	.79829	1.25268	.82727	1.20879	24
37	.71637	1.39593	.74312	1.34568	.77057	1.29775	.79877	1.25193	.82776	1.20808	23
38	.71681	1.39507	.74357	1.34487	.77103	1.29696	.79924	1.25118	.82825	1.20736	22
39	.71725	1.39421	.74402	1.34405	.77149	1.29618	.79972	1.25044	.82874	1.20665	21
40	.71769	1.39336	.74447	1.34323	.77196	1.29541	.80020	1.24969	.82923	1.20593	20
41	.71813	1.39250	.74492	1.34242	.77242	1.29463	.80067	1.24895	.82972	1.20522	19
42	.71857	1.39165	.74538	1.34160	.77289	1.29385	.80115	1.24820	.83022	1.20451	18
43	.71901	1.39079	.74583	1.34079	.77335	1.29307	.80163	1.24746	.83071	1.20379	17
44	.71946	1.38994	.74628	1.33998	.77382	1.29229	.80211	1.24672	.83120	1.20308	16
45	.71990	1.38909	.74674	1.33916	.77428	1.29152	.80258	1.24597	.83169	1.20237	15
46	.72034	1.38824	.74719	1.33835	.77475	1.29074	.80306	1.24523	.83218	1.20166	14
47	.72078	1.38738	.74764	1.33754	.77521	1.28997	.80354	1.24449	.83268	1.20095	13
48	.72122	1.38653	.74810	1.33673	.77568	1.28919	.80402	1.24375	.83317	1.20024	12
49	.72167	1.38568	.74855	1.33592	.77615	1.28842	.80450	1.24301	.83366	1.19953	11
50	.72211	1.38484	.74900	1.33511	.77661	1.28764	.80498	1.24227	.83415	1.19882	10
51	.72255	1.38399	.74946	1.33430	.77708	1.28687	.80546	1.24153	.83465	1.19811	9
52	.72299	1.38314	.74991	1.33349	.77754	1.28610	.80594	1.24079	.83514	1.19740	8
53	.72344	1.38229	.75037	1.33268	.77801	1.28533	.80642	1.24005	.83564	1.19669	7
54	.72388	1.38145	.75082	1.33187	.77848	1.28456	.80690	1.23931	.83613	1.19599	6
55	.72432	1.38060	.75128	1.33107	.77895	1.28379	.80738	1.23858	.83662	1.19528	5
56	.72477	1.37976	.75173	1.33026	.77941	1.28302	.80786	1.23784	.83712	1.19457	4
57	.72521	1.37891	.75219	1.32946	.77988	1.28225	.80834	1.23710	.83761	1.19387	3
58	.72565	1.37807	.75264	1.32865	.78035	1.28148	.80882	1.23637	.83811	1.19316	2
59	.72610	1.37722	.75310	1.32785	.78082	1.28071	.80930	1.23563	.83860	1.19246	1
60	.72654	1.37638	.75355	1.32704	.78129	1.27994	.80978	1.23490	.83910	1.19175	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	54°		53°		52°		51°		50°		

′	40°		41°		42°		43°		44°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.83910	1.19175	.86929	1.15037	.90040	1.11001	.93252	1.07237	.96569	1.03553	60
1	.83960	1.19105	.86980	1.14969	.90093	1.10996	.93306	1.07174	.96625	1.03493	59
2	.84009	1.19035	.87031	1.14902	.90146	1.10931	.93360	1.07112	.96681	1.03433	58
3	.84059	1.18964	.87082	1.14834	.90190	1.10867	.93415	1.07049	.96738	1.03372	57
4	.84108	1.18894	.87133	1.14767	.90251	1.10802	.93469	1.06987	.96794	1.03312	56
5	.84158	1.18824	.87184	1.14699	.90304	1.10737	.93524	1.06925	.96850	1.03252	55
6	.84208	1.18754	.87236	1.14632	.90357	1.10672	.93578	1.06862	.96907	1.03192	54
7	.84258	1.18684	.87287	1.14565	.90410	1.10607	.93633	1.06800	.96963	1.03132	53
8	.84307	1.18614	.87338	1.14498	.90463	1.10543	.93688	1.06738	.97020	1.03072	52
9	.84357	1.18544	.87389	1.14430	.90516	1.10478	.93742	1.06676	.97076	1.03012	51
10	.84407	1.18474	.87441	1.14363	.90569	1.10414	.93797	1.06613	.97133	1.02952	50
11	.84457	1.18404	.87492	1.14296	.90621	1.10349	.93852	1.06551	.97189	1.02892	49
12	.84507	1.18334	.87543	1.14229	.90674	1.10285	.93906	1.06489	.97246	1.02832	48
13	.84556	1.18264	.87595	1.14162	.90727	1.10220	.93961	1.06427	.97302	1.02772	47
14	.84606	1.18194	.87646	1.14095	.90781	1.10156	.94016	1.06365	.97359	1.02713	46
15	.84656	1.18125	.87698	1.14028	.90834	1.10091	.94071	1.06303	.97416	1.02653	45
16	.84706	1.18055	.87749	1.13961	.90887	1.10027	.94125	1.06241	.97472	1.02593	44
17	.84756	1.17986	.87801	1.13894	.90940	1.09963	.94180	1.06179	.97529	1.02533	43
18	.84806	1.17916	.87852	1.13828	.90993	1.09899	.94235	1.06117	.97586	1.02474	42
19	.84856	1.17846	.87904	1.13761	.91046	1.09834	.94290	1.06056	.97643	1.02414	41
20	.84906	1.17777	.87955	1.13694	.91099	1.09770	.94345	1.05994	.97700	1.02355	40
21	.84956	1.17708	.88007	1.13627	.91153	1.09706	.94400	1.05932	.97756	1.02295	39
22	.85006	1.17638	.88059	1.13561	.91206	1.09642	.94455	1.05870	.97813	1.02236	38
23	.85057	1.17569	.88110	1.13494	.91259	1.09578	.94510	1.05809	.97870	1.02176	37
24	.85107	1.17500	.88162	1.13428	.91313	1.09514	.94565	1.05747	.97927	1.02117	36
25	.85157	1.17430	.88214	1.13361	.91366	1.09450	.94620	1.05685	.97984	1.02057	35
26	.85207	1.17361	.88265	1.13295	.91419	1.09386	.94676	1.05624	.98041	1.01998	34
27	.85257	1.17292	.88317	1.13228	.91473	1.09322	.94731	1.05562	.98098	1.01939	33
28	.85308	1.17223	.88369	1.13162	.91526	1.09258	.94786	1.05501	.98155	1.01879	32
29	.85358	1.17154	.88421	1.13096	.91580	1.09195	.94841	1.05439	.98213	1.01820	31
30	.85408	1.17085	.88473	1.13029	.91633	1.09131	.94896	1.05378	.98270	1.01761	30
31	.85458	1.17016	.88524	1.12963	.91687	1.09067	.94952	1.05317	.98327	1.01702	29
32	.85509	1.16947	.88576	1.12897	.91740	1.09003	.95007	1.05255	.98384	1.01642	28
33	.85559	1.16878	.88628	1.12831	.91794	1.08940	.95062	1.05194	.98441	1.01583	27
34	.85609	1.16809	.88680	1.12765	.91847	1.08876	.95118	1.05133	.98499	1.01524	26
35	.85660	1.16741	.88732	1.12699	.91901	1.08813	.95173	1.05072	.98556	1.01465	25
36	.85710	1.16672	.88784	1.12633	.91955	1.08749	.95229	1.05010	.98613	1.01406	24
37	.85761	1.16603	.88836	1.12567	.92008	1.08686	.95284	1.04949	.98671	1.01347	23
38	.85811	1.16535	.88888	1.12501	.92062	1.08622	.95340	1.04888	.98728	1.01288	22
39	.85862	1.16466	.88940	1.12435	.92116	1.08559	.95395	1.04827	.98786	1.01229	21
40	.85912	1.16398	.88992	1.12369	.92170	1.08496	.95451	1.04766	.98843	1.01170	20
41	.85963	1.16329	.89045	1.12303	.92224	1.08432	.95506	1.04705	.98901	1.01112	19
42	.86014	1.16261	.89097	1.12238	.92277	1.08369	.95562	1.04644	.98958	1.01053	18
43	.86064	1.16192	.89149	1.12172	.92331	1.08306	.95618	1.04583	.99016	1.00994	17
44	.86115	1.16124	.89201	1.12106	.92385	1.08243	.95673	1.04522	.99073	1.00935	16
45	.86166	1.16056	.89253	1.12041	.92439	1.08179	.95729	1.04461	.99131	1.00876	15
46	.86216	1.15987	.89306	1.11975	.92493	1.08116	.95785	1.04401	.99189	1.00818	14
47	.86267	1.15919	.89358	1.11909	.92547	1.08053	.95841	1.04340	.99247	1.00759	13
48	.86318	1.15851	.89410	1.11844	.92601	1.07990	.95897	1.04279	.99304	1.00701	12
49	.86368	1.15783	.89463	1.11778	.92655	1.07927	.95952	1.04218	.99362	1.00642	11
50	.86419	1.15715	.89515	1.11713	.92709	1.07864	.96008	1.04158	.99420	1.00583	10
51	.86470	1.15647	.89567	1.11648	.92763	1.07801	.96064	1.04097	.99478	1.00525	9
52	.86521	1.15579	.89620	1.11582	.92817	1.07738	.96120	1.04036	.99536	1.00467	8
53	.86572	1.15511	.89672	1.11517	.92872	1.07676	.96176	1.03976	.99594	1.00408	7
54	.86623	1.15443	.89725	1.11452	.92926	1.07613	.96232	1.03915	.99652	1.00350	6
55	.86674	1.15375	.89777	1.11387	.92980	1.07550	.96288	1.03855	.99710	1.00291	5
56	.86725	1.15308	.89830	1.11321	.93034	1.07487	.96344	1.03794	.99768	1.00233	4
57	.86776	1.15240	.89883	1.11256	.93088	1.07425	.96400	1.03734	.99826	1.00175	3
58	.86827	1.15172	.89935	1.11191	.93143	1.07362	.96457	1.03674	.99884	1.00116	2
59	.86878	1.15104	.89988	1.11126	.93197	1.07299	.96513	1.03613	.99942	1.00058	1
60	.86929	1.15037	.90040	1.11061	.93252	1.07237	.96569	1.03553	1.00000	1.00000	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	49°		48°		47°		46°		45°		

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